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A distinctive Sumudu treatment of trigonometric functions

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ABSTRACT

The Sumudu transform integral equation is solved by continuous integration by parts, to obtain its definition for trigonometric functions. The transform variable, u, is included as a factor in the argument of $f(t)$, and summing the integrated coefficients evaluated at zero yields the image of trigonometric functions. The obtained result is inverted to show the expansion of trigonometric functions as an infinite series. Maple graphs, tables of extended Sumudu properties, and infinite series expansions of trigonometric functions Sumudi images are given.

1. Introduction

Integral transforms of function $f(t)$ in time t are calculated by differentiating the function continuously and summing up the series obtained using He's HPM (Homotopy Perturbation Method) for Laplace transform [1–6] in [7], Fourier transform in [8] and Sumudu transform in [9]. ADM (Adomain Decomposition Method) for Laplace transform in [10] and Sumudu transform in [11]. Without applying HPM (or) ADM, Laplace transform is calculated by differentiating the function in [12] (equation (1) in [12]) and the same was extended for wide properties of Laplace transform in [12]. Again in [12] Laplace transform of trigonometric functions are computed by integrating the function continuously and summing up the series obtained (Proposition 3 in [12]). By Laplace inverting the said proposition, the trigonometric functions are expressed as infinite series (Proposition 4 in [12]), [13]. Along with the proof of infinite series, list of trigonometric functions with their infinite series are given in [12] (Table 2 in [12]), while the properties relating the Laplace transform said proposition are given in [12] (Table 3 in [12]).

Sumudu transform the successor of Laplace transform and distinguished from other integral transforms by having unit, scale and dimension preserving properties are given with extensive list of functions with corresponding Sumudu transform in [14] (Table 3.1 [14]). Sumudu transform with number theory in [15] (Corollary 2.5, Corollary 4.4, Theorem 4.3 equation (4.14) [15]). Inverse Sumudu transform is applied to solve Bessel's differential equation in [16] along with certain relations among Laplace transform, Sumudu transform and Bessel's functions are shown in [16] (Theorem 5.1, equation (5.8) [16]). Sumudu transform application to polynomials in [17] where Sumudu transform is applied to calculate the integer zeros of Krawtchouk polynomial (Theorem 2, [17]). Applications to ordinary differential equations by Sumudu transform are shown in [18–22], to the partial differential equations in [19,23,24] and to fractional differential equations in [25–33]. Relations among Laplace, Fourier and Mellin transforms with Sumudu transform is shown in [34]. HPM with Sumudu application to solve Klein–Gordon equations given in [35,22].

Throughout this research article, the Sumudu transform integral equation is solved to compute the images of trig functions, by integration against the exponential kernel, $\exp(-t)$, with the Sumudu variable, u, being a factor in the argument.

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Table 1
Sumudu transform properties of trigonometric functions.

S.No	$f(t)$	$\mathbb{S}[f(t)]$
1	$\frac{d^i f(t)}{dt^i}$	$-\left[e^{-t}\sum_{n=0}^{\infty}\frac{f_{(n+1)}(t)}{u^{n+i+1}}\right]_0^{\infty}-\sum_{k=0}^{i-1}\frac{f^{(k)}(0)}{u^{i-k}}$
2	$\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}} f(\tau)(d\tau)^i$	$-\left[e^{-t}\sum_{n=0}^{\infty}\frac{f_{(n+1)}(t)}{u^{n-i+1}}\right]_0^{\infty}$
3	$t^m f(t)$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{(n-j)f_{(n+1)}(t)}{u^{n-m+1}}\right]_0^{\infty}$
4	$\frac{f(t)}{t^m}$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{f_{(n+1)}(t)}{(n+j)u^{n+m+1}}\right]_0^{\infty}$
5	$t^m\frac{d^i f(t)}{dt^i}$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{(n+i-j)f_{(n+1)}(t)}{u^{n+i-m+1}}\right]_0^{\infty}-\sum_{k=0}^{i-1}\frac{(k-i+j)f^{(k)}(0)}{u^{i-k+m}}$
6	$\frac{1}{t^m}\frac{d^i f(t)}{dt^i}$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{f_{(n+1)}(t)}{(n+i+j)u^{n+i+m+1}}\right]_0^{\infty}-\sum_{k=0}^{i-1}\frac{f^{(k)}(0)}{(k-i-j)u^{i-k+m}}$
7	$t^m\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}} f(\tau)(d\tau)^i$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{(n-i-j)f_{(n+1)}(t)}{u^{n-i-m+1}}\right]_0^{\infty}$
8	$\frac{1}{t^m}\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}} f(\tau)(d\tau)^i$	$\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{f_{(n+1)}(t)}{(n-i+j)u^{n-i+m+1}}\right]_0^{\infty}$
9	$\frac{d^i}{dt^i}\left(t^m f(t)\right)$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{(n-j)f_{(n+1)}(t)}{u^{n+i-m+1}}\right]_0^{\infty}-\sum_{k=0}^{i-1}\frac{t^m f^{(k)}(0)}{u^{i-k}}$
10	$\frac{d^i}{dt^i}\left(\frac{f(t)}{t^m}\right)$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{f_{(n+1)}(t)}{(n+j)u^{n+i+m+1}}\right]_0^{\infty}-\sum_{k=0}^{i-1}\frac{f^{(k)}(0)}{t^m u^{i-k}}$
11	$\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}} \tau^m f(\tau)(d\tau)^i$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=0}^{m-1}\frac{(n-j)f_{(n+1)}(t)}{u^{n-i-m+1}}\right]_0^{\infty}$
12	$\underbrace{\int_0^t\cdots\int_0^t}_{i\text{ times}} \frac{f(\tau)}{\tau^m}(d\tau)^i$	$(-1)^m\left[e^{-t}\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{f_{(n+1)}(t)}{(n+j)u^{n-i+m+1}}\right]_0^{\infty}$
13	$\int_0^t f(t-\xi)g(\xi)d\xi$	$\left[e^{-t}\sum_{n=0}^{\infty}\frac{f_{(n+1)}(t)\times g_{(n+1)}(t)}{u^{2n+1}}\right]_0^{\infty}$

Following the process in [12,13,36], by obtaining the inverse Sumudu, the trigonometric functions are expressed as infinite series where the coefficients are obtained by integrating the trigonometric functions evaluated at origin. An attemptively exhaustive list of trigonometric functions with their infinite series is given in Table 2, while some related crucial Sumudu properties in Table 1.

2. Sumudu transform of trigonometric functions

Sumudu transform of function $f(t)$ defined in the set $A=\{f(t)|\exists M,\tau_1,\tau_2>0, |f(t)|<Me^{\frac{|t|}{\tau_j}},\ \text{if}\ t\in(-1)^j\times[0,\infty)\}$ given,

$$\mathbb{S}[f(t)]=\int_0^{\infty}e^{-t}f(ut)dt=1/u\int_0^{\infty}e^{-t/u}f(t)dt;\quad u\in(-\tau_1,\tau_2). \tag{1}$$

To note that, Sumudu transform of trigonometric function such as $\cos(at)$ is calculated by applying $\cos(aut)$ in Eq. (1) and performing by parts by taking $u=\cos(aut)$, $dv=e^{-t}dt$ gives $\frac{1}{1+a^2u^2}$, which on the other hand can also be calculated by taking $u=e^{-t}$, $dv=\cos(aut)dt$ results in same Sumudu transform. Therefore letting the function $f(t)$ in Eq. (1) be trigonometric function, solving by parts, taking $u=e^{-t}$, $dv=f(ut)dt$ and substituting in the continuous integration by parts formula of Bernoulli $\sum_{n=0}^{\infty}(-1)^n u^{(n)}v_{(n+1)}$ leads to the following definition.

Definition 1. The Sumudu transform of Taylor seriezable trigonometric function $f(t)$ contained in set A defined by,

$$\mathbb{S}[f(t)]=\left[e^{-t}\sum_{n=0}^{\infty}\frac{1}{u^{n+1}}f_{(n+1)}(t)\right]_0^{\infty};\quad u\in(-\tau_1,\tau_2) \tag{2}$$

Table 2
New infinite series representation of trigonometric functions.

S.No	$f(t)$	$-\sum_{n=0}^{\infty} f_{(n+1)}(t)\vert_{t\to 0} \int_0^\infty \frac{d^n}{dt^n} J_0(2\sqrt{vt})dv$
1	$\frac{\sin\alpha t-\alpha t\cos\alpha t}{2\alpha^3}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{\alpha^{2(n+2)}} \int_0^\infty \frac{d^{2n}}{dt^{2n}} J_0(2\sqrt{vt})dv$
2	$\frac{\sin\alpha t+\alpha t\cos\alpha t}{2\alpha}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{\alpha^{2(n+2)}} \int_0^\infty \frac{d^{2(n+1)}}{dt^{2(n+1)}} J_0(2\sqrt{vt})dv$
3	$\cos\alpha t - \frac{\alpha t}{2}\sin\alpha t$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{\alpha^{2(n+2)}} \int_0^\infty \frac{d^{2n+3}}{dt^{2n+3}} J_0(2\sqrt{vt})dv$
4	$\frac{\alpha t\cosh\alpha t-\sinh\alpha t}{2\alpha^3}$	$\sum_{n=0}^{\infty} \frac{(n+1)}{\alpha^{2(n+2)}} \int_0^\infty \frac{d^{2n}}{dt^{2n}} J_0(2\sqrt{vt})dv$
5	$\frac{\sinh\alpha t+\alpha t\cosh\alpha t}{2\alpha}$	$\sum_{n=0}^{\infty} \frac{(n+1)}{\alpha^{2(n+2)}} \int_0^\infty \frac{d^{2(n+1)}}{dt^{2(n+1)}} J_0(2\sqrt{vt})dv$
6	$\cosh\alpha t + \frac{\alpha t}{2}\sinh\alpha t$	$\sum_{n=0}^{\infty} \frac{(n+1)}{\alpha^{2(n+2)}} \int_0^\infty \frac{d^{2n+3}}{dt^{2n+3}} J_0(2\sqrt{vt})dv$
7	$\frac{(3-\alpha^2t^2)\sin\alpha t-3\alpha t\cos\alpha t}{8\alpha^5}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n}}{dt^{2n}} J_0(2\sqrt{vt})dv$
8	$\frac{t\sin\alpha t-\alpha t^2\cos\alpha t}{8\alpha^3}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n+1}}{dt^{2n+1}} J_0(2\sqrt{vt})dv$
9	$\frac{(1+\alpha^2t^2)\sin\alpha t-\alpha t\cos\alpha t}{8\alpha^3}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2(n+1)}}{dt^{2(n+1)}} J_0(2\sqrt{vt})dv$
10	$\frac{3t\sin\alpha t+\alpha t^2\cos\alpha t}{8\alpha}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n+3}}{dt^{2n+3}} J_0(2\sqrt{vt})dv$
11	$\frac{(3-\alpha^2t^2)\sin\alpha t+5\alpha t\cos\alpha t}{8\alpha}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2(n+2)}}{dt^{2(n+2)}} J_0(2\sqrt{vt})dv$
12	$\frac{(8-\alpha^2t^2)\cos\alpha t-7\alpha t\sin\alpha t}{8}$	$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n+5}}{dt^{2n+5}} J_0(2\sqrt{vt})dv$
13	$(1+\alpha t)e^{\alpha t}$	$\sum_{n=0}^{\infty} \frac{(n+1)}{\alpha^{n+2}} \int_0^\infty \frac{d^{n+1}}{dt^{n+1}} J_0(2\sqrt{vt})dv$
14	$\frac{\alpha\sin\beta t-\beta\sin\alpha t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{(-1)^n\left(\alpha^{2(n+1)}-\beta^{2(n+1)}\right)}{(\alpha^2-\beta^2)(\alpha\beta)^{2n+1}} \int_0^\infty \frac{d^{2n}}{dt^{2n}} J_0(2\sqrt{vt})dv$
15	$\frac{\cos\beta t-\cos\alpha t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{(-1)^n\left(\alpha^{2(n+1)}-\beta^{2(n+1)}\right)}{(\alpha^2-\beta^2)(\alpha\beta)^{2n+1}} \int_0^\infty \frac{d^{2n+1}}{dt^{2n+1}} J_0(2\sqrt{vt})dv$
16	$\frac{\alpha\sin\alpha t-\beta\sin\beta t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{(-1)^n\left(\alpha^{2(n+1)}-\beta^{2(n+1)}\right)}{(\alpha^2-\beta^2)(\alpha\beta)^{2(n+1)}} \int_0^\infty \frac{d^{2(n+1)}}{dt^{2(n+1)}} J_0(2\sqrt{vt})dv$
17	$\frac{\alpha^2\cos\alpha t-\beta^2\cos\beta t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{(-1)^n\left(\alpha^{2(n+1)}-\beta^{2(n+1)}\right)}{(\alpha^2-\beta^2)(\alpha\beta)^{2(n+1)}} \int_0^\infty \frac{d^{2n+3}}{dt^{2n+3}} J_0(2\sqrt{vt})dv$
18	$\frac{\beta\sinh\alpha t-\alpha\sinh\beta t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{\alpha^{2(n+1)}-\beta^{2(n+1)}}{(\alpha^2-\beta^2)(\alpha\beta)^{2n+1}} \int_0^\infty \frac{d^{2n}}{dt^{2n}} J_0(2\sqrt{vt})dv$
19	$\frac{\cosh\alpha t-\cosh\beta t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{\alpha^{2(n+1)}-\beta^{2(n+1)}}{(\alpha^2-\beta^2)(\alpha\beta)^{2(n+1)}} \int_0^\infty \frac{d^{2n+1}}{dt^{2n+1}} J_0(2\sqrt{vt})dv$
20	$\frac{\alpha\sinh\alpha t-\beta\sinh\beta t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{\alpha^{2(n+1)}-\beta^{2(n+1)}}{(\alpha^2-\beta^2)(\alpha\beta)^{2(n+1)}} \int_0^\infty \frac{d^{2(n+1)}}{dt^{2(n+1)}} J_0(2\sqrt{vt})dv$
21	$\frac{\alpha^2\cosh\alpha t-\beta^2\cosh\beta t}{\alpha^2-\beta^2}$	$\sum_{n=0}^{\infty} \frac{\alpha^{2(n+1)}-\beta^{2(n+1)}}{(\alpha^2-\beta^2)(\alpha\beta)^{2(n+1)}} \int_0^\infty \frac{d^{2n+3}}{dt^{2n+3}} J_0(2\sqrt{vt})dv$
22	$\frac{\sin\alpha t\sinh\alpha t}{2\alpha^2}$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}\alpha^{4(n+1)}} \int_0^\infty \frac{d^{4n+1}}{dt^{4n+1}} J_0(2\sqrt{vt})dv$
23	$\cos\alpha t\cosh\alpha t$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}\alpha^{4(n+1)}} \int_0^\infty \frac{d^{4n+3}}{dt^{4n+3}} J_0(2\sqrt{vt})dv$
24	$\frac{(3+\alpha^2t^2)\sinh\alpha t-3\alpha t\cosh\alpha t}{8\alpha^5}$	$-\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n}}{dt^{2n}} J_0(2\sqrt{vt})dv$
25	$\frac{\alpha t^2\cosh\alpha t-t\sinh\alpha t}{8\alpha^3}$	$-\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n+1}}{dt^{2n+1}} J_0(2\sqrt{vt})dv$
26	$\frac{\alpha t\cosh\alpha t+(\alpha^2t^2-1)\sinh\alpha t}{8a^3}$	$-\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2(n+1)}}{dt^{2(n+1)}} J_0(2\sqrt{vt})dv$
27	$\frac{3t\sinh\alpha t+\alpha t^2\cosh\alpha t}{8\alpha}$	$-\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n+3}}{dt^{2n+3}} J_0(2\sqrt{vt})dv$
28	$\frac{(3+\alpha^2t^2)\sinh\alpha t+5\alpha t\cosh\alpha t}{8\alpha}$	$-\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2(n+2)}}{dt^{2(n+2)}} J_0(2\sqrt{vt})dv$
29	$\frac{(8+\alpha^2t^2)\cosh\alpha t+7\alpha t\sinh\alpha t}{8}$	$-\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2\alpha^{2(n+3)}} \int_0^\infty \frac{d^{2n+5}}{dt^{2n+5}} J_0(2\sqrt{vt})dv$
30	$\frac{1}{4\alpha^3}[\sin\alpha t\cosh\alpha t - \cos\alpha t\sinh\alpha t]$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}\alpha^{4(n+1)}} \int_0^\infty \frac{d^{4n}}{dt^{4n}} J_0(2\sqrt{vt})dv$
31	$\frac{1}{2\alpha}[\sin\alpha t\cosh\alpha t + \cos\alpha t\sinh\alpha t]$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}\alpha^{4(n+1)}} \int_0^\infty \frac{d^{4n+2}}{dt^{4n+2}} J_0(2\sqrt{vt})dv$

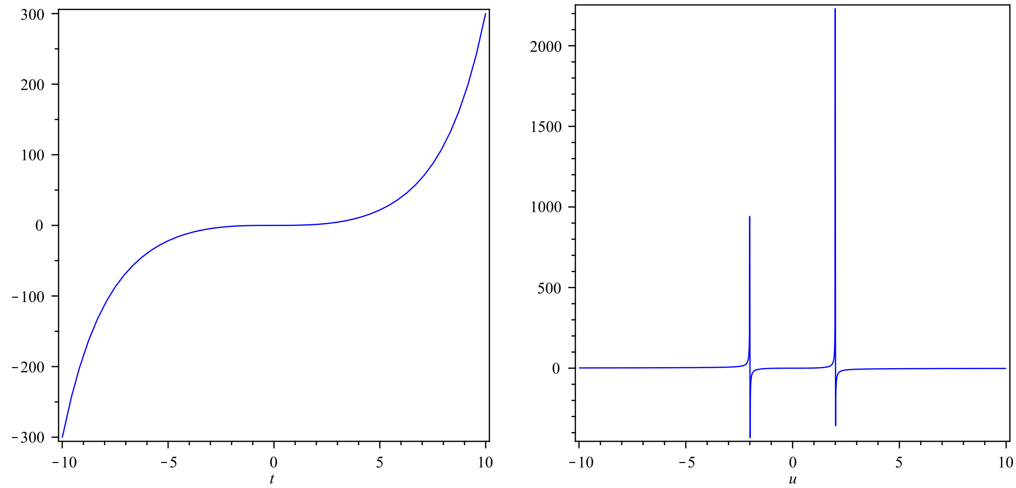
Table 2 (*continued*)

S.No	$f(t)$	$-\sum_{n=0}^{\infty} f_{(n+1)}(t)\|_{t\to 0}\int_0^{\infty}\frac{d^n}{dt^n}J_0(2\sqrt{vt})dv$
32	$\frac{1}{2\alpha^3}[\sinh\alpha t-\sin\alpha t]$	$-\sum_{n=0}^{\infty}\frac{1}{\alpha^{4(n+1)}}\int_0^{\infty}\frac{d^{4n}}{dt^{4n}}J_0(2\sqrt{vt})dv$
33	$\frac{1}{2\alpha^2}[\cosh\alpha t-\cos\alpha t]$	$-\sum_{n=0}^{\infty}\frac{1}{\alpha^{4(n+1)}}\int_0^{\infty}\frac{d^{4n+1}}{dt^{4n+1}}J_0(2\sqrt{vt})dv$
34	$\frac{1}{2\alpha}[\sinh\alpha t+\sin\alpha t]$	$-\sum_{n=0}^{\infty}\frac{1}{\alpha^{4(n+1)}}\int_0^{\infty}\frac{d^{4n+2}}{dt^{4n+2}}J_0(2\sqrt{vt})dv$
35	$\frac{1}{2}[\cosh\alpha t+\cos\alpha t]$	$-\sum_{n=0}^{\infty}\frac{1}{\alpha^{4(n+1)}}\int_0^{\infty}\frac{d^{4n+3}}{dt^{4n+3}}J_0(2\sqrt{vt})dv$
36	$\frac{e^{\frac{\alpha t}{2}}}{3\alpha^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}\alpha t}{2}-\cos\frac{\sqrt{3}\alpha t}{2}+e^{-\frac{3\alpha t}{2}}\right\}$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{3(n+1)}}\int_0^{\infty}\frac{d^{3n}}{dt^{3n}}J_0(2\sqrt{vt})dv$
37	$\frac{e^{\frac{\alpha t}{2}}}{3\alpha}\left\{\cos\frac{\sqrt{3}\alpha t}{2}+\sqrt{3}\sin\frac{\sqrt{3}\alpha t}{2}-e^{-\frac{3\alpha t}{2}}\right\}$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{3(n+1)}}\int_0^{\infty}\frac{d^{3n+1}}{dt^{3n+1}}J_0(2\sqrt{vt})dv$
38	$\frac{1}{3}\left\{e^{-\alpha t}+2e^{\frac{\alpha t}{2}}\cos\frac{\sqrt{3}\alpha t}{2}\right\}$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{3(n+1)}}\int_0^{\infty}\frac{d^{3n+2}}{dt^{3n+2}}J_0(2\sqrt{vt})dv$
39	$\frac{e^{-\frac{\alpha t}{2}}}{3\alpha^2}\left\{e^{\frac{3\alpha t}{2}}-\cos\frac{\sqrt{3}\alpha t}{2}-\sqrt{3}\sin\frac{\sqrt{3}\alpha t}{2}\right\}$	$\sum_{n=0}^{\infty}\frac{-1}{\alpha^{3(n+1)}}\int_0^{\infty}\frac{d^{3n}}{dt^{3n}}J_0(2\sqrt{vt})dv$
40	$\frac{e^{-\frac{\alpha t}{2}}}{3\alpha}\left\{\sqrt{3}\sin\frac{\sqrt{3}\alpha t}{2}-\cos\frac{\sqrt{3}\alpha t}{2}+e^{\frac{3\alpha t}{2}}\right\}$	$\sum_{n=0}^{\infty}\frac{-1}{\alpha^{3(n+1)}}\int_0^{\infty}\frac{d^{3n+1}}{dt^{3n+1}}J_0(2\sqrt{vt})dv$
41	$\frac{1}{3}\left\{e^{\alpha t}+2e^{-\frac{\alpha t}{2}}\cos\frac{\sqrt{3}\alpha t}{2}\right\}$	$\sum_{n=0}^{\infty}\frac{-1}{\alpha^{3(n+1)}}\int_0^{\infty}\frac{d^{3n+2}}{dt^{3n+2}}J_0(2\sqrt{vt})dv$
42	$\frac{e^{\beta t}-e^{\alpha t}}{\beta-\alpha};\ \alpha\neq\beta$	$-\sum_{n=0}^{\infty}\left(\frac{1}{\beta-\alpha}\right)\left(\frac{1}{\beta^{n+1}}-\frac{1}{\alpha^{n+1}}\right)\int_0^{\infty}\frac{d^n}{dt^n}J_0(2\sqrt{vt})dv$
43	$\frac{\beta e^{\beta t}-\alpha e^{\alpha t}}{\beta-\alpha};\ \alpha\neq\beta$	$-\sum_{n=0}^{\infty}\left(\frac{1}{\beta-\alpha}\right)\left(\frac{1}{\beta^{n}}-\frac{1}{\alpha^{n}}\right)\int_0^{\infty}\frac{d^n}{dt^n}J_0(2\sqrt{vt})dv$
44	$\frac{t\sin\alpha t}{2\alpha}$	$\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{\alpha^{2(n+2)}}\int_0^{\infty}\frac{d^{2n+1}}{dt^{2n+1}}J_0(2\sqrt{vt})dv$
45	$\frac{t\sinh\alpha t}{2\alpha}$	$\sum_{n=0}^{\infty}\frac{(n+1)}{\alpha^{2(n+2)}}\int_0^{\infty}\frac{d^{2n+1}}{dt^{2n+1}}J_0(2\sqrt{vt})dv$
46	$t\cos\alpha t$	$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)}{\alpha^{2(n+1)}}\int_0^{\infty}\frac{d^{2n}}{dt^{2n}}J_0(2\sqrt{vt})dv$
47	$t\cosh\alpha t$	$\sum_{n=0}^{\infty}\frac{(2n+1)}{\alpha^{2n+4}}\int_0^{\infty}\frac{d^{2n+1}}{dt^{2n+1}}J_0(2\sqrt{vt})dv$
48	$\frac{t^2\sin\alpha t}{2\alpha}$	$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+1)(2n+1)}{\alpha^{2n+4}}\int_0^{\infty}\frac{d^{2n}}{dt^{2n}}J_0(2\sqrt{vt})dv$
49	$\frac{t^2\cos\alpha t}{2}$	$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+1)(2n+3)}{\alpha^{2n+4}}\int_0^{\infty}\frac{d^{2n+1}}{dt^{2n+1}}J_0(2\sqrt{vt})dv$
50	$\frac{t^3\cos\alpha t}{6}$	$\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(4(n+1)^2-1)}{3\alpha^{2n+4}}\int_0^{\infty}\frac{d^{2n}}{dt^{2n}}J_0(2\sqrt{vt})dv$
51	$\frac{t^3\sin\alpha t}{24\alpha}$	$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+1)(n+2)(2n+3)}{6\alpha^{2n+6}}\int_0^{\infty}\frac{d^{2n+1}}{dt^{2n+1}}J_0(2\sqrt{vt})dv$
52	$\frac{t^2\sinh\alpha t}{2\alpha}$	$-\sum_{n=0}^{\infty}\frac{(n+1)(2n+1)}{\alpha^{2n+4}}\int_0^{\infty}\frac{d^{2n}}{dt^{2n}}J_0(2\sqrt{vt})dv$
53	$\frac{t^2\cosh\alpha t}{2}$	$-\sum_{n=0}^{\infty}\frac{(n+1)(2n+3)}{\alpha^{2n+4}}\int_0^{\infty}\frac{d^{2n+1}}{dt^{2n+1}}J_0(2\sqrt{vt})dv$
54	$\frac{t^3\cosh\alpha t}{6}$	$\sum_{n=0}^{\infty}\frac{(n+1)(4(n+1)^2-1)}{3\alpha^{2n+4}}\int_0^{\infty}\frac{d^{2n}}{dt^{2n}}J_0(2\sqrt{vt})dv$
55	$\frac{t^3\sinh\alpha t}{24\alpha}$	$\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(2n+3)}{6\alpha^{2n+6}}\int_0^{\infty}\frac{d^{2n+1}}{dt^{2n+1}}J_0(2\sqrt{vt})dv$
56	$\frac{e^{-t/\alpha}}{\alpha}$	$\sum_{n=0}^{\infty}(-1)^n\alpha^n\int_0^{\infty}\frac{d^n}{dt^n}J_0(2\sqrt{vt})dv$
57	$\frac{e^{-t/\alpha}-e^{-t/\beta}}{\alpha-\beta}$	$\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta}\int_0^{\infty}\frac{d^n}{dt^n}J_0(2\sqrt{vt})dv$

where

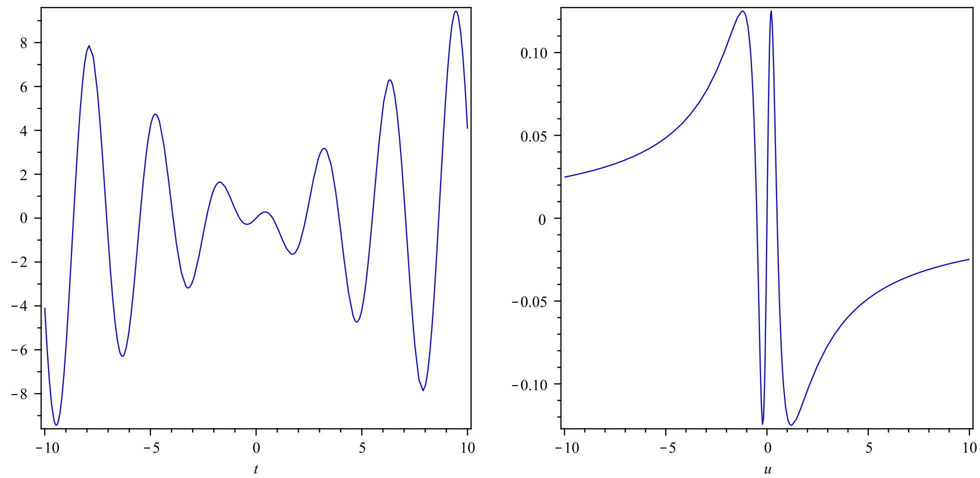
$$f_{(n+1)}(t)=\underbrace{\int\cdots\int}_{(n+1)\ \text{times}} f(t)(dt)^{n+1}. \tag{3}$$

Remark 1. Definition 1 can be obtained by applying the Laplace–Sumudu Duality (LSD) established in [18] (Theorem 1.1, equation (1.8), [18]) to the proposition in [12] (Proposition 3, [12]).



(a) $f(t) = \frac{1}{2\alpha^3}(\sinh(\alpha t) - \sin(\alpha t));\ \alpha = 0.5.$ (b) $G(u) = \frac{u^3}{1-\alpha^4 u^4};\ \alpha = 0.5.$

Fig. 1. Graph of Example 1.



(a) $f(t) = t\cos(\alpha t);\ \alpha = 2.$ (b) $G(u) = \frac{u(1-\alpha^2u^2)}{(1+\alpha^2u^2)^2};\ \alpha = 2.$

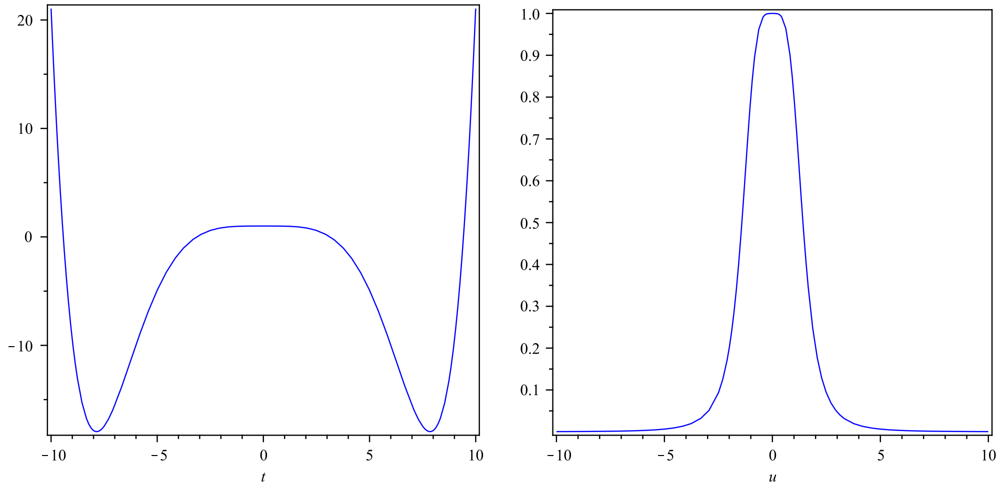
Fig. 2. Graph of Example 2.

Remark 2. From Eq. (2), the power of Sumudu transform variable u is same as order of integration of function $f(t)$ which shows Sumudu transform preserves the unit (and hence dimension) in both time and image domain which is referred to as Sumudu reciprocity defined in [24]. When the non-zero constant is factor of function $f(t)$ then transformed Sumudu image also has the same factor unlike its reciprocal as in the case of Laplace transform, thus the Sumudu transform possess the scale preserving property (Theorem 2.4, equation (2.21), [18]).

Example 1. For $f(t) = \frac{1}{2\alpha^3}(\sinh(\alpha t) - \sin(\alpha t))$ when substituted in Eq. (2), $f_{(4n+1)} = \frac{1}{\alpha^{4n+4}u^{4n+1}};\ n \geq 0$ and remaining terms are zero, the Sumudu transform is given by $\frac{u^3}{1-\alpha^4u^4}$ whose plots are shown in Fig. 1.

Example 2. For $f(t) = t\cos(\alpha t)$ when substituted in Eq. (2), $f_{(2n+1)} = \frac{(-1)^n(2n+1)}{\alpha^{2n+2}};\ n \geq 0$ and remaining terms are zero, the Sumudu transform is given by $\frac{u(1-\alpha^2u^2)}{(1+\alpha^2u^2)^2}$ and graphed in Fig. 2.

Some of the properties of Sumudu transform are given in Table 1 provided the function $f(t)$ satisfies the conditions of Section 3.



(a) $f(t) = \cos(\alpha t)\cosh(\alpha t);\ \alpha = 0.5$.

(b) $G(u) = \frac{1}{(1+4\alpha^4 u^4)};\ \alpha = 0.5$.

Fig. 3. Graph of Example 3.

3. Infinite series expansion of trigonometric functions

The necessary condition for existence of Definition 1, Eq. (2) is derived by making Eq. (2) as,

$$\mathbb{S}[f(t)] = -\sum_{n=0}^{\infty} f_{(n+1)}(t)\Big|_{t\to 0} \frac{1}{u^{n+1}}; \quad u \in (-\tau_1.\tau_2). \tag{4}$$

Inverse Sumudu transform of Eq. (4) leads to the following theorem [12].

Theorem 1. *The necessary condition for existence of Eq. (2) is, Taylor seriezable trigonometric functions expand to the infinite series,*

$$f(t) = -\sum_{n=0}^{\infty} f_{(n+1)}(t)\Big|_{t\to 0} \int_0^{\infty} \frac{d^n}{dt^n} J_0(2\sqrt{vt})dv. \tag{5}$$

Proof. Inverting both the sides of Eq. (4) by Sumudu,

$$f(t) = -\sum_{n=0}^{\infty} f_{(n+1)}(t)\Big|_{t\to 0} \mathbb{S}^{-1}\left[\frac{1}{u^{n+1}}\right]. \tag{6}$$

The inverse Laplace transform [16] of 1 is $\int_0^{\infty} J_0(2\sqrt{vt})dv$ where $J_0(2\sqrt{vt}) = \sum_{m=0}^{\infty} \frac{(-1)^m (vt)^m}{(m!)^2}$. Using the Laplace–Sumudu Duality (LSD), Sumudu transform of $\int_0^{\infty} J_0(2\sqrt{vt})dv$ is $\frac{1}{u}$. Therefore in general, following the process established in [12] and applying the LSD,

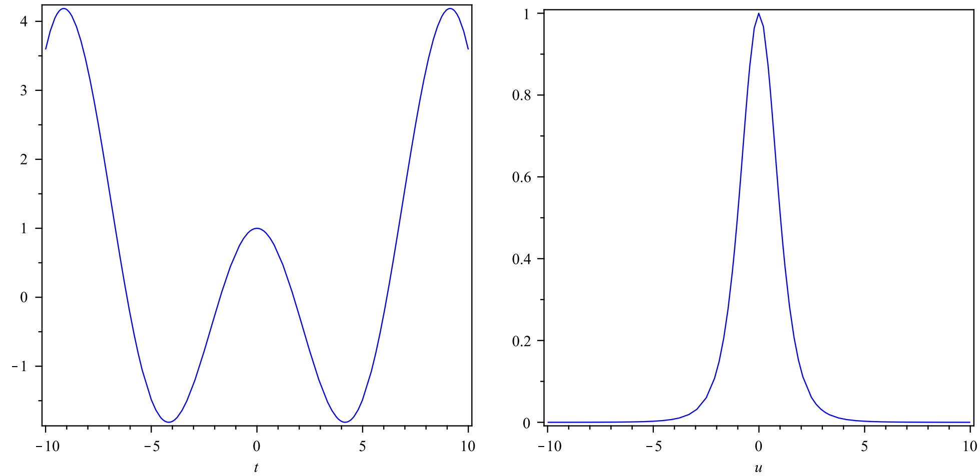
$$\mathbb{S}^{-1}\left[\frac{1}{u^{n+1}}\right] = \int_0^{\infty} \frac{d^n}{dt^n} J_0(2\sqrt{vt})dv. \tag{7}$$

Substituting Eq. (7) in Eq. (6) completes the proof. □

Remark 3. From the result $\mathbb{S}^{-1}[\frac{1}{u}] = \delta(t),\ \mathbb{S}^{-1}[\frac{1}{u^2}] = \delta^{(1)}(t),\ \mathbb{S}^{-1}[\frac{1}{u^3}] = \delta^{(2)}(t),\ \cdots, \mathbb{S}^{-1}[\frac{1}{u^{n+1}}] = \delta^{(n)}(t)$, Theorem 1 Eq. (5) can be defined as,

$$f(t) = -\sum_{n=0}^{\infty} f_{(n+1)}(t)\Big|_{t\to 0} \delta^{(n)}(t). \tag{8}$$

Infinite series for some trigonometric functions are given in Table 2.



(a) $f(t) = \frac{(8-\alpha^2t^2)\cos(\alpha t)-7\alpha t\sin(\alpha t)}{8}$; $\alpha = 0.5$.
(b) $G(u) = \frac{1}{(1+\alpha^2u^2)^3}$; $\alpha = 0.5$.

Fig. 4. Graph of Example 4.

Example 3. Substituting $f(t) = \cos(\alpha t)\cosh(\alpha t)$, in Eq. (5) and integrating continuously, $f_{(4n+4)}(t) = \frac{(-1)^{n+1}}{4^{n+1}\alpha^{4n+4}}$; $n \geq 0$, therefore,

$$\cos(\alpha t)\cosh(\alpha t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}a^{4n+4}}\int_{n=0}^{\infty}\frac{d^{4n+3}}{dt^{4n+3}}J_0(2\sqrt{vt})dv. \tag{9}$$

Now the Sumudu transform of Eq. (9) and the plot in Fig. 3 is,

$$\mathbb{S}[\cos(\alpha t)\cosh(\alpha t)] = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}a^{4n+4}}\frac{1}{u^{4n+4}} = \frac{1}{1+4(au)^4}. \tag{10}$$

Example 4. For $f(t) = \frac{(8-\alpha^2t^2)\cos(\alpha t)-7\alpha t\sin(\alpha t)}{8}$, integrating continuously gives the coefficients $f_{(2n+7)} = \frac{(-1)^{n+1}(n+1)(n+2)}{2\alpha^{2n+6}}$, therefore,

$$\frac{(8-\alpha^2t^2)\cos(\alpha t)-7\alpha t\sin(\alpha t)}{8} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2\alpha^{2n+6}}\int_0^{\infty}\frac{d^{2n+5}}{dt^{2n+5}}J_0(2\sqrt{vt})dv. \tag{11}$$

Hence, the resulting Sumudu transform of the function $f(t)$ in Eq. (11) shown in Fig. 4(a), is plotted in Fig. 4(b) and given by,

$$\mathbb{S}\left[\frac{(8-\alpha^2t^2)\cos(\alpha t)-7\alpha t\sin(\alpha t)}{8}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2(\alpha u)^{2n+6}} = \frac{1}{(1+\alpha^2u^2)^3}. \tag{12}$$

4. Conclusion

Among the obtained results it is shown that inverse Sumudu transform of a singular function satisfies the Tauberian theorem, where the dirac delta function fails. Comparison of our results with those in the literature, ensures that the proposed method in this research article, yields the sought Sumudi for trigonometric functions with a minimal number of iterations. Maple graphs show the simulative transfer between the original trig functions, and their respective transformed images.

We trust that the attemptively extensive list of trigonometric series expansions provided in Table 2, along with the Sumudu properties given in Table 1, will help open up new horizons in integral transform theory and applications. Furthermore, we hope that our results will turn out to be useful for future studies whether by us or by others. In this regard, we remain open to communications from readers, for their comments and any inquisitive queries shall only be most welcome!

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