Applied Mathematical Modelling 37 (2013) 5743–5755

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A finite difference method for singularly perturbed differential-difference equations with layer and oscillatory behavior

R. Nageshwar Rao, P. Pramod Chakravarthy *

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India

ARTICLE INFO

Article history:
Received 27 November 2011
Received in revised form 21 September 2012
Accepted 12 November 2012
Available online 14 December 2012

Keywords:
Differential-difference equation
Singular perturbations
Boundary layer
Oscillations
Finite difference method

ABSTRACT

In this paper, we present a finite difference method for singularly perturbed linear second order differential-difference equations of convection–diffusion type with a small shift, i.e., where the second order derivative is multiplied by a small parameter and the shift depends on the small parameter. Similar boundary value problems are associated with expected first-exit times of the membrane potential in models of neurons. Here, the study focuses on the effect of shift on the boundary layer behavior or oscillatory behavior of the solution via finite difference approach. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameter on the boundary layer behavior and oscillatory behavior of the solution of the problem.

1. Introduction

A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one shift or delay term. The determination of the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites can be modeled as a first-exit time problem. The case of inputs distributed as a Poisson process with exponential decay between the inputs was formulated by Stein [1] and studied by Tuckwell [2,3] and by Wilbur and Rinzel [4]. If, in addition, there are inputs that can be modeled as a Wiener process with variance parameter σ and drift parameter μ, then the problem for the expected first-exit time y, given the initial membrane potential $x \in (x_1, x_2)$, can be formulated as a general boundary-value problem for the linear second-order differential-difference equation:

$$\frac{\sigma^2}{2} y''(x) + (\mu - x)y'(x) + \lambda_E y(x + a_E) + \lambda_I y(x - a_I) - (\lambda_E + \lambda_I)y(x) = -1,$$

where the values $x = x_1$ and $x = x_2$ correspond to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively. The first-order derivative term $-xy'$ corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs modeled as Poisson processes with mean rates λ_E and λ_I, respectively, and produce jumps in the membrane potential of amounts a_E and a_I, respectively, which are small quantities and could depend on voltage. The boundary condition is $y(x) = 0$, $x \notin (x_1, x_2)$. The singular perturbation analysis of boundary-value problem for differential-difference equations with small shifts has been given by Lange and Miura [5,6]. In recent years, there has been a growing interest in the numerical study of such problems owing to its applications in areas such as neurobiology [5], optimal control theory [7,8], in the study of an optically

* Corresponding author.
E-mail address: pramodpodila@yahoo.co.in (P. Pramod Chakravarthy).

0307-904X/$ - see front matter
http://dx.doi.org/10.1016/j.apm.2012.11.004

bistable devices [9], in describing the human pupil-light reflex [10], in variety of models for physiological processes or diseases [11,12]. The numerical study of second order singularly perturbed differential-difference equation with small shift or delay has been given in [13–19] and references therein. Amiraliyev and Cimen [20] have given an exponentially fitted difference scheme on a uniform mesh for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term.

In this paper, we present a finite difference method for singularly perturbed differential-difference equations of convection–diffusion type with a small shift. When the shift parameter is smaller than the perturbation parameter, the term containing the shift is expanded in Taylor series and an exponentially fitted tridiagonal finite difference method is developed. It is analyzed for convergence. When the shift parameter is larger than perturbation parameter a special type of mesh is used, so that the term containing shift lies on nodal points after discretization and a fourth order finite difference method is applied. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameter on the boundary layer behavior and oscillatory behavior of the solution of the problem.

2. Statement of the problem

We consider a linear singularly perturbed differential-difference equation, which contains only negative shift in the convection term

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \tag{1}$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, subject to the interval and boundary conditions

$$\begin{aligned} &y(x) = \phi(x), \quad x \leqslant 0,\\ &y(1) = \beta, \end{aligned} \tag{2}$$

where $a(x), b(x), f(x)$ and $\phi(x)$ are known analytic functions and, further, that each function is simple enough so that analytic differentiation is feasible, β is a constant and $\delta(\varepsilon)$ is a small shifting parameter. For $\delta = 0$ the corresponding singular perturbation problem has boundary layer on left side when $a(x) > 0$ or on right side when $a(x) < 0$ on the interval [0, 1]. The layer is maintained at the same end for sufficiently small δ, i.e., when $\delta = o(\varepsilon)$. The layer behavior can change its character and even be destroyed as the shifts increase, i.e., when $\delta = O(\varepsilon)$ [5].

3. Layer behavior

When $\delta = o(\varepsilon)$, the use of Taylor's series expansion for the term containing delay is valid [21]. By using the Taylor approximation to the term containing the delay, the boundary value problem (1) and (2) reduces to

$$(\varepsilon - \delta a(x))u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{3}$$

subject to

$$u(0) = \phi(0) = \phi_0(\text{say}), \quad u(1) = \beta. \tag{4}$$

We assume that $(\varepsilon - \delta a(x)) > 0$, $b(x) \leqslant -\theta < 0$, $a(x) \geqslant M > 0$ throughout the interval [0, 1]. Under these assumptions, (3) has a unique solution $u(x)$ which in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε. Since $u \in C^2[0, 1]$ and the delay argument is sufficiently small, the solution $u(x)$ of the problem (3) and (4) provide a good approximation to the solution $u(x)$ of the problem (1) and (2). We denote by L_ε the differential operator for the above problem (3) and (4) which is defined for any function $\psi(x) \in C^2[0, 1]$ as $L_\varepsilon\psi(x) = (\varepsilon - \delta a(x))\psi''(x) + a(x)\psi'(x) + b(x)\psi(x)$.

Throughout the paper θ and M denote generic positive constants that are independent of ε and in the case of discrete problems, also independent of the mesh parameter N. $\|\cdot\|$ denotes the global maximum norm over the appropriate domain of the independent variable, i.e., $\|f\| = \max_{x\in[0,1]}|f(x)|$.

Lemma 1. Let $u(x)$ be the solution of the problem (3) and (4), then we have

$$\|u\| \leqslant \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|).$$

Proof. Let us construct the two barrier functions $\psi^{\pm}$ defined by

$$\psi^{\pm}(x) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm u(x).$$

Then we have

$$\psi^{\pm}(0) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm u(0) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm \phi_0, \quad \text{since } u(0) = \phi_0 \geqslant 0,$$

$$\psi^{\pm}(1) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm u(1) = \theta^{-1}\|f\| + \max(|\phi_0|, |\beta|) \pm \beta, \quad \text{since } u(1) = \beta \geqslant 0,$$

and we have

$$L_\varepsilon\psi^\pm(x) = (\varepsilon - \delta a(x))\left(\psi^\pm(x)\right)'' + a(x)\left(\psi^\pm(x)\right)' + b(x)\psi^\pm(x) = b(x)(\theta^{-1}\|f\| + \max(|\phi_0|,|\beta|)) \pm L_\varepsilon u(x)$$
$$= b(x)(\theta^{-1}\|f\| + \max(|\phi_0|,|\beta|)) \pm f(x).$$

We have $b(x)\theta^{-1} \leqslant -1$, since $b(x) \leqslant -\theta < 0$. □

Using this inequality in the above inequality, we get

$$L_\varepsilon\psi^\pm(x) \leqslant (-\|f\| \pm f(x)) + b(x)\max(|\phi_0|,|\beta|) \leqslant 0 \;\forall x \in (0,1), \quad \text{since } \|f\| \geqslant f(x).$$

Therefore by the minimum principle [22], we obtain $\psi^\pm(x) \geqslant 0$ for all $x \in [0, 1]$, which gives the required estimate.

3.1. Exponentially fitted tridiagonal finite difference method

From the theory of singular perturbations it is known that the solution of (3) and (4) is of the form [23, pp. 22–26]

$$u(x) = u_0(x) + \frac{a(0)}{a(x)}(\phi(0) - u_0(0))\exp\left\{-\int_0^x\left(\frac{a(x)}{\varepsilon - \delta a(x)}\right)dx\right\} + O(\varepsilon), \tag{5}$$

where $u_0(x)$ is the solution of the reduced problem

$$a(x)u_0'(x) + b(x)u_0(x) = f(x), \quad u_0(1) = \beta. \tag{6}$$

By taking the Taylor's series expansion for $a(x)$ about the point '0' and restricting to their first terms, (5) becomes,

$$u(x) = u_0(x) + (\phi(0) - u_0(0))\exp\left\{-\left(\frac{a(0)}{\varepsilon - \delta a(0)}\right)x\right\} + O(\varepsilon). \tag{7}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots, x_n = 1$ be the mesh points. Then we have $x_i = ih$, $i = 0, 1, 2, \ldots, N$.

From (7), we have

$$u(x_i) = u_0(x_i) + (\phi(0) - u_0(0))\exp\left\{-\left(\frac{a(0)}{\varepsilon - \delta a(0)}\right)x_i\right\} + O(\varepsilon),$$

i.e.,

$$u(ih) = u_0(ih) + (\phi(0) - u_0(0))\exp\left\{-\left(\frac{a(0)}{\varepsilon - \delta a(0)}\right)ih\right\} + O(\varepsilon).$$

Therefore

$$\lim_{h\to 0} u(ih) = u_0(0) + (\phi(0) - u_0(0))\exp\{-a(0)i\rho\}, \tag{8}$$

where $\rho = \frac{h}{\varepsilon - \delta a(0)}, \quad \varepsilon - \delta a(0) \neq 0.$

Remark. It may be noted that if $\varepsilon - \delta a(0) = 0$, then $\rho \to \infty$ and $u(x)$ reduces to $u_0(x)$, which is the solution of reduced problem.

We consider

$$g(x, u, u') = f(x) - a(x)u'(x) - b(x)u(x). \tag{9}$$

Then Eq. (3) reduces to

$$(\varepsilon - \delta a(x))u''(x) = g(x, u, u'), \quad 0 \leqslant x \leqslant 1. \tag{10}$$

Now, we consider the fourth order finite difference method by Chawla [24] as follows:

$$\bar{u}_i' = \frac{u_{i+1} - u_{i-1}}{2h},$$

$$\bar{u}_{i+1}' = \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h},$$

$$\bar{u}_{i-1}' = \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h},$$

$$\bar{\bar{u}}_i' = \bar{u}_i' - \frac{h}{20}(\bar{g}_{i+1} - \bar{g}_{i-1})$$

and

$$(\varepsilon-\delta a(x_i)(\sigma(\rho))\left(\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}\right)=\frac{1}{12}(\bar{g}_{i+1}+10\bar{\bar{g}}_i+\bar{g}_{i-1}),\tag{11}$$

where $\bar{\bar{g}}_i=g(x_i,u_i,\bar{\bar{u}}')$ and $\bar{g}_{i\pm1}=g(x_{i\pm1},u_{i\pm1},\bar{u}'_{i\pm1})$.

Here $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (11) converges uniformly in ε to the solution of (3) and (4).

Now, multiplying Eq. (11) by h and taking limit as $h\to0$, we get

$$\lim_{h\to0}\left[\frac{\sigma(\rho)}{\rho}(u_{i+1}-2u_i+u_{i-1})+\frac{1}{2}a(ih)(u_{i+1}-u_{i-1})\right]=0,\quad \text{since } f(x_i)-b(x_i)u_i \text{ is bounded.}$$

$$\therefore\lim_{h\to0}\left[\frac{\sigma(\rho)}{\rho}(u(ih+h)-2u(ih)+u(ih-h))+\frac{1}{2}a(ih)(u(ih+h)-u(ih-h))\right]=0.\tag{12}$$

Substituting (8) in (12) and simplifying, we get the fitting factor as

$$\sigma(\rho)=a(0)\frac{\rho}{2}\coth\left(\frac{a(0)\rho}{2}\right),$$

which is a constant fitting factor.

In general we take a variable fitting factor as

$$\sigma_i(\rho_i)=a(x_i)\frac{\rho_i}{2}\coth\left(\frac{a(x_i)\rho_i}{2}\right),\tag{13}$$

where $\rho_i=\dfrac{h}{\varepsilon-\delta a(x_i)}$.

Eq. (11) is a fourth order tridiagonal finite difference scheme and it can be written as

$$E_iu_{i-1}-F_iu_i+G_iu_{i+1}=H_i,\quad i=1,2,3,\ldots,N-1,\tag{14}$$

where

$$E_i=\frac{a_i}{2h}\coth\left(\frac{a_i\rho_i}{2}\right)+\frac{1}{12}\left[b_{i-1}+\frac{1}{2h}a_{i+1}-\frac{5}{h}a_i-\frac{h}{2}a_ib_{i-1}-\frac{3}{2h}a_{i-1}+\frac{1}{4}a_i(a_{i+1}+3a_{i-1})\right],$$

$$F_i=\frac{a_i}{h}\coth\left(\frac{a_i\rho_i}{2}\right)+\left[-\frac{5}{6}b_i+\frac{1}{6h}(a_{i+1}-a_{i-1})+\frac{1}{12}a_i(a_{i+1}+a_{i-1})\right],$$

$$G_i=\frac{a_i}{2h}\coth\left(\frac{a_i\rho_i}{2}\right)+\frac{1}{12}\left[b_{i+1}+\frac{3}{2h}a_{i+1}+\frac{5}{h}a_i+\frac{h}{2}a_ib_{i+1}-\frac{1}{2h}a_{i-1}+\frac{1}{4}a_i(3a_{i+1}+a_{i-1})\right],$$

$$H_i=\frac{1}{12}\left(f_{i+1}+10f_i+f_{i-1}+\frac{h}{2}a_i(f_{i+1}-f_{i-1})\right)$$

and $a(x_i)=a_i$, $b(x_i)=b_i$, $f(x_i)=f_i$.

We solve the tridiagonal system (14) where σ is given by (13) subject to the boundary conditions (4) by using Thomas Algorithm.

Remark. When $\delta=o(\varepsilon)$, $0<(\varepsilon-\delta a(x))\ll1$, $a(x)\leqslant M<0$, $b(x)<0$ throughout the interval [0, 1], where M is some negative constant, the boundary value problem (3) and (4) displays a boundary layer at $x=1$. It can be observed that the same variable fitting factor can be obtained in this case also.

3.2. Convergence analysis

Multiplying Eq. (14) by h and incorporating the boundary conditions we obtain the system of equations in the matrix form as

$$(D+P)U+Q+T(h)=0,\tag{15}$$

where

$$D=\left[\frac{a_i}{2}\coth\left(\frac{a_i\rho_i}{2}\right),-a_i\coth\left(\frac{a_i\rho_i}{2}\right),\frac{a_i}{2}\coth\left(\frac{a_i\rho_i}{2}\right)\right]=\begin{bmatrix}-a_1\coth\left(\frac{a_1\rho_1}{2}\right) & \frac{a_1}{2}\coth\left(\frac{a_1\rho_1}{2}\right) & 0 & \cdots & 0\\ \frac{a_2}{2}\coth\left(\frac{a_2\rho_2}{2}\right) & -a_2\coth\left(\frac{a_2\rho_2}{2}\right) & \frac{a_2}{2}\coth\left(\frac{a_2\rho_2}{2}\right) & \cdots & 0\\ 0 & . & . & \cdots & .\\ . & . & . & \cdots & .\\ 0 & \cdots & 0 & \frac{a_{N-1}}{2}\coth\left(\frac{a_{N-1}\rho_{N-1}}{2}\right) & -a_{N-1}\coth\left(\frac{a_{N-1}\rho_{N-1}}{2}\right)\end{bmatrix}$$

and

$$P = [z_i, v_i, w_i] = \begin{bmatrix} v_1 & w_1 & 0 & \dots & 0 \\ z_2 & v_2 & w_2 & \dots & 0 \\ 0 & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & \dots & 0 & z_{N-1} & v_{N-1}, \end{bmatrix},$$

where

$$z_i = \frac{1}{12}\left[hb_{i-1} + \frac{1}{2}a_{i+1} - 5a_i - \frac{h^2}{2}a_i b_{i-1} - \frac{3}{2}a_{i-1} + \frac{h}{4}a_i(a_{i+1} + 3a_{i-1})\right],$$

$$v_i = \left[\frac{5h}{6}b_i - \frac{1}{6}(a_{i+1} - a_{i-1}) - \frac{h}{12}a_i(a_{i+1} + a_{i-1})\right],$$

$$w_i = \frac{1}{12}[hb_{i+1} + \frac{3}{2}a_{i+1} + 5a_i + \frac{h^2}{2}a_i b_{i+1} - \frac{1}{2}a_{i-1} + \frac{h}{4}a_i(3a_{i+1} + a_{i-1})]$$

and

$$Q = \left[q_1 + \left(\frac{a_1}{2}\coth\frac{a_1\rho_1}{2} + z_1\right)\phi(0), q_2, q_3, \dots, q_{N-2}, q_{N-1} + \left(\frac{a_{N-1}}{2}\coth\frac{a_{N-1}\rho_{N-1}}{2} + w_{N-1}\right)\beta\right]^T,$$

where $q_i = -\frac{1}{12}\left[h(f_{i+1} + 10f_i + f_{i-1}) + \frac{h^2}{2}a_i(f_{i+1} - f_{i-1})\right], \quad i = 1, 2, \dots, N-1.$

$T(h) = O(h^4)$ and $U = [U_1, U_2, \dots, U_{N-1}]^T$, $T(h) = [T_1, T_2, \dots, T_{N-1}]^T$, $0 = [0, 0, \dots, 0]^T$ are the associated vectors of Eq. (15).

Let $u = [u_1, u_2, \dots, u_{N-1}]^T \cong U$ which satisfies the equation

$$(D + P)U + Q = 0, \tag{16}$$

Let $e_i = u_i - U_i, \quad i = 1, 2, \dots, N-1$ be the discretization error so that $E = [e_1, e_2, \dots, e_{N-1}]^T = u - U$.

Subtracting Eq. (15) from Eq. (16) we get

$$(D + P)E = T(h). \tag{17}$$

Let $|a(x)| \leqslant C_1; |b(x)| \leqslant C_2$.

Let $p_{i,j}$ be the (i, j)th element of the matrix P, then

$$|p_{i,i+1}| = |w_i| \leqslant \frac{1}{12}\left(hC_2 + 6C_1 + \frac{h^2}{2}C_1C_2 + hC_1^2\right); \quad i = 1, 2, \dots, N-2.$$

$$|p_{i,i-1}| = |z_i| \leqslant \frac{1}{12}|hC_2 - 6C_1 - \frac{h^2}{2}C_1C_2 + hC_1^2|; \quad i = 2, \dots, N-1.$$

Thus for sufficiently small h,

$$\frac{a_i}{2}\coth\frac{a_i\rho_i}{2} + |p_{i,i+1}| \leqslant \frac{C_1}{2}\left(\coth\frac{C_1\rho_i}{2} + 1\right) \neq 0, \quad i = 1, 2, \dots, N-2 \text{ since}$$

$$|a_i| \leqslant C_1 \cdot \frac{a_i}{2}\coth\frac{a_i\rho_i}{2} + |p_{i,i-1}| \leqslant \frac{C_1}{2}\left(\coth\frac{C_1\rho_i}{2} + 1\right) \neq 0, \quad i = 1, 2, \dots, N-1.$$

Hence, the matrix $(D + P)$ is irreducible [25].

Let S_i be the sum of the elements of the ith row of the matrix $(D + P)$, then we have

$$S_i = -\frac{a_i}{2}\coth\frac{a_i\rho_i}{2} + \frac{5h}{6}b_i - \frac{h}{24}a_{i+1} + \frac{5}{12}a_i + \frac{1}{8}a_{i-1} + \frac{h}{12}b_{i+1} - \frac{h}{48}a_ia_{i+1} - \frac{h}{16}a_ia_{i-1} + \frac{h^2}{24}a_ib_{i+1} \quad \text{for } i = 1,$$

$$S_i = -\frac{a_i}{2}\coth\left(\frac{a_i\rho_i}{2}\right) + \frac{h}{12}b_{i-1} - \frac{1}{8}a_{i+1} - \frac{5}{12}a_i + \frac{1}{24}a_{i-1} - \frac{h}{16}a_ia_{i+1} - \frac{h}{48}a_ia_{i-1} - \frac{h^2}{24}a_ib_{i-1} + \frac{5h}{6}b_i \quad \text{for } i = N-1,$$

$$S_i = \frac{h}{12}(b_{i-1} + 10b_i + b_{i+1}) + \frac{h^2}{24}a_i(b_{i+1} - b_{i-1}) \quad \text{for } i = 2, 3, 4, \dots, N-2.$$

Let $C_{1^*} = \min|a(x)|$, $C_1^* = \max|a(x)|$, $C_{2^*} = \min|b(x)|$, $C_2^* = \max|b(x)|$.

Then $0 < C_{1^*} \leqslant C_1 \leqslant C_1^*$, $0 < C_{2^*} \leqslant C_2 \leqslant C_2^*$.

It is easy to verify that for sufficiently small h, $(D + P)$ is monotone [25,26]. Hence $(D + P)^{-1}$exists and $(D+P)^{-1} \geqslant 0$.

From the error equation (17) we have $||E|| = ||(D + P)^{-1}||\cdot||T||$.

For sufficiently small h, we have

$$S_i > \frac{h^2}{24}C_1C_2 \quad \text{for } i = 1,$$

$$S_i > \frac{h^2}{24}C_1C_2 \quad \text{for } i = N-1 \text{ and}$$

$$S_i \geqslant \frac{h^2}{24}C_1C \quad \text{for } i = 2,3,4,\ldots,N-2. \tag{18}$$

where $C = |b_{i+1} - b_{i-1}|$.

Let $(D+P)^{-1}_{i,k}$ be the (i, k)th element of $(D + P)^{-1}$ and we define

$$\|(D+P)^{-1}\| = \max_{1\leqslant i\leqslant N-1}\sum_{k=1}^{N-1}(D+P)^{-1}_{i,k} \text{ and } \|T(h)\| = \max_{1\leqslant i\leqslant N-1}|T_i|,$$

since $(D+P)^{-1}_{i,k} \geqslant 0$ and $\sum_{k=1}^{N-1}(D+P)^{-1}_{i,k}\cdot S_k = 1$ for i = 1, 2, 3, 4, . . ., $N - 1$.

Hence, $(D+P)^{-1}_{i,1} \leqslant \frac{1}{S_1} < \frac{24}{h^2C_1C_2}$,

$$(D+P)^{-1}_{i,N-1} \leqslant \frac{1}{S_{N-1}} < \frac{24}{h^2C_1C_2}.$$

Further $\sum_{k=2}^{N-2}(D+P)^{-1}_{i,k} \leqslant \frac{1}{\min\limits_{2\leqslant k\leqslant N-2} S_k} \leqslant \frac{24}{h^2C_1C}$ for i = 1, 2, 3, 4, . . ., $N - 1$.

Hence from Eqs. (17) and (18), we get

$$\|E\| = \frac{24}{h^2}\left[\frac{1}{C_1C_2} + \frac{1}{CC_1} + \frac{1}{C_1C_2}\right] \times T(h) = \frac{24}{h^2}[\frac{1}{C_1C_2} + \frac{1}{CC_1} + \frac{1}{C_1C_2}] \times O(h^4). \tag{19}$$

This establishes the convergence of the finite difference scheme (14) and the rate of convergence of the scheme is 2.

From Eq. (19), it is observed that the proposed method is ε uniform convergent since the error is of the form $||E|| = C^*h^2$ where C^* is independent of perturbation parameter ε.

3.3. Numerical examples (boundary layer behavior)

To demonstrate the applicability of the method we consider two boundary value problems of singularly perturbed linear differential difference equations exhibiting boundary layer at the left of the interval [0, 1], and one problem exhibiting boundary layer at the right end of the under lying interval. These examples were widely discussed in the literature [14–16]. Since the exact solutions of the problems for different values of δ are not known, the maximum absolute errors for the examples are calculated using the double mesh principle $E_N = \max_{0\leqslant i\leqslant N}|y_i^N - y_{2i}^{2N}|$. The maximum absolute errors are tabulated in the form of Table 1 for considered examples. From the numerical results, it can be observed that proposed method is ε uniform convergent. Our numerical results are compared with the results given in [14–16]. It has been observed that the proposed method gives high accurate numerical results and higher order of convergence than the methods proposed in [14–16]. From the results, it also can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution.

Example 1 [15, p. 700]. $\varepsilon y''(x) + 0.25y'(x-\delta) - y(x) = 0$, subject to the interval and boundary conditions $y(x) = 1;\ -\delta \leqslant x \leqslant 0$, $y(1) = -1$.

Example 2 [14, p. 195]. $\varepsilon y''(x) + e^{-x}y'(x-\delta) - xy(x) = 0$, subject to the interval and boundary conditions $y(x) = 1;\ -\delta \leqslant x \leqslant 0$, $y(1) = 1$.

Example 3 [16, p. 808]. $\varepsilon y''(x) - (1+x)y'(x-\delta) - e^{-x}y(x) = 1$, subject to the interval and boundary conditions $y(x) = 1;\ -\delta \leqslant x \leqslant 0$, $y(1) = -1$.

4. Oscillatory behavior

To discuss the finite difference method, we consider a linear singularly perturbed differential-difference equation of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x)$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, subject to the interval and boundary conditions

Table 1
The maximum absolute errors E_N for $\delta = 0.5\varepsilon$ using the exponentially fitted finite difference method (14).

ε	N				
	100	200	300	400	500
Example 1					
10^{-1}	1.5441e−005	1.7156e−006	1.7156e−006	9.6503e−007	6.1762e−007
10^{-2}	2.4266e−004	6.0723e−005	2.6941e−005	1.5157e−005	9.6976e−006
10^{-3}	2.7351e−003	7.9548e−004	3.3437e−004	1.7441e−004	1.1457e−004
10^{-4}	3.8338e−003	1.9348e−003	1.2825e−003	9.4102e−004	7.2670e−004
10^{-5}	3.8338e−003	1.9367e−003	1.2956e−003	9.7346e−004	7.7960e−004
10^{-10}	3.8338e−003	1.9367e−003	1.2956e−003	9.7346e−004	7.7960e−004
10^{-20}	3.8338e−003	1.9367e−003	1.2956e−003	9.7346e−004	7.7960e−004
Example 2					
10^{-1}	2.4266e−004	6.0723e−005	2.6941e−005	1.5157e−005	9.6976e−006
10^{-2}	2.4266e−004	6.0723e−005	2.6941e−005	1.5157e−005	9.6976e−006
10^{-3}	3.5343e−003	1.3266e−003	6.7418e−004	4.0273e−004	2.6620e−004
10^{-4}	3.8857e−003	1.9634e−003	1.3133e−003	9.8501e−004	7.8493e−004
10^{-5}	3.8857e−003	1.9634e−003	1.3136e−003	9.8694e−004	7.9039e−004
10^{-10}	3.8857e−003	1.9634e−003	1.3136e−003	9.8694e−004	7.9039e−004
10^{-20}	3.8857e−003	1.9634e−003	1.3136e−003	9.8694e−004	7.9039e−004
Example 3					
10^{-1}	2.9324e−005	7.3329e−006	3.2593e−006	1.8334e−006	1.1734e−006
10^{-2}	5.0780e−004	1.2843e−004	5.7199e−005	3.2198e−005	2.0614e−005
10^{-3}	2.6123e−003	1.0331e−003	5.2760e−004	3.1404e−004	2.0667e−004
10^{-4}	2.7246e−003	1.3749e−003	9.1940e−004	6.9050e−004	5.5243e−004
10^{-5}	2.7246e−003	1.3749e−003	9.1940e−004	6.9061e−004	5.5300e−004
10^{-10}	2.7246e−003	1.3749e−003	9.1940e−004	6.9061e−004	5.5300e−004
10^{-20}	2.7246e−003	1.3749e−003	9.1940e−004	6.9061e−004	5.5300e−004

$$y(x) = \phi(x), \quad x \leqslant 0,$$
$$y(1) = \beta,$$

where $a(x),\ b(x),\ f(x)$ and $\phi(x)$are known analytic functions and, further, that each function is simple enough so that analytic differentiation is feasible, β is a constant and $\delta(\varepsilon)$is a small shifting parameter.

When the shift parameter is bigger one, i.e., $\delta = O(\varepsilon)$, the use of Taylor's series expansion for the term containing the delay may lead to a bad approximation. In this case a special type of mesh is used, so that the term containing shift lies on nodal points after discretization and fourth order finite difference method is applied.

We consider $g(x, y, y') = f(x) - a(x)y'(x - \delta) - b(x)y(x)$, then Eq. (1) reduces to

$$\varepsilon y''(x) = g(x, y, y'), \quad 0 \leqslant x \leqslant 1.$$

Now we consider the fourth order finite difference method by Chawla [24] as follows:

$$\bar{y}'_i = \frac{y_{i+1} - y_{i-1}}{2h},$$

$$\bar{y}'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h},$$

$$\bar{y}'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h},$$

$$\bar{\bar{y}}'_i = \bar{y}'_i - \frac{h}{20}(\bar{g}_{i+1} - \bar{g}_{i-1}),$$

$$\varepsilon\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) = \frac{1}{12}(\bar{g}_{i+1} + 10\bar{\bar{g}}_i + \bar{g}_{i-1}), \tag{20}$$

where $\bar{\bar{g}}_i = g(x_i, y_i, \bar{\bar{y}}'_i)$ and $\bar{g}_{i\pm1} = g(x_{i\pm1}, y_{i\pm1}, \bar{y}'_{i\pm1})$.

To handle the delay argument, we construct a special type of mesh, so that the term containing delay lies on nodal points after discretization. We divide the interval [0, 1] into N equal parts by choosing the mesh parameter $h = \frac{\delta}{m}$, where m is a positive integer chosen such that $1 < m < N$.

We have

$$\varepsilon y''(x_i) = g(x_i, y_i, y'_i) = f(x_i) - a(x_i)y'(x_{i-m}) - b(x_i)y(x_i). \tag{21}$$

The boundary conditions can be written as

$$\begin{aligned} &y_i = \phi_i, \text{ for } i \text{ a non positive integer,} \\ &y_N = \beta, \end{aligned} \tag{22}$$

where $\phi_i = \phi(x_i)$.

We consider the notation

$$g(x_i, y_i, y_i') = g_i, a(x_i) = a_i, b(x_i) = b_i \text{ and } f(x_i) = f_i.$$

Then we get

$$g_i = f_i - a_i y'_{i-m} - b_i y_i,$$

$$\bar{g}_{i+1} = f_{i+1} - \frac{3}{2h} a_{i+1} y_{i-m+1} + \frac{2}{h} a_{i+1} y_{i-m} - \frac{1}{2h} a_{i+1} y_{i-m-1} - b_{i+1} y_{i+1},$$

$$\bar{g}_{i-1} = f_{i-1} + \frac{1}{2h} a_{i-1} y_{i-m+1} - \frac{2}{h} a_{i-1} y_{i-m} + \frac{3}{2h} a_{i-1} y_{i-m-1} - b_{i-1} y_{i-1},$$

$$\begin{aligned} \bar{\bar{g}}_i = {} & f_i + \frac{h}{20} a_i f_{i-m+1} - \frac{h}{20} a_i f_{i-m-1} - \frac{1}{2h} a_i y_{i-m+1} + \frac{1}{2h} a_i y_{i-m-1} - \frac{3}{40} a_i a_{i-m+1} y_{i-2m+1} + \frac{1}{10} a_i a_{i-m+1} y_{i-2m} - \frac{1}{40} a_i a_{i-m+1} y_{i-2m-1} \\ & - \frac{h}{20} a_i b_{i-m+1} y_{i-m+1} - \frac{1}{40} a_i a_{i-m-1} y_{i-2m+1} + \frac{1}{10} a_i a_{i-m-1} y_{i-2m} - \frac{3}{40} a_i a_{i-m-1} y_{i-2m-1} + \frac{h}{20} a_i b_{i-m-1} y_{i-m-1} - b_i y_i. \end{aligned}$$

Substituting these in (20) we obtain the difference scheme as follows:

$$p_i y_{i+1} + q_i y_i + r_i y_{i-1} + u_i y_{i-m+1} + v_i y_{i-m} + w_i y_{i-m-1} + \xi_i y_{i-2m+1} + \eta_i y_{i-2m} + \zeta_i y_{i-2m-1} = R_i, \tag{23}$$

where

$$p_i = \frac{\varepsilon}{h^2} + \frac{1}{12} b_{i+1}, q_i = -\frac{2\varepsilon}{h^2} + \frac{5}{6} b_i, r_i = \frac{\varepsilon}{h^2} + \frac{1}{12} b_{i-1},$$

$$u_i = \frac{1}{12}\left(\frac{3}{2h} a_{i+1} + \frac{5}{h} a_i + \frac{h}{2} a_i b_{i-m+1} - \frac{1}{2h} a_{i-1}\right),$$

$$v_i = -\frac{1}{6h}(a_{i+1} - a_{i-1}),$$

$$w_i = \frac{1}{12}\left(\frac{1}{2h} a_{i+1} - \frac{5}{h} a_i - \frac{h}{2} a_i b_{i-m-1} - \frac{3}{2h} a_{i-1}\right),$$

$$\xi_i = \frac{1}{48} a_i (3a_{i-m+1} + a_{i-m-1}),$$

$$\eta_i = -\frac{1}{12} a_i (a_{i-m+1} + a_{i-m-1}),$$

$$\zeta_i = \frac{1}{48} a_i (a_{i-m+1} + 3a_{i-m-1}),$$

$$R_i = \frac{1}{12}(f_{i+1} + 10f_i + f_{i-1}) + \frac{h}{24} a_i (f_{i-m+1} - f_{i-m-1}).$$

By using the boundary conditions (22), the difference scheme (23) can be written as

$$\begin{aligned} & p_i y_{i+1} + q_i y_i + r_i y_{i-1} = R_i - u_i \phi_{i-m+1} - v_i \phi_{i-m} - w_i \phi_{i-m-1} - \xi_i \phi_{i-2m+1} - \eta_i \phi_{i-2m} - \zeta_i \phi_{i-2m-1} \quad \text{for } 1 \leqslant i \leqslant m-1, \\ & p_i y_{i+1} + q_i y_i + r_i y_{i-1} + u_i y_{i-m+1} = R_i - v_i \phi_{i-m} - w_i \phi_{i-m-1} - \xi_i \phi_{i-2m+1} - \eta_i \phi_{i-2m} - \zeta_i \phi_{i-2m-1} \quad \text{for } i = m, \\ & p_i y_{i+1} + q_i y_i + r_i y_{i-1} + u_i y_{i-m+1} + v_i y_{i-m} = R_i - w_i \phi_{i-m-1} - \xi_i \phi_{i-2m+1} - \eta_i \phi_{i-2m} - \zeta_i \phi_{i-2m-1} \quad \text{for } i = m+1, \\ & p_i y_{i+1} + q_i y_i + r_i y_{i-1} + u_i y_{i-m+1} + v_i y_{i-m} + w_i y_{i-m-1} = R_i - \xi_i \phi_{i-2m+1} - \eta_i \phi_{i-2m} - \zeta_i \phi_{i-2m-1} \quad \text{for } m+2 \leqslant i \leqslant 2m-1, \\ & p_i y_{i+1} + q_i y_i + r_i y_{i-1} + u_i y_{i-m+1} + v_i y_{i-m} + w_i y_{i-m-1} + \xi_i y_{i-2m+1} = R_i - \eta_i \phi_{i-2m} - \zeta_i \phi_{i-2m-1}, \quad \text{for } i = 2m, \\ & p_i y_{i+1} + q_i y_i + r_i y_{i-1} + u_i y_{i-m+1} + v_i y_{i-m} + w_i y_{i-m-1} + \xi_i y_{i-2m+1} + \eta_i y_{i-2m} = R_i - \zeta_i \phi_{i-2m-1} \quad \text{for } i = 2m+1, \\ & p_i y_{i+1} + q_i y_i + r_i y_{i-1} + u_i y_{i-m+1} + v_i y_{i-m} + w_i y_{i-m-1} + \xi_i y_{i-2m+1} + \eta_i y_{i-2m} + \zeta_i y_{i-2m-1} = R_i \quad \text{for } 2m+2 \leqslant i \leqslant n-1. \end{aligned} \tag{24}$$

The above system of equations is solved by Gauss elimination method with partial pivoting. In fact, any numerical method or analytical method can be used. An extensive amount of computational work has been carried out to demonstrate the proposed method to show the effect of shift parameter on the oscillatory behavior of the solution of the problem.

4.1. Calculation of truncation error

From the Taylor series expansions, we have

$$\bar{y}'_i = y'_i + \frac{h^2}{6}y^{(3)}(\xi_1^{(i)}) = y'_i + \frac{h^2}{6}y_i^{(3)} + \frac{h^4}{120}y^{(5)}(\xi_2^{(i)}), \tag{25}$$

and

$$\bar{y}'_{i\pm1} = y'_{i\pm1} - \frac{h^2}{3}y^{(3)}\left(\xi_{\pm3}^{(i)}\right) = y'_{i\pm1} - \left[\frac{h^2}{3}y_i^{(3)} \pm \frac{h^3}{12}y^{(4)}\left(\xi_{\pm4}^{(i)}\right)\right], \tag{26}$$

$$= y'_{i\pm1} - \left[\frac{h^2}{3}y_i^{(3)} \pm \frac{h^3}{12}y_i^{(4)} + \frac{h^4}{30}y^{(5)}\left(\xi_{\pm5}^{(i)}\right)\right], \tag{27}$$

where all ξs $\in (x_{i-1}, x_{i+1})$.

From (21), we have

$$\bar{g}_{i\pm1} = f_{i\pm1} - a_{i\pm1}\bar{y}'_{i-m\pm1} - b_{i\pm1}y_{i\pm1}.$$

Now using (26) in the above expression, we get

$$\bar{g}_{i\pm1} = \varepsilon y''_{i\pm1} + a_{i\pm1}\left[\frac{h^2}{3}y_{i-m}^{(3)} \pm \frac{h^3}{12}y^{(4)}\left(\xi_{\pm4}^{(i-m)}\right)\right]. \tag{28}$$

and hence,

$$\bar{g}_{i+1} - \bar{g}_{i-1} = \varepsilon(y''_{i+1} - y''_{i-1}) + \frac{h^2}{3}(a_{i+1} - a_{i-1})y_{i-m}^{(3)} + \frac{h^3}{12}a_{i+1}y^{(4)}\left(\xi_4^{(i-m)}\right) + \frac{h^3}{12}a_{i-1}y^{(4)}\left(\xi_{-4}^{(i-m)}\right). \tag{29}$$

By Taylor's series expansion we have

$$\varepsilon(y''_{i+1} - y''_{i-1}) = 2h\varepsilon y_i^{(3)} + \frac{h^3}{3}\varepsilon y^{(5)}\left(\xi_8^{(i)}\right)$$

and by mean value theorem we have $a_{i+1} - a_{i-1} = 2ha'(\eta_1)$ where $\eta_1 \in (x_{i-1}, x_{i+1})$.

Using these in (29) we get,

$$\bar{g}_{i+1} - \bar{g}_{i-1} = 2h\varepsilon y_i^{(3)} + \frac{h^3}{12}\tau_4^{(i)}, \tag{30}$$

where $\tau_4^{(i)} = 4\varepsilon y^{(5)}\left(\xi_8^{(i)}\right) + 8a'(\eta_1)y_{i-m}^{(3)} + a_{i+1}y^{(4)}\left(\xi_4^{(i-m)}\right) + a_{i-1}y^{(4)}\left(\xi_{-4}^{(i-m)}\right)$.

Using (25) and (30) in $\bar{\bar{y}}'_i = \bar{y}'_i - \frac{h}{20}(\bar{g}_{i+1} - \bar{g}_{i-1})$, we get

$$\bar{\bar{y}}'_i = y'_i + \frac{h^2}{30}(5 - 3\varepsilon)y_i^{(3)} + \frac{h^4}{240}\tau_5^{(i)}, \tag{31}$$

where $\tau_5^{(i)} = 2y^{(5)}(\xi_2^{(i)}) - \tau_4^{(i)}$.

Using (31) in $\bar{\bar{g}}_i = f_i - a_i\bar{\bar{y}}'_{i-m} - b_iy_i$, we get

$$\bar{\bar{g}}_i = \varepsilon y''_i - \frac{a_ih^2}{30}(5 - 3\varepsilon)y_{i-m}^{(3)} - \frac{a_ih^4}{240}\tau_5^{(i-m)}. \tag{32}$$

Using (27) in $\bar{g}_{i\pm1} = f_{i\pm1} - a_{i\pm1}\bar{y}'_{i-m\pm1} - b_{i\pm1}y_{i\pm1}$, we get

$$\bar{g}_{i\pm1} = \varepsilon y''_{i\pm1} + a_{i\pm1}\left[\frac{h^2}{3}y_{i-m}^{(3)} \pm \frac{h^3}{12}y_{i-m}^{(4)} + \frac{h^4}{30}y^{(5)}\left(\xi_{\pm5}^{(i-m)}\right)\right]. \tag{33}$$

Therefore, we have

$$\bar{g}_{i+1} + \bar{g}_{i-1} = \varepsilon(y''_{i+1} + y''_{i-1}) + \frac{2h^2}{3}a_iy_{i-m}^{(3)} + \frac{h^4}{30}\tau_6^{(i)}, \tag{34}$$

where $\tau_6^{(i)} = 10a''(\eta_2)y_{i-m}^{(3)} + 5a''(\eta_1) + a_{i+1}y^{(5)}\left(\xi_5^{(i-m)}\right) + a_{i-1}y^{(5)}\left(\xi_{-5}^{(i-m)}\right)$.

From (32) and (34) we get,

$$\bar{g}_{i+1} + 10\bar{\bar{g}}_i + \bar{g}_{i-1} = \varepsilon(y''_{i+1} + 10y''_i + y''_{i-1}) + a_i h^2(\varepsilon - 1)y^{(3)}_{i-m} + \frac{h^4}{120}\left(4\tau_6^{(i)} - 5\tau_5^{(i-m)}\right). \tag{35}$$

Substituting (35) in the scheme

$$(y_{i-1} - 2y_i + y_{i+1}) = \frac{h^2}{12}(y''_{i-1} + 10y''_i + y''_{i+1}) - \frac{h^6}{240}y^{(6)}\left(\xi_9^{(i)}\right),$$

we obtain

$$\varepsilon(y_{i-1} - 2y_i + y_{i+1}) = \frac{h^2}{12}(\bar{g}_{i+1} + 10\bar{\bar{g}}_i + \bar{g}_{i-1}) - \frac{h^4}{12}a_i(\varepsilon - 1)y^{(3)}_{i-m} - \frac{h^6}{1440}\left(4\tau_6^{(i)} - 5\tau_5^{(i-m)} - 6\varepsilon y^{(6)}\left(\xi_9^{(i)}\right)\right). \tag{36}$$

Hence the truncation error is

$$|T(h)| \leqslant \frac{h^4}{12}|a_i(\varepsilon - 1)y^{(3)}_{i-m}| - \frac{h^6}{1440}\left|\left(4\tau_6^{(i)} - 5\tau_5^{(i-m)} - 6\varepsilon y^{(6)}\left(\xi_9^{(i)}\right)\right)\right|. \tag{37}$$

It can be observed that the truncation error of the proposed finite difference scheme is $O(h^4)$.

4.2. Numerical examples (oscillatory behavior)

To demonstrate the applicability of the method we consider three boundary value problems of singularly perturbed linear differential difference equations exhibiting boundary layer and oscillatory behavior for different values of the delay parameter. These examples were discussed widely in the literature [5,6,15,16]. Since the exact solutions of the problems for different values of δ are not known, the maximum absolute errors for the examples are calculated using the double mesh principle $E_N = \max_{0\leqslant i\leqslant N}|y_i^N - y_{2i}^{2N}|$. The maximum absolute error is tabulated in the form of Table 2 for considered examples. The graphs of the solution of the considered examples for different values of delay parameter are plotted in Figs. 1–8 to examine the effect of delay on the boundary layer behavior of the solution. The numerical results of the proposed method

Table 2
The maximum absolute error E_N for $\varepsilon = 0.1$ using the difference scheme (23).

δ	N				
	100	200	300	400	500
Example 4					
0.03	6.5852e−004	1.6479e−004	7.3238e−005	4.1192e−005	2.6365e−005
0.05	8.8337e−004	2.2131e−004	9.8347e−005	5.5325e−005	3.5409e−005
0.07	1.0907e−003	2.7253e−004	1.2111e−004	6.8122e−005	4.3598e−005
0.09	2.1240e−003	5.3078e−004	2.3588e−004	1.3268e−004	8.4918e−005
Example 5					
0.03	1.0569e−003	2.6408e−004	1.1736e−004	6.6010e−005	4.2246e−005
0.05	1.3788e−003	3.4466e−004	1.5318e−004	8.6162e−005	5.5144e−005
0.07	1.6367e−003	4.1000e−004	1.8219e−004	1.0250e−004	6.5596e−005
0.09	1.6937e−003	4.2460e−004	1.8874e−004	1.0618e−004	6.7958e−005
Example 6					
0.03	2.8135e−004	7.0382e−005	3.1277e−005	1.7592e−005	1.1259e−005
0.05	1.8387e−004	4.5948e−005	2.0427e−005	1.1490e−005	7.3535e−006
0.07	1.3304e−004	3.3253e−005	1.4778e−005	8.3127e−006	5.3201e−006
0.09	1.0237e−004	2.5588e−005	1.1372e−005	6.3968e−006	4.0940e−006

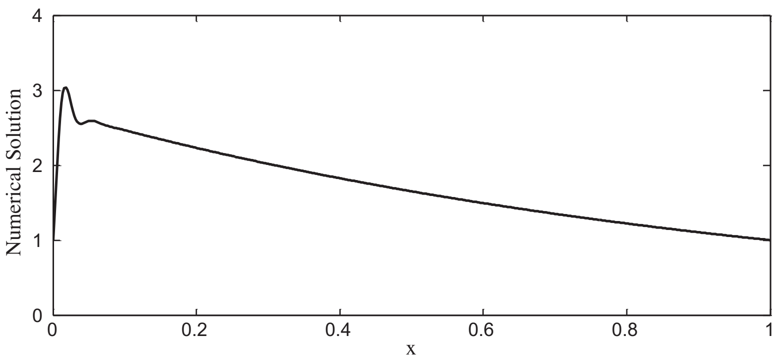

Fig. 1. Numerical solution of example 4 for $\varepsilon = 0.01$ and $\delta = 0.7\varepsilon$.

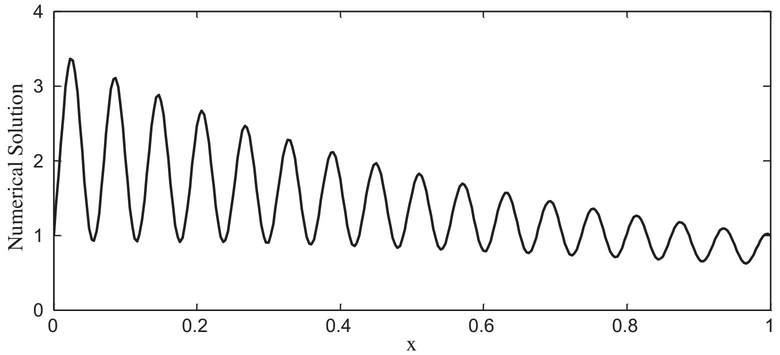

Fig. 2. Numerical solution of example 4 for $\varepsilon = 0.01$ and $\delta = 1.5\varepsilon$.

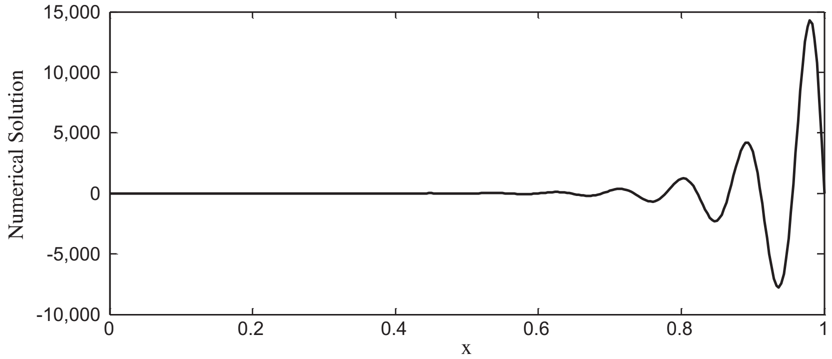

Fig. 3. Numerical solution of example 4 for $\varepsilon = 0.01$ and $\delta = 2.5\varepsilon$.

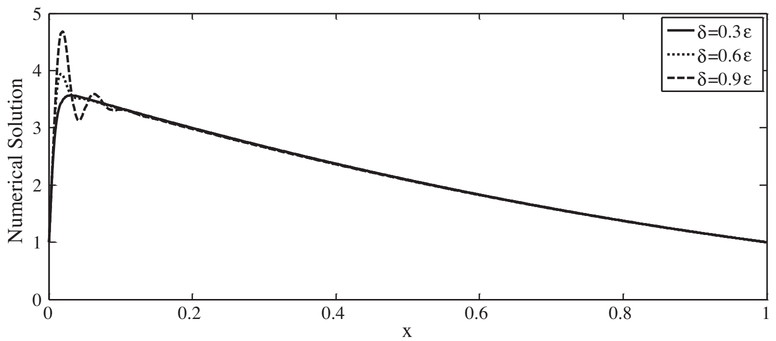

Fig. 4. Numerical solution of example 5 for $\varepsilon = 0.01$ for different values of δ.

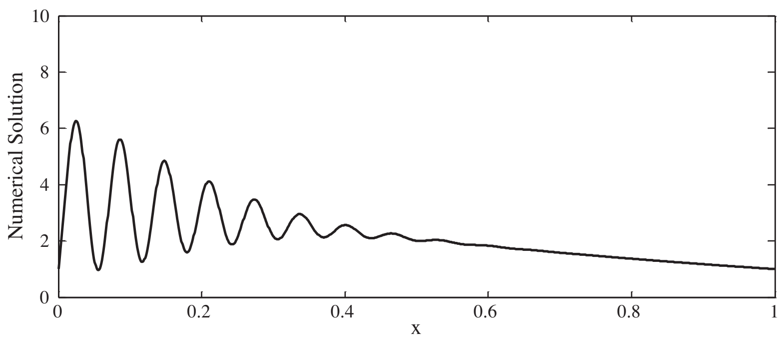

Fig. 5. Numerical solution of example 5 for $\varepsilon = 0.01$ for $\delta = 1.5\varepsilon$.

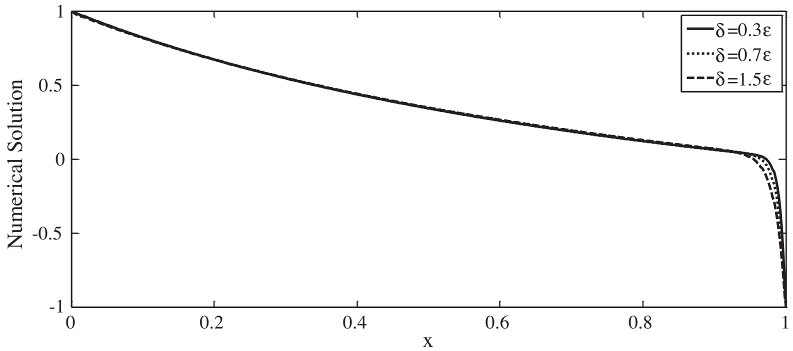

Fig. 7. Numerical solution of example 6 for $\varepsilon = 0.01$ for different values of δ.

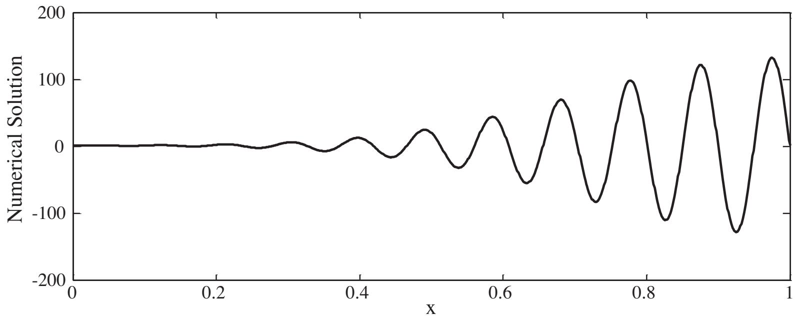

Fig. 6. Numerical solution of example 5 for $\varepsilon = 0.01$ for $\delta = 2.5\varepsilon$.

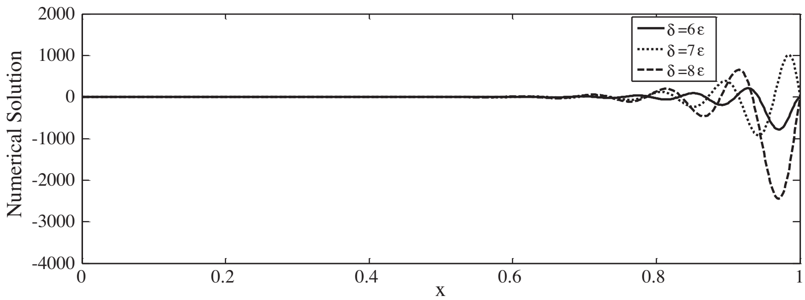

Fig. 8. Numerical solution of example 6 for $\varepsilon = 0.01$ for different values of δ.

are compared with the results given in [15,16]. It has been observed that the proposed method gives high accurate numerical results and higher order of convergence than the methods proposed in [15,16]. We compared the graphs with the graphs presented in [5,6].

The numerical rate of convergence for all the examples have been calculated by the formula $R_N = \frac{\log|E_N/E_{2N}|}{\log 2}$[27] and it observed that for all the examples cited below $R_N \approx 2$.

Example 4 [5, p. 254]. $\varepsilon y''(x) + y''(x-\delta) + y(x) = 0$, subject to the interval and boundary conditions $y(x) = 1;\ -\delta \leqslant x \leqslant 0$, $y(1) = 1$.

Example 5 [6, p. 275]. $\varepsilon y''(x) + e^{-0.5x} y'(x-\delta) + y(x) = 0$, subject to the interval and boundary conditions $y(x) = 1;\ -\delta \leqslant x \leqslant 0$, $y(1) = 1$.

Example 6 [16, p. 808]. $\varepsilon y''(x) - (1+x)y'(x-\delta) - e^{-x}y(x) = 1$, subject to the interval and boundary conditions $y(x) = 1;\ -\delta \leqslant x \leqslant 0$, $y(1) = -1$.

5. Conclusions

Boundary value problems for linear second order singularly perturbed differential-difference equations of convection–diffusion type with a small shift in the convection term is considered. To obtain an approximate solution for such type of boundary value problems a finite difference method is presented. When the shift parameter is smaller than the perturbation parameter, the term containing the shift is expanded in Taylor series and an exponentially fitted tridiagonal finite difference method is developed. It is also analyzed for convergence. The proposed method converges uniformly in ε. When the shift parameter is larger than perturbation parameter a special type of mesh is used, so that the term containing shift lies on nodal points after discretization and a fourth order finite difference method is applied. The truncation error of the finite difference scheme is calculated. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameter on the boundary layer behavior and oscillatory behavior of the solution of the problem.

The maximum absolute error is tabulated in the form of Tables 1 and 2 for the considered examples in support of the predicted theory. The graphs of the solution of the considered examples for different values of shift parameter are plotted in Figs. 1–8 to examine the effect of shift on the boundary layer and oscillatory behavior of the solution.

It is observed that when the shift parameter is smaller than the perturbation parameter, the layer behavior is maintained. As the delay increases, thickness of the layer decreases in the case when the solution exhibits layer behavior on the left side while in the case of the right side boundary layer, it increases. When the shift parameter is greater than the perturbation parameter, it is observed that the layer behavior of the solution is no longer maintained and the solution exhibits oscillatory behavior. Also when the delay further increases the oscillations previously confined to the layer region are extended throughout the interval. From the results, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. On the basis of the numerical results of a variety of examples, it is concluded that the present method offers significant advantage for the linear singularly perturbed differential difference equations.

Acknowledgments

The authors wish to thank the Department of Science & Technology, Government of India, for their financial support under the project No. SR/S4/MS: 598/09.

Authors are grateful to the referees for their valuable suggestions and comments.

References

[1] R.B. Stein, A theoretical analysis of neuronal variability, Biophys. J. 5 (1965) 173–194.
[2] H.C. Tuckwell, On the first-exit time problem for temporally homogeneous Markov processes, J. Appl. Probab. 13 (1976) 39–48.
[3] H.C. Tuckwell, Introduction to Theoretical Neurobiology, vol. 2, Cambridge University Press, Cam-bridge, UK, 1988.
[4] W.J. Wilbur, J. Rinzel, An analysis of Stein's model for stochastic neuronal excitation, Biol. Cybernet. 45 (1982) 107–114.
[5] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary-value problems for differential difference equations. V. Small shifts with layer behavior, SIAM J. Appl. Math. 54 (1994) 249–272.
[6] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary-value problems for differential difference equations. VI. Small shifts with rapid oscillations, SIAM J. Appl. Math. 54 (1994) 273–283.
[7] V.Y. Glizer, Asymptotic solution of a boundary-value problem for linear singularly-perturbed functional differential equations arising in optimal control theory, J. Optim. Theory Appl. 106 (2000) 49–85.
[8] V.Y. Glizer, Block wise estimate of the fundamental matrix of linear singularly perturbed differential systems with small delay and its application to uniform asymptotic solution, J. Math. Anal. 278 (2003) 409–433.
[9] M.W. Derstine, F.A.H.H.M. Gibbs, D.L. Kaplan, Bifurcation gap in a hybrid optical system, Phys. Rev. A 26 (1982) 3720–3722.
[10] A. Longtin, J. Milton, Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci. 90 (1988) 183–199.
[11] M. Wazewska-Czyzewska, A. Lasota, Mathematical models of the red cell system, Mat. Stosow. 6 (1976) 25–40.
[12] M.C. Mackey, L. Glass, Oscillations and chaos in physiological control systems, Science 197 (1977) 287–289.
[13] M.K. Kadalbajoo, K.K. Sharma, Numerical analysis of singularly perturbed delay differential equations with layer behavior, Appl. Math. Comput. 157 (2004) 11–28.
[14] M.K. Kadalbajoo, K.K. Sharma, Parameter-uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior, Electron. Trans. Numer. Anal. 23 (2006) 180–201.
[15] M.K. Kadalbajoo, K.K. Sharma, A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations, Appl. Math. Comput. 197 (2008) 692–707.
[16] M.K. Kadalbajoo, V.P. Ramesh, Hybrid method for numerical solution of singularly perturbed delay differential equations, Appl. Math. Comput. 187 (2007) 797–814.
[17] M.K. Kadalbajoo, D. Kumar, A computational method for singularly perturbed nonlinear differential-difference equations with small shift, Appl. Math. Model. 34 (2010) 2584–2596.
[18] J.I. Ramos, Exponential methods for singularly perturbed ordinary differential-difference equations, Appl. Math. Comput. 182 (2006) 1528–1541.
[19] J. Mohapatra, S. Natesan, Uniformly convergent numerical method for singularly perturbed differential-difference equation using grid equidistribution, Int. J. Numer. Methods Biomed. Eng. 27 (9) (2011) 1427–1445.
[20] G.M. Amiraliyev, E. Cimen, Numerical method for a singularly perturbed convection–diffusion problem with delay, Appl. Math. Comput. 216 (2010) 2351–2359.
[21] H. Tian, Numerical Treatment of Singularly Perturbed Delay Differential Equations, Ph.D. Thesis, University of Manchester, 2000.
[22] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, Robust Computational Techniques for Boundary Layers, CRC Press LLC, Florida, 2000.
[23] R.E. O'Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.
[24] M.M. Chawla, A fourth-order tridiagonal finite difference method for general non-linear two-point boundary value problems with mixed boundary conditions, J. Inst. Math. Appl. 21 (1978) 83–93.
[25] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.
[26] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.
[27] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.