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A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data

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ABSTRACT

A singularly perturbed second order ordinary differential equation having two parameters with a discontinuous source term is presented for numerical analysis. Theoretical bounds on the derivatives, regular and singular components of the solution are derived. A hybrid monotone difference scheme with the method of averaging at the discontinuous point is constructed on Shishkin mesh. Parameter-uniform error bounds for the numerical approximation are established. Numerical results are presented which support the theoretical results.

1. Introduction

Singularly perturbed differential equations arise in many areas of applied mathematics and mathematical physics such as fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory, meteorology, oceanography, rarefied gas dynamics, diffraction theory, reaction–diffusion process, non-equilibrium and radiating flows, Navier–Stokes equations of fluid flow at high Reynolds number, etc. The differential equation depends on a small positive parameter (ε), multiplying the highest derivative term. When the parameter tends to zero ($\varepsilon \to 0$) the problem has a limiting solution which is called the solution of the reduced problem [1] and the regions of non-uniform convergence lie near the boundary, which are known as boundary layers. These problems have steep gradients in the narrow layer regions of the domain in consideration. This causes severe hurdles in the computations for classical numerical methods. In order to capture the layers, a large number of special purpose methods have been developed by the researchers to provide accurate numerical solutions which cover second order equations with single parameter for smooth [1–3] and non smooth data [4–8]. In recent years, authors have considered singularly perturbed second order ordinary differential equation with two small parameters (ε, μ) in smooth data [9–12] and have considered non-smooth data [13,14]. These types of problems are widely found in many applications, for example the model transport phenomena in chemistry [15], Lubrication theory [16], Chemical reactor theory [17] and also in DC motor analysis [18].

In [19] Vigo Aguiar et al. considered a two point boundary value problem for second order ordinary differential equation. A boundary value technique is used in parallel computers to reduce the computation time and showed the reliability and

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performance of the proposed parallel schemes. In [20], the authors proposed a method for numerical solution of singularly perturbed two point boundary value problems, in which the second order BVP is converted into a system of IVPs and second order convergence is shown using exponentially fitted finite difference schemes. P. Das and V. Mehrmann discussed a singularly perturbed parabolic initial boundary value problem for 1-D convection–diffusion–reaction equation containing two small parameters. A moving mesh technique by the equidistribution of a positive monitor function is taken to generate meshes and it shows first order accuracy [21]. A system of coupled singularly perturbed reaction–diffusion problems having diffusion parameters with different magnitudes is considered in [22]. Central difference scheme is used to discretize the problem on equidistribution mesh to obtain an optimal second-order parameter uniform convergence.

Motivated by the works of [5,14,23], we have considered a singularly perturbed reaction–convection–diffusion equation in one dimension with a discontinuous source term of the form:

$$Ly(x) \equiv \varepsilon y''(x) + \mu a(x)y'(x) - b(x)y(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \tag{1}$$

$$y(0) = y_0, \quad y(1) = y_1 \tag{2}$$

$$|[f(d)]| \le C.$$

It is convenient to introduce the notations $\overline{\Omega} = [0, 1]$, $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$, $a(x)$ and $b(x)$ are sufficiently smooth functions in $\overline{\Omega}$ and $f(x)$ is sufficiently smooth in $\Omega^- \cup \Omega^+ \cup \{0, 1\}$. Also $f(x)$ and its derivatives have a jump discontinuity at $d \in \Omega = (0, 1)$ (denoted by $[w](d) = w(d^+) - w(d^-)$), $0 < \varepsilon << 1$, $0 \le \mu \le 1$, $a(x) \ge \alpha > 0$, $b(x) \ge \beta > 0$ and $\rho = \min_{\overline{\Omega}} \left\{\frac{b}{a}\right\}$.

Under these assumptions, the SPP (1)–(2) has a solution $y(x) \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, when $\mu = 1$ the problem is a well known convection–diffusion problem [7] and when $\mu = 0$, we get the reaction–diffusion problem [5,8]. In the present article the following cases are considered $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ and $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$.

Throughout this article C denotes a generic positive constant independent of nodal points, mesh size (N) and the perturbation parameters ε, μ. We measure all functions in the supremum norm, denoted by

$$\|w\|_{\overline{\Omega}} = \sup_{x \in \overline{\Omega}} |w(x)|.$$

The structure of the paper is as follows. In Section 2, we establish an existence theorem for (1)–(2), minimum principle, stability result and some priori estimates on the solution and its derivatives. Section 3 presents a decomposition of the discrete solution to solve the problem, which generates robust numerical approximation to the solution. Truncation error analysis is estimated in Section 4. This analysis gears the main theoretical results presented in Section 5, ε–μ uniform convergence in the maximum norm of the approximations is generated by the numerical method. Numerical examples are provided in Section 6 to illustrate the applicability of the method with maximum pointwise errors, and rate of convergence in the form of tables.

2. A priori bounds on the solution and its derivatives

We commence this section by the following existence theorem.

Theorem 1. *The SPP (1)–(2) has a solution $y(x) \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.*

Proof. The proof is by construction. Let $y_1(x)$, $y_2(x)$ be particular solutions of the differential equations

$$\varepsilon y_1''(x) + \mu a(x)y_1'(x) - b(x)y_1(x) = f(x), \quad x \in \Omega^- \quad \text{and}$$
$$\varepsilon y_2''(x) + \mu a(x)y_2'(x) - b(x)y_2(x) = f(x), \quad x \in \Omega^+.$$

Consider the function

$$y(x) = \begin{cases} y_1(x) + (y(0) - y_1(0))\phi_1(x) + A\phi_2(x), & x \in \Omega^- \\ y_2(x) + B\phi_1(x) + (y(1) - y_2(1))\phi_2(x), & x \in \Omega^+ \end{cases}$$

where $\phi_1(x)$, $\phi_2(x)$ are the solutions of the boundary value problems

$$\varepsilon\phi_1''(x) + \mu a(x)\phi_1'(x) - b(x)\phi_1(x) = 0, \quad x \in \Omega,\ \phi_1(0) = 1,\ \phi_1(1) = 0$$
$$\varepsilon\phi_2''(x) + \mu a(x)\phi_2'(x) - b(x)\phi_2(x) = 0, \quad x \in \Omega,\ \phi_2(0) = 0,\ \phi_2(1) = 1$$

and A, B are constants to be chosen so that $y(x) \in C^1(\Omega)$.

Note that on the open interval $(0, 1)$, $0 < \phi_i < 1$, $i = 1, 2$. Thus ϕ_1, ϕ_2 cannot have an internal maximum or minimum and hence

$$\phi_1'(x) < 0, \qquad \phi_2'(x) > 0, \quad x \in (0, 1).$$

We wish to choose the constants A, B so that, $y(x) \in C^1(\Omega)$. That is, we impose

$$y(d-) = y(d+) \quad \text{and} \quad y'(d-) = y'(d+).$$

For the constants A, B to exist we require that

$$\begin{vmatrix} \phi_2(d) & -\phi_1(d) \\ \phi_2'(d) & -\phi_1'(d) \end{vmatrix} \neq 0.$$

This follows from $\phi_2'(d)\phi_1(d) - \phi_2(d)\phi_1'(d) > 0$. □

The operator L of (1) satisfies the following minimum principle on $\overline{\Omega}$.

Lemma 1 (*Minimum Principle*)**.** *Let us suppose that a function $y(x) \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfy $y(0) \geq 0$, $y(1) \geq 0$, $Ly(x) \leq 0$, $\forall x \in \Omega^- \cup \Omega^+$, $[y](d) = 0$ and $[y'](d) \leq 0$. Then $y(x) \geq 0$, $\forall x \in \overline{\Omega}$.*

Proof. Let x_k be any point at which $y(x_k)$ attains its minimum value in $\overline{\Omega}$. If $y(x_k) \geq 0$, then the result is obvious. Suppose that $y(x_k) < 0$, with the assumptions considered on the boundary value, we have either $x_k \in \Omega^- \cup \Omega^+$ or $x_k = d$. If $x_k \in \Omega^- \cup \Omega^+$ then $y'(x_k) = 0, y''(x_k) \geq 0$ and hence $Ly(x_k) = \varepsilon y''(x_k) + \mu a(x_k)y'(x_k) - b(x_k)y(x_k) > 0$, which is a contradiction.

The only possibility remaining is that $x_k = d$. Our assumption $y(x_k) < 0$ shows $y'(d-) \leq 0$ and $y'(d+) \geq 0$. Hence we have $[y'](d) > 0$, which is a contradiction.

Hence it is clear that $y(x) \geq 0 \; \forall x \in \overline{\Omega}$. □

An immediate consequence of the minimum principle is the following stability result.

Lemma 2 (*Stability Result*)**.** *Let $y(x)$ be a solution of (1)–(2), then*

$$\|y(x)\|_{\overline{\Omega}} \leq \max \left\{ |y_0|, |y_1|, \frac{1}{\beta}\|f(x)\|_{\overline{\Omega}} \right\}.$$

Lemma 3. *Let $y(x)$ be the solution of the problem (1)–(2), where $|y(0)| \leq C$, $|y(1)| \leq C$. Then*

$$\|y'(x)\|_{\overline{\Omega}} \leq \tfrac{C}{\sqrt{\varepsilon}} \left\{1 + \left(\tfrac{\mu}{\sqrt{\varepsilon}}\right)\right\}$$

$$\|y^{(k)}(x)\|_{\overline{\Omega}\setminus\{d\}} \leq \tfrac{C}{\sqrt{\varepsilon}^k} \left\{1 + \left(\tfrac{\mu}{\sqrt{\varepsilon}}\right)^k\right\}, \quad 2 \leq k \leq 4.$$

Proof. The bounds on the derivatives are proved from [9]. □

Before going into the details about decomposition of $y(x)$ into regular $(v(x))$ and singular $(w(x))$ components, we consider the following observations. Let F be a smooth function in $\Omega^- \cup \Omega^+$, F and its derivatives have a jump discontinuity at $d \in \Omega$. Find $y(x) \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, such that

$$\begin{cases} Ly(x) = F(x), & x \in \Omega^- \cup \Omega^+ \\ y(0) = \alpha, & y(1) = \beta. \end{cases} \tag{3}$$

It can be proved that the problem (3) has a unique solution [7]. Let

$$F^{*(k)}(x) = \begin{cases} F^{(k)}(x), & x \in (0, d) \\ F^{(k)}(d^-) & \text{at } x = d, \end{cases}$$

where $F^{(k)}$ stands for kth derivative of F. Further let $y_l^*(x)$ be the solution of

$$\begin{cases} Ly_l^*(x) = F^*(x), & x \in (0, d) \\ y_l^*(0) = \alpha, & y_l^*(d) = y(d). \end{cases} \tag{4}$$

Similarly one can define $y_r^*(x)$ on the interval $[d, 1]$. It can be verified that

$$y(x) = \begin{cases} y_l^*(x), & x \in [0, d) \\ y_l^*(d) = y_r^*(d), & \\ y_r^*(x), & x \in (d, 1]. \end{cases}$$

It can be verified that the solution $y(x)$ of the BVP (1)–(2), can be decomposed as $y(x) = v(x) + w_l(x) + w_r(x)$, where

$$Lv = f, \quad x \in \Omega^- \cup \Omega^+, \tag{5}$$

$$v(0) = y(0), \qquad v(1) = y(1) \quad \text{and} \quad v(d^-), v(d^+) \text{ are chosen suitably} \tag{6}$$

and

$$Lw_l(x) = 0, \quad x \in \Omega^- \cup \Omega^+, \tag{7}$$
$$w_l(0) = y(0) - v(0) - w_r(0), \qquad w_l(1) = 0,$$
$$Lw_r(x) = 0, \quad x \in \Omega^- \cup \Omega^+, \tag{8}$$
$$\text{and } w_r(0) \text{ is suitably chosen, } w_r(1) = y(1) - v(1),$$
$$[w_r(d)] = -[v(d)] - [w_l(d)], \qquad [w_r'(d)] = -[v'(d)] - [w_l'(d)]. \tag{9}$$

Hence, $v(x)$, $w_l(x)$ and $w_r(x)$ are discontinuous at $x = d$, but by (9) their sum is in $C^1(\Omega)$.

Note that,

$$v(x) = \begin{cases} v^-(x), & x \in \Omega^- \\ v^+(x), & x \in \Omega^+, \end{cases} \qquad w_l(x) = \begin{cases} w_l^-(x), & x \in \Omega^- \\ w_l^+(x), & x \in \Omega^+, \end{cases}$$
$$\text{and} \quad w_r(x) = \begin{cases} w_r^-(x), & x \in \Omega^- \\ w_r^+(x), & x \in \Omega^+. \end{cases}$$

Case(i): Consider $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$.

Let $v(x) = v_0(x) + \sqrt{\varepsilon} v_1(x) + \sqrt{\varepsilon}^2 v_2(x) + \sqrt{\varepsilon}^3 v_3(x)$, where $v_0(x),\ v_1(x)$ and $v_2(x)$ are the solutions of the following problems.

$$-b(x)v_0(x) = f(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$b(x)v_1(x) = \frac{\mu}{\sqrt{\varepsilon}} a(x)v_0'(x) + \sqrt{\varepsilon} v_0''(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$b(x)v_2(x) = \frac{\mu}{\sqrt{\varepsilon}} a(x)v_1'(x) + \sqrt{\varepsilon} v_1''(x), \quad x \in \Omega^- \cup \Omega^+.$$

Choose $v_3(x) \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, such that

$$\begin{cases} Lv_3(x) = \dfrac{-\mu}{\sqrt{\varepsilon}} a(x)v_2'(x) - \sqrt{\varepsilon} v_2''(x), & x \in \Omega^- \cup \Omega^+, \\ v_3(0) = v_3(1) = 0. \end{cases} \tag{10}$$

Lemma 4. *When $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ and for each integer k satisfying $0 \le k \le 4$, the smooth component $v(x)$ satisfies the following bounds*

$$\|v^{(k)}(x)\|_{\overline{\Omega}\setminus\{d\}} \le C\left(1 + \frac{1}{(\sqrt{\varepsilon})^{k-3}}\right). \tag{11}$$

Proof. With sufficient smoothness on the co-efficient $a(x), b(x)$ in $\overline{\Omega}$ and $f(x)$ in $\Omega^- \cup \Omega^+$, we observed that $v_0(x),\ v_1(x),\ v_2(x)$ are bounded independently. To bound $v_3(x)$ define the barrier function

$$\varphi^{\pm}(x) = \max\{|v_3(0)|, |v_3(1)|\} + \frac{1}{\beta}\left(\|v_2''(x)\| + \|v_2'(x)\|\right) \pm v_3(x).$$

Clearly $\varphi^{\pm}(0) \ge 0$, $\varphi^{\pm}(1) \ge 0$ and $L\varphi^{\pm}(x) \le 0$. Therefore, by applying minimum principle to $\varphi^{\pm}(x)$, we obtain

$$\|v_3(x)\| \le \max\{|v_3(0)|, |v_3(1)|\} + \frac{1}{\beta}\left(\|v_2''(x)\| + \|v_2'(x)\|\right).$$

Using the bounds on $v_2(x)$ we say $\|v_3(x)\| \le C$. Using mean value theorem and (10), we obtain

$$\|v_3^k(x)\| \le \frac{C}{\left(\sqrt{\varepsilon}\right)^k} \max\left\{\|v_3(x)\|, \|v_2'(x)\|, \|v_2''(x)\|\right\}$$
$$\le \frac{C}{\left(\sqrt{\varepsilon}\right)^k}, \quad k = 1, 2.$$
$$\text{Also, } \|v_3^k(x)\| \le \frac{C}{\left(\sqrt{\varepsilon}\right)^k}, \quad k = 3, 4$$

by using all the bounds derived above in $v(x)$, we have

$$\|v^k(x)\| \le C\left(1 + \frac{1}{\left(\sqrt{\varepsilon}\right)^{k-3}}\right), \quad \text{for } 0 \le k \le 4. \quad \square$$

Using the defined problem (7), (8) and the methods discussed in [9,11], we have the following lemma.

Lemma 5. *When $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$, the singular components $w_l(x)$ and $w_r(x)$ satisfy the bounds*

$$\|w_l^{(k)}(x)\|_{\overline{\Omega}\setminus\{d\}} \le \frac{C}{\sqrt{\varepsilon}^k}\begin{cases} e^{-\theta_1 x}, & x \in \Omega^-, \\ e^{-\theta_1 (x-d)}, & x \in \Omega^+, \end{cases} \quad 0 \le k \le 4$$

$$\|w_r^{(k)}(x)\|_{\overline{\Omega}\setminus\{d\}} \le \frac{C}{\sqrt{\varepsilon}^k}\begin{cases} e^{-\theta_2 (d-x)}, & x \in \Omega^-, \\ e^{-\theta_2 (1-x)}, & x \in \Omega^+, \end{cases} \quad 0 \le k \le 4$$

where

$$\theta_1 = \frac{\sqrt{\rho\alpha}}{2\sqrt{\varepsilon}} \quad \text{and} \quad \theta_2 = \frac{\sqrt{\rho\alpha}}{2\sqrt{\varepsilon}}. \tag{12}$$

Case(ii): Consider $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$.
Let $v(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \varepsilon^3 v_3(x)$, where $v_0(x)$, $v_1(x)$ and $v_2(x)$ are the solutions of the problems.

$$\begin{aligned} &L_\mu v_0(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \\ &v_0(d-,\mu), \qquad v_0(1,\mu), \text{ are chosen}, \\ &L_\mu v_1(x) = -v_0''(x), \quad x \in \Omega^- \cup \Omega^+, \\ &v_1(d-,\mu), \qquad v_1(1,\mu), \text{ are chosen}, \\ &\text{and } L_\mu v_2(x) = -v_1''(x), \quad x \in \Omega^- \cup \Omega^+, \\ &v_2(d-,\mu), \quad v_2(1,\mu), \text{ are chosen}, \end{aligned}$$

where $L_\mu z(x) = \mu a(x) z'(x) - b(x) z(x)$. Choose $v_3(x) \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, such that

$$\begin{cases} L v_3(x) = -v_2''(x), & x \in \Omega^- \cup \Omega^+, \\ v_3(0) = v_3(1) = 0. \end{cases} \tag{13}$$

Lemmas 6 and 7 can be proved following [9].

Lemma 6. *When $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$ and for each integer k satisfying $0 \le k \le 4$, the smooth component $v(x)$ satisfies the following bounds*

$$\|v^{(k)}(x)\|_{\overline{\Omega}\setminus\{d\}} \le C\left(1 + \left(\frac{\varepsilon}{\mu}\right)^{3-k}\right). \tag{14}$$

Lemma 7. *When $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$, the singular components $w_l(x)$ and $w_r(x)$ satisfy the bounds*

$$\|w_l^{(k)}(x)\|_{\overline{\Omega}\setminus\{d\}} \le C\left(\frac{\mu}{\varepsilon}\right)^k \begin{cases} e^{-\theta_1 x}, & x \in \Omega^-, \\ e^{-\theta_1 (x-d)}, & x \in \Omega^+, \end{cases} \quad 0 \le k \le 4$$

$$\|w_r^{(k)}(x)\|_{\overline{\Omega}\setminus\{d\}} \le \left(\frac{C}{\mu^k}\right) \begin{cases} e^{-\theta_2 (d-x)}, & x \in \Omega^-, \\ e^{-\theta_2 (1-x)}, & x \in \Omega^+, \end{cases} \quad 0 \le k \le 4$$

where

$$\theta_1 = \frac{\alpha\mu}{2\varepsilon} \quad \text{and} \quad \theta_2 = \frac{\rho}{2\mu}. \tag{15}$$

3. Discrete problem

In this section, an appropriate piecewise uniform mesh for the BVP (1)–(2) is introduced, and classical upwind finite difference schemes are used on this mesh to obtain the numerical solution. On $\overline{\Omega}$ a piecewise uniform mesh of N mesh intervals is constructed as follows. The domain $\overline{\Omega}$ is divided into six subintervals as $\overline{\Omega} = [0, \tau_1] \cup [\tau_1, d - \tau_2] \cup [d - \tau_2, d] \cup [d, d + \tau_3] \cup [d + \tau_3, 1 - \tau_4] \cup [1 - \tau_4, 1]$, for some τ_1, τ_2, τ_3 and τ_4, such that, $0 < \tau_1, \tau_2 \le d/4$ and $0 < \tau_3, \tau_4 \le (1-d)/4$. The interior points of the mesh are denoted by

$$\Omega^N_{\varepsilon/\mu} = \left\{x_i : 1 \le i \le \frac{N}{2} - 1\right\} \cup \left\{x_i : \frac{N}{2} + 1 \le i \le N - 1\right\}.$$

Clearly $x_{N/2} = d$ and $\overline{\Omega}^N_{\varepsilon/\mu} = \{x_i\}_0^N \cup \{d\}$. It is fitted to the problem by choosing $\tau_1,\ \tau_2,\ \tau_3$ and τ_4 to be the following functions of $N,\ \varepsilon$ and μ.

$$\begin{cases} \tau_1 = \min\left\{\dfrac{d}{4}, \dfrac{2}{\theta_1}\ln N\right\}, \qquad \tau_2 = \min\left\{\dfrac{d}{4}, \dfrac{2}{\theta_2}\ln N\right\}, \\ \tau_3 = \min\left\{\dfrac{1-d}{4}, \dfrac{2}{\theta_1}\ln N\right\}, \qquad \tau_4 = \min\left\{\dfrac{1-d}{4}, \dfrac{2}{\theta_2}\ln N\right\}, \end{cases} \tag{16}$$

where θ_1 and θ_2 are defined in the previous section.

On the subintervals $[0, \tau_1]$, $[d - \tau_2, d]$, $[d, d + \tau_3]$ and $[1 - \tau_4, 1]$ a uniform mesh with $N/8$ mesh intervals is placed, whereas $[\tau_1, d - \tau_2]$ and $[d + \tau_3, 1 - \tau_4]$ have a uniform mesh with $N/4$ mesh intervals. The interior points of the mesh are denoted by the step sizes in each subinterval by $h_1 = 8\tau_1/N$, $h_2 = 4(d - \tau_1 - \tau_2)/N$, $h_3 = 8\tau_2/N$, $h_4 = 8\tau_3/N$, $h_5 = 4(1 - d - \tau_3 - \tau_4)/N$, and $h_6 = 8\tau_4/N$. The mesh points are given by

$$x_i = \begin{cases} ih_1, & \text{if } 0 \le i \le \dfrac{N}{8}, \\ \tau_1 + ih_2, & \text{if } \dfrac{N}{8} \le i \le \dfrac{3N}{8}, \\ d - \tau_2 + ih_3, & \text{if } \dfrac{3N}{8} \le i \le \dfrac{N}{2}, \\ d + ih_4, & \text{if } \dfrac{N}{2} \le i \le \dfrac{5N}{8}, \\ d + \tau_3 + ih_5, & \text{if } \dfrac{5N}{8} \le i \le \dfrac{3N}{4}, \\ 1 - \tau_4 + ih_6, & \text{if } \dfrac{3N}{4} \le i \le N - 1. \end{cases}$$

Set $h_{i+1} = x_{i+1} - x_i$ and $\overline{h}_i = (h_i + h_{i+1})/2$ for $i = 0, 1, 2, \dots, N - 1$. On the piecewise uniform mesh $\overline{\Omega}^N_{\varepsilon/\mu}$, we discretize the BVP (1)–(2) as

$$\begin{aligned} L^N \equiv \varepsilon\delta^2 Y(x_i) + \mu a(x_i) D^+ Y(x_i) - b(x_i)Y(x_i) = f(x_i), \quad x_i \in \Omega^N_{\varepsilon/\mu} \\ Y_0 = y(0), \qquad Y_N = y(1) \\ D^+Y(x_{N/2}) - D^-Y(x_{N/2}) = 0. \end{aligned} \tag{17}$$

The above discrete problem (17) is discretized using the combinations of mid-point scheme, upwind scheme and second order central difference scheme. That is,

$$L_c^N Y(x_i) = \varepsilon\delta^2 Y(x_i) + \mu a(x_i) D^0 Y(x_i) - b(x_i)Y(x_i) = f(x_i), \tag{18}$$

$$L_u^N Y(x_i) = \varepsilon\delta^2 Y(x_i) + \mu a(x_i) D^+ Y(x_i) - b(x_i)Y(x_i) = f(x_i), \tag{19}$$

$$L_m^N Y(x_i) = \varepsilon\delta^2 Y(x_i) + \mu \bar{a}(x_i) D^+ Y(x_i) - \overline{b(x_i)Y(x_i)} = \bar{f}(x_i), \tag{20}$$

$$L_T^N Y(x_{N/2}) = \varepsilon\delta^2 Y(x_{N/2}) + \mu a(x_{N/2}) D^+ Y(x_{N/2}) - b(x_{N/2})Y(x_{N/2}) = \hat{f}(x_{N/2}) \tag{21}$$

where,

$$\bar{z}(x_i) = z\left((x_i + x_{i+1})/2\right), \qquad D^+Y(x_i) = \frac{Y(x_{i+1}) - Y(x_i)}{h_{i+1}},$$

$$D^-Y(x_i) = \frac{Y(x_i) - Y(x_{i-1})}{h_i}, \qquad D^0Y(x_i) = \frac{Y(x_{i+1}) - Y(x_{i-1})}{h_i + h_{i+1}},$$

$$\delta^2 Y(x_i) = \frac{1}{\bar{h}_i}\left(D^+Y(x_i) - D^-Y(x_i)\right) \quad \text{and} \quad \hat{f}(x_{N/2}) = \frac{f(x_{N/2-1}) + f(x_{N/2+1})}{2}.$$

For $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$,

$$L^N Y(x_i) \equiv \begin{cases} L_c^N Y(x_i), & \text{if } x_i \in \{(0, \tau_1), (d, d + \tau_3)\}, \\ L_c^N Y(x_i), & \text{if } x_i \in \{(\tau_1, d - \tau_2), (d - \tau_2, d), (d + \tau_3, 1 - \tau_4) \\ & \quad \text{and } (1 - \tau_4, 1)\}, \quad \text{for } \mu h_k \|a\| < 2\varepsilon\ (k = 2, 3, 5, 6) \\ L_T^N Y(x_i), & \text{if } x_i = d. \end{cases}$$

For $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$,

$$L^NY(x_i) \equiv \begin{cases} L_c^NY(x_i), & \text{if } x_i \in \{(0,\tau_1),\,(d,d+\tau_3)\},\\ L_m^NY(x_i), & \text{if } x_i \in \{(\tau_1, d-\tau_2),\ \text{for } \mu h_2\|a\| \geq 2\varepsilon,\ \|b\|h_2 < 2\mu\alpha \\ \quad \text{and } (d+\tau_3, 1-\tau_4) \text{ for } \mu h_5\|a\| \geq 2\varepsilon,\ \|b\|h_5 < 2\mu\alpha\},\\ L_u^NY(x_i), & \text{if } x_i \in \{(\tau_1, d-\tau_2),\ \text{for } \mu h_2\|a\| \geq 2\varepsilon,\ \|b\|h_2 \geq 2\mu\alpha \\ \quad \text{and } (d+\tau_3, 1-\tau_4) \text{ for } \mu h_5\|a\| \geq 2\varepsilon,\ \|b\|h_5 \geq 2\mu\alpha\},\\ L_m^NY(x_i), & \text{if } x_i \in \{(d-\tau_2, d),\ \text{for } \mu h_3\|a\| \geq 2\varepsilon,\ \text{and } (1-\tau_4, 1) \text{ for } \mu h_6\|a\| \geq 2\varepsilon\},\\ L_T^NY(x_i), & \text{if } x_i = d. \end{cases}$$

At the transition points τ_1 and $d+\tau_3$ the scheme is given by

$$L^NY(x_i) \equiv \begin{cases} L_c^NY(x_i), & \text{if } x_i = \tau_1 = \dfrac{d}{4},\ d+\tau_3 = \dfrac{5d}{4},\\ L_m^NY(x_i), & \text{if } x_i = \tau_1 < \dfrac{d}{4},\ \text{for } \|b\|h_2 < 2\mu\alpha \text{ and } d+\tau_3 < \dfrac{5d}{4},\ \text{for } \|b\|h_5 < 2\mu\alpha,\\ L_u^NY(x_i), & \text{otherwise.} \end{cases}$$

At the transition points $(d-\tau_2)$ and $(1-\tau_4)$ the scheme is given by

$$L^NY(x_i) \equiv \begin{cases} L_c^NY(x_i), & \text{if } x_i = d-\tau_2 = \dfrac{3d}{4},\ 1-\tau_4 = 1-\dfrac{d}{4},\\ L_m^NY(x_i), & \text{if } x_i = d-\tau_2 < \dfrac{3d}{4},\ 1-\tau_4 < 1-\dfrac{d}{4},\\ L_u^NY(x_i), & \text{otherwise.} \end{cases}$$

On the piecewise uniform mesh $\overline{\Omega}^N_{\varepsilon/\mu}$ the scheme is given by

$$L^NY(x_i) = r_i^-Y(x_{i-1}) + r_i^cY(x_i) + r_i^+Y(x_{i+1}) = Q^Nf(x_i), \tag{22}$$

where

$$\begin{aligned} &r_i^- = \frac{\varepsilon}{h_i\bar{h}_i} - \frac{\mu a(x_i)}{2\bar{h}_i}, \qquad r_i^+ = \frac{\varepsilon}{h_{i+1}\bar{h}_i} + \frac{\mu a(x_i)}{2\bar{h}_i},\\ &r_i^c = -r_i^+ - r_i^- - b(x_i), \quad \text{if } L^N \equiv L_c^N, && (23)\\ &r_i^- = \frac{\varepsilon}{h_i\bar{h}_i}, \qquad r_i^+ = \frac{\varepsilon}{h_{i+1}\bar{h}_i} + \frac{\mu a(x_i)}{h_{i+1}},\\ &r_i^c = -r_i^+ - r_i^- - b(x_i), \quad \text{if } L^N \equiv L_u^N, && (24)\\ &r_i^- = \frac{\varepsilon}{h_i\bar{h}_i}, \qquad r_i^+ = \frac{\varepsilon}{h_{i+1}\bar{h}_i} + \frac{\mu \bar{a}(x_i)}{h_{i+1}} - \frac{b(x_{i+1})}{2},\\ &r_i^c = -r_i^+ - r_i^- - \bar{b}(x_i), \quad \text{if } L^N \equiv L_m^N, && (25)\\ &r_i^- = \frac{\varepsilon}{h_i\bar{h}_i}, \qquad r_i^+ = \frac{\varepsilon}{h_{i+1}\bar{h}_i} + \frac{\mu a(x_i)}{h_{i+1}},\\ &r_i^c = -r_i^+ - r_i^- - b(x_i), \quad \text{if } L^N \equiv L_T^N, && (26) \end{aligned}$$

and

$$Q^Nf(x_i) = \begin{cases} f_i, & \text{if } L^N \equiv L_c^N \ \text{ or } L_u^N,\\ \bar{f}(x_i), & \text{if } L^N \equiv L_m^N,\\ \hat{f}(x_i), & \text{if } L^N \equiv L_T^N. \end{cases}$$

In $(0,\tau_1)$ and $(d, d+\tau_3)$ note that

$$\begin{aligned} &\mu\|a\|h_1/2\varepsilon = 4\mu\|a\|\tau_1/\varepsilon N \leq 16\|a\|\ln N/\alpha N, && (27)\\ &\mu\|a\|h_4/2\varepsilon = 4\mu\|a\|\tau_3/\varepsilon N \leq 16\|a\|\ln N/\alpha N. \end{aligned}$$

For $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$ in $(d-\tau_2, d)$ and $(1-\tau_4, 1)$,

$$\begin{aligned} &\mu\|a\|h_3/2\varepsilon = 4\mu\|a\|\tau_2/\varepsilon N \leq 16\|a\|\ln N/\alpha N, && (28)\\ &\mu\|a\|h_6/2\varepsilon = 4\mu\|a\|\tau_4/\varepsilon N \leq 16\|a\|\ln N/\alpha N, \end{aligned}$$

and for $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$ in $(d-\tau_2, d)$ and $(1-\tau_4, 1)$,

$$\|b\|h_3/2\alpha\mu = 4\|b\|\tau_2/\alpha\mu N \leq 16\|b\|\ln N/\alpha N,$$
$$\|b\|h_6/2\alpha\mu = 4\|b\|\tau_4/\alpha\mu N \leq 16\|b\|\ln N/\alpha N. \tag{29}$$

To guarantee a monotone difference operator L^N, the following mild assumption is imposed on the minimum number of mesh points

$$N(\ln N)^{-1} > 16\max\{\|a\|/\alpha, \|b\|/\alpha\rho\}. \tag{30}$$

Thus the discrete problem is

$$L^N Y(x_i) = Q^N f(x_i), \quad x_i \in \Omega^N_{\varepsilon/\mu} \cup \{d\}, \tag{31}$$
$$Y(0) = y(0), \qquad Y(1) = y(1). \tag{32}$$

The following lemma shows that the finite difference operator L^N has properties analogous to those of the differential operator L.

Lemma 8 (*Discrete Minimum Principle*). *Suppose that a mesh function $Y(x_i)$ satisfies*

$$Y(0) \geq 0, \qquad Y(1) \geq 0, \qquad L^N Y(x_i) \leq 0, \quad \forall\, x_i \in \Omega^N_{\varepsilon/\mu} \cup \{d\},$$

then $Y(x_i) \geq 0, \quad \forall\, x_i \in \overline{\Omega}^N_{\varepsilon/\mu}$.

Proof. Here we will show that the operator defined in (22) guarantee a M-matrix. Hence it is important to check the conditions

$$r_i^- > 0,\ r_i^+ > 0, \quad r_i^- + r_i^c + r_i^+ < 0, \tag{33}$$

for all the operators defined in $L^N Y(x_i)$.

The central difference operator L_c^N satisfies the conditions (33), if

$$r_i^- = \frac{\varepsilon}{h_i \bar{h}_i} - \frac{\mu\|a\|}{2\bar{h}_i} > 0.$$

Consider the cases $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$ or $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$. For both the cases, L_c^N operator is applied in the left layer region $(0, \tau_1)$ and $(d, d+\tau_3)$, $r_i^- > 0$ is guaranteed here with the definition on h_1 and h_4. Consider the remaining regions in the case of $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$, L_c^N operator is used, if $\mu h_k\|a\| < 2\varepsilon$ $(k = 2, 3, 5, 6)$ respectively to guarantee $r_i^- > 0$. This can be proved using (28)–(30).

When $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$, the mid point operator L_m^N is used in the right layer region $(d-\tau_2, d)$ and $(1-\tau_4, 1)$ if $\|b\|h_3 < 2\mu\alpha$ and $\|b\|h_6 < 2\mu\alpha$. Hence $r_i^+ > 0$ is satisfied. In the coarse mesh region $(\tau_1, d-\tau_2)$ and $(d+\tau_3, 1-\tau_4)$ mid point operator L_m^N is applied if $\|b\|h_2 < 2\mu\alpha$ and $\|b\|h_5 < 2\mu\alpha$ on each interval respectively, providing $r_i^+ > 0$. If $\|b\|h_2, \|b\|h_5 \geq 2\mu\alpha$, to guarantee $r_i^+ > 0$ the upwind operator L_u^N is used.

At $x_i = d$, the mid-point operator L_m^N is used. For $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$, $r_i^+ > 0$ is obvious, $r_i^- > 0$ is guaranteed, since $\|b\|h_3^2, \|b\|h_4^2 \leq \varepsilon/4$. When $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$, $r_i^+ > 0$ is natural and $r_i^- > 0$ is true since $\|b\|h_3, \|b\|h_4 \leq \mu\alpha/2$, hence (33) is satisfied for both the cases.

Combining all the above operators defined in various mesh points we say that these operators guarantee M-matrix and hence $L^N Y(x_i)$ satisfies discrete minimum principle [2]. □

Lemma 9 (*Discrete Stability Result*). *If $Y(x_i)$ is the solution of (31)–(32), then $|Y(x_i)| \leq C$, for all $x_i \in \overline{\Omega}^N_{\varepsilon/\mu}$.*

Proof. Define the mesh function

$$\psi(x_i) = M \pm Y(x_i)$$

where $M = \max\left\{|Y(0)|, |Y(1)|, \frac{1}{\beta}\|Q^N f(x_i)\|_{\overline{\Omega}^N_{\varepsilon/\mu}}\right\}$.

Clearly $\psi(x_i) \in C^0(\overline{\Omega})$, $\psi(0) \geq 0$, $\psi(1) \geq 0$. Now for each $x_i \in \Omega^N_{\varepsilon/\mu}$

$$L_c^N \psi(x_i) = -b(x_i)M \pm L_c^N Y(x_i) \leq 0,$$

similarly,

$$L_m^N \psi(x_i) \leq 0 \quad \text{and} \quad L_u^N \psi(x_i) \leq 0.$$

At the point $x_{N/2} = d$,

$$L_T^N \psi(x_{N/2}) = -b(x_{N/2})\, M \pm L_T^N Y(x_{N/2}) \leq 0.$$

Applying the discrete minimum principle of Lemma 8, it follows that

$$\psi(x_i) \geq 0 \quad \forall\, x_i \in \overline{\Omega}^N_{\varepsilon/\mu}.$$

This leads to the required result

$$|Y(x_i)| \leq C, \quad \forall\, x_i \in \overline{\Omega}^N_{\varepsilon/\mu}. \quad \square$$

4. Truncation error analysis

The solution of the discrete problem (31)–(32) can be decomposed as $Y(x_i) = V(x_i) + W_l(x_i) + W_r(x_i)$. Let us denote the error at each mesh point $x_i \in \overline{\Omega}^N_{\varepsilon/\mu}$ by $e(x_i) = |Y(x_i) - y(x_i)|$. To bound the nodal error $|e(x_i)|$, the argument is divided into two main parts. Initially, we define mesh functions $V^-(x_i)$ and $V^+(x_i)$ which approximate $V(x_i)$ respectively to the left and right sides of the point of discontinuity $x_i = d$. Then, we construct mesh functions $W_l^-(x_i)$, $W_l^+(x_i)$ and $W_r^-(x_i)$, $W_r^+(x_i)$ to approximate respectively $W_l(x_i)$ and $W_r(x_i)$ on each side of $x_i = d$. Using these mesh functions the nodal error $|e(x_i)|$ is bounded separately outside and inside the layers.

Let $V^-(x_i)$ and $V^+(x_i)$ be respectively, the solutions of the following discrete problems

$$\begin{aligned} &L^N V^-(x_i) = f(x_i), \quad \forall\, x_i \in \Omega^N_{\varepsilon/\mu} \cap \Omega^-, \\ &V^-(0) = v(0), \qquad V^-(d) = v(d^-), \end{aligned}$$

and

$$\begin{aligned} &L^N V^+(x_i) = f(x_i), \quad \forall\, x_i \in \Omega^N_{\varepsilon/\mu} \cap \Omega^+, \\ &V^+(d) = v(d^+), \qquad V(1) = v(1). \end{aligned}$$

Define $V(x_i)$ as

$$V(x_i) = \begin{cases} V^-(x_i), & 1 \leq i \leq N/2 - 1, \\ V^+(x_i), & N/2 + 1 \leq i \leq N - 1. \end{cases} \tag{34}$$

Define $W(x_i)$ as

$$W(x_i) = W_l(x_i) + W_r(x_i). \tag{35}$$

Further, let $W_l^-(x_i)$, $W_l^+(x_i)$, $W_r^-(x_i)$ and $W_r^+(x_i)$ be respectively, the solutions of the following problems:

$$\begin{aligned} &L^N W_l^-(x_i) = 0 \quad \text{on } 1 \leq i \leq N/2 - 1, \\ &W_l^-(0) = w_l^-(0), \qquad W_l^-(d) = w_l^-(d), \\ &L^N W_l^+(x_i) = 0 \quad \text{on } N/2 + 1 \leq i \leq N - 1, \\ &W_l^+(d) = w_l^+(d), \qquad W_l^+(1) = 0, \\ &L^N W_r^-(x_i) = 0 \quad \text{on } 1 \leq i \leq N/2 - 1, \\ &W_r^-(0) = 0, \qquad W_r^-(d) = w_r^-(d), \\ &L^N W_r^+(x_i) = 0 \quad \text{on } N/2 + 1 \leq i \leq N - 1, \\ &W_r^+(d) = 0, \qquad W_r^+(1) = w_r^+(1). \\ &\left(V^- + W_l^- + W_r^-\right)(d) = \left(V^+ + W_l^+ + W_r^+\right)(d), \\ &L_T^N \left(V^- + W_l^- + W_r^-\right)(d) = L_T^N \left(V^+ + W_l^+ + W_r^+\right)(d). \end{aligned}$$

Note that we can define $Y(x_i)$ to be

$$Y(x_i) = \begin{cases} \left(V^- + W_l^- + W_r^-\right)(x_i), & x_i \in \Omega^N_{\varepsilon/\mu} \cap \Omega^-, \\ \left(V^- + W_l^- + W_r^-\right)(d) = \left(V^+ + W_l^+ + W_r^+\right)(d), \\ \left(V^+ + W_l^+ + W_r^+\right)(x_i), & x_i \in \Omega^N_{\varepsilon/\mu} \cap \Omega^+. \end{cases}$$

Lemma 10. *We have the following bounds on $W_l^-(x_i)$, $W_l^+(x_i)$, $W_r^-(x_i)$ and $W_r^+(x_i)$*

$$|W_l^-(x_i)| \leq C \prod_{j=1}^{k_1} (1 + \theta_1 h_j)^{-1} = \psi_{li}^-, \qquad \psi_{l0}^- = C,$$

$$|W_l^+(x_i)| \le C \prod_{j=N/2+1}^{k_2} (1+\theta_1 h_j)^{-1} = \psi_{li}^+, \qquad \psi_{lN/2}^+ = C,$$

$$|W_r^-(x_i)| \le C \prod_{j=k_1+1}^{N/2} (1+\theta_2 h_j)^{-1} = \psi_{ri}^-, \qquad \psi_{rN/2}^- = C,$$

$$|W_r^+(x_i)| \le C \prod_{j=k_2+1}^{N} (1+\theta_2 h_j)^{-1} = \psi_{ri}^+, \qquad \psi_{rN}^+ = C,$$

where $W_l^-(x_i)$, $W_l^+(x_i)$, $W_r^-(x_i)$ and $W_r^+(x_i)$ are defined above. θ_1 and θ_2 are chosen from (12) and (15).

Proof. Define the barrier functions

$$\psi_{li}^- = \begin{cases} \prod_{j=1}^{k_1} \left(1+\theta_1 h_j\right)^{-1}, & 1 \le k_1 \le N/2, \\ 1, & k_1 = 0, \end{cases}$$

$$\psi_{ri}^- = \begin{cases} \prod_{j=k_1+1}^{N/2} \left(1+\theta_2 h_j\right), & 1 \le k_1 < N/2, \\ 1, & k_1 = N/2, \end{cases}$$

where,

$$\theta_1 = \begin{cases} \dfrac{\sqrt{\rho\alpha}}{2\sqrt{\varepsilon}}, & \text{if } \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}, \\ \dfrac{\mu\alpha}{2\varepsilon}, & \text{if } \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}, \end{cases} \tag{36}$$

$$\theta_2 = \begin{cases} \dfrac{\sqrt{\rho\alpha}}{2\sqrt{\varepsilon}}, & \text{if } \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}, \\ \dfrac{\rho}{2\mu}, & \text{if } \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}. \end{cases} \tag{37}$$

To prove, $L^N\psi_{li}^- \le 0$ and $L^N\psi_{ri}^- \le 0$. Applying discrete operator (22) on ψ_{li}^- we have,

$$\begin{aligned} L^N\psi_{li}^- &= \psi_{li}^- \left((1+\theta_1 h_i)\, r_i^- + r_i^c + \frac{1}{1+\theta_1 h_{i+1}} r_i^+ \right) \\ &= \psi_{li}^- \left(r_i^- + r_i^c + r_i^+ - \theta_1 \left(\frac{h_{i+1} r_i^+}{1+\theta_1 h_{i+1}} - h_i r_i^- \right) \right). \end{aligned}$$

As defined earlier, various discretization methods used in the operator L^N are discussed here. Consider the central difference operator.

$$L_c^N\psi_{li}^- = \psi_{li+1}^- \left[2\varepsilon\theta_1^2 \left(\tfrac{h_{i+1}}{2\bar{h}_i} - 1 \right) + \left(2\varepsilon\theta_1^2 - \mu a_i\theta_1 - 2b_i \right) + \mu a_i\theta_1 \left(1-\theta_1 h_i\right) \tfrac{h_{i+1}}{2\bar{h}_i} + b_i\theta_1 h_{i+1} \right] \le 0.$$

Hence,

$$L_c^N\psi_{li}^- \le \psi_{li+1}^- \left(2\varepsilon\theta_1^2 - \mu a_i\theta_1 - b_i \right) \le 0.$$

Applying (36) for both the cases we get

$$\begin{aligned} L_c^N\psi_{li}^- &\le \psi_{li+1}^- \left(\frac{\rho\alpha}{2} - 2b_i - \mu a_i \frac{\sqrt{\rho\alpha}}{2\sqrt{\varepsilon}} \right) \\ &\le \psi_{li+1}^- \left(\frac{\rho\alpha}{2} - 2b_i \right) \le 0 \end{aligned}$$

and

$$L_c^N\psi_{li}^- \le \psi_{li+1}^- \left(\frac{\mu^2\alpha}{2\varepsilon}(\alpha - a_i) - 2b_i \right) \le 0.$$

For the upwind scheme, we can show that

$$L_u^N\psi_{li}^- \le \psi_{li+1}^- \left(\varepsilon\theta_1^2 - \mu a_i\theta_1 - b_i \right) \le 0$$

for the mid point scheme, we can prove

$$L_m^N \psi_{li}^- \le \psi_{li+1}^- \left(\varepsilon\theta_1^2 - \mu\bar{a}_i\theta_1 - \bar{b}_i\right) \le 0$$

we can complete the argument similarly as done for the central difference scheme.

Now consider the right layer barrier function ψ_{ri}^-, operating the discrete operator of (22) on ψ_{ri}^- we find

$$L^N\psi_{ri}^- = \psi_{ri}^- \left[r_i^- + r_i^c + r_i^+ - \theta_2 \left(\frac{h_i r_i^-}{1+\theta_2 h_i} - h_{i+1} r_i^+ \right) \right].$$

Applying the central difference scheme to $L^N\psi_{ri}^-$, we get

$$\begin{aligned} L_c^N\psi_{ri}^- &= \frac{\psi_{ri}^-}{1+\theta_2 h_i} \left[2\varepsilon\theta_2^2 \left(h_i/\bar{h}_i - 1\right) + \left(2\varepsilon\theta_2^2 + \mu a_i\theta_2 - 2b_i\right)(1+\theta_2 h_i) - 2\varepsilon\theta_2^3 h_i \right] \\ &\le \psi_{ri}^- \left(2\varepsilon\theta_2^2 + \mu a_i\theta_2 - 2b_i\right) \le 0, \end{aligned}$$

for the upwind scheme, we can show that

$$L_u^N\psi_{ri}^- \le \psi_{ri}^- \left(2\varepsilon\theta_2^2 + \mu a_i\theta_2 - b_i\right) \le 0.$$

Applying the two cases for (37) we obtain

$$L_u^N\psi_{ri}^- \le \psi_{ri}^- \left(\rho a_i - 2b_i\right) \le \psi_{ri}^- \left(\rho - \frac{2b_i}{a_i}\right) \le 0.$$

When we apply the mid point scheme,

$$L_m^N\psi_{ri}^- \le \psi_{ri}^- \left(\varepsilon\theta_2^2 + \mu\bar{a}_i\theta_2 - \bar{b}_i\right) \le 0.$$

In both cases of (37) we have $L_m^N\psi_{ri}^- \le 0$. Applying Lemma 8 we can prove that $\psi_{ri}^- \ge 0$. Similarly we can prove for $W_{l,i}^+$ and $W_{r,i}^+$ in the interval $(N/2+1, N)$ and obtain the required result. □

Now, examining the truncation error at the interior mesh points $x_i \in \Omega_{\varepsilon/\mu}^N \setminus \{d\}$. The standard upwinded operator is always monotone and has a second order truncation error. In fact, we have on arbitrary mesh points

$$L^N e(x_i) = \begin{cases} \|(L_c^N - L)y(x_i)\| \le \varepsilon\bar{h}_i\|y^{(3)}\| + \mu\bar{h}_i\|a\|\,\|y^{(2)}\|, \\ \|(L_u^N - L)y(x_i)\| \le \varepsilon\bar{h}_i\|y^{(3)}\| + \mu\bar{h}_{i+1}\|a\|\,\|y^{(2)}\|, \end{cases}$$

and on a uniform mesh with step size h

$$L^N e(x_i) = \begin{cases} \|(L_c^N - L)y(x_i)\| \le \varepsilon h^2\|y^{(4)}\| + \mu h^2\|a\|\,\|y^{(3)}\|, \\ \|(L_u^N - L)y(x_i)\| \le \varepsilon h^2\|y^{(4)}\| + \mu h\|a\|\,\|y^{(2)}\|. \end{cases}$$

Lemma 11. *At each mesh point $x_i \in \Omega_{\varepsilon/\mu}^N$ the regular component of the truncation error satisfies the following estimate*

$$\|(V - v)(x_i)\| \le CN^{-2}.$$

Proof. When the mesh is uniform, then

$$\begin{aligned} |L^N(V-v)(x_i)| &= |L^N V(x_i) - Q^N f(x_i)| \\ &\le \left|\varepsilon\left(\delta^2 - \frac{d^2}{dx^2}\right)V(x_i)\right| - \left|\mu a(x_i)\left(D^+ - \frac{d}{dx}\right)V(x_i)\right| \\ &\le C\varepsilon\,(x_{i+1}-x_i)^2\,\|v\|_4 - \mu a(x_i)\,(x_{i+1}-x_i)\,\|v\|_2 \\ \|(V-v)(x_i)\| &\le CN^{-2}. \end{aligned}$$

When $\tau_1 = d/4$, it is seen that away from the transition points the mesh is uniform. In all the above cases $|L^N(V-v)(x_i)| \le CN^{-2}$. When the mesh is non-uniform we employ the mid-point scheme to obtain the above result for both the cases $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ and $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$. □

Lemma 12. *Assume (30). The left singular component of the error satisfies the following estimate for $x_i \in \Omega_{\varepsilon/\mu}^N$.*

$$\|(W_l - w_l)(x_i)\| \le \begin{cases} C(N^{-1}\ln N)^2, & \text{if } \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}, \\ CN^{-2}\ln^3 N, & \text{if } \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}. \end{cases}$$

Proof. Consider the uniform mesh and for $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ in the interval $1 \le i \le N/2-1$, we have

$$\begin{aligned}|L^N(W_l^- - w_l^-)(x_i)| &= |L_c^N W_l^- - L^N w_l^-| \\ &\le \left|CN^{-2}\varepsilon w_l^{-(4)}(x)\right| + \left|CN^{-2}\mu w_l^{-(3)}(x)\right| \\ &\le CN^{-2}(\varepsilon\|w_l^-\|_4 + \mu\|w_l^-\|_3) \\ |L^N(W_l^- - w_l^-)(x_i)| &\le CN^{-2}/\varepsilon \le C(N^{-1}\ln N)^2.\end{aligned}$$

Similarly for the interval $N/2+1 \le i \le N-1$,

$$|L^N(W_l^+ - w_l^+)(x_i)| \le C(N^{-1}\ln N)^2.$$

When $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$,

$$\begin{aligned}|L^N(W_l^- - w_l^-)(x_i)| &= |L_c^N W_l^- - L^N w_l^-| \\ &\le \left|Ch^2\varepsilon w_l^{-(4)}(x)\right| + \left|Ch^2\mu w_l^{-(3)}(x)\right| \\ &\le CN^{-2}(\varepsilon\|w_l^-\|_4 + \mu\|w_l^-\|_3) \\ |L^N(W_l^- - w_l^-)(x_i)| &\le CN^{-2}\mu^4/\varepsilon^3 \le C\mu N^{-2}\ln^3 N.\end{aligned}$$

Similarly,

$$|L^N(W_l^+ - w_l^+)(x_i)| \le C\mu N^{-2}\ln^3 N.$$

In the case of non-uniform mesh over the domain $x_i \in [\tau_1, d)$. From Lemmas 5 and 7, it is found that,

$$\begin{aligned}|L^N(W_l^- - w_l^-)(x_i)| &\le |W_l^-(x_i)| + |w_l^-(x_i)| \\ &\le C\left(e^{-\theta_1 x_i} + N^{-2}\right) \\ &\le C\left(e^{-\theta_1 \tau_1} + N^{-2}\right) \\ |L^N(W_l^- - w_l^-)(x_i)| &\le CN^{-2}.\end{aligned}$$

Similarly, in $x_i \in [d+\tau_3, 1-\tau_4)$,

$$\begin{aligned}|L^N(W_l^+ - w_l^+)(x_i)| &\le |W_l^+(x_i)| + |w_l^+(x_i)| \\ &\le C\left(e^{-\theta_1(x_i-d)} + N^{-2}\right) \\ &\le C\left(e^{-\theta_1(d+\tau_3-d)} + N^{-2}\right) \\ |L^N(W_l^+ - w_l^+)(x_i)| &\le CN^{-2}.\end{aligned}$$

When $x_i \in (0, \tau_1)$ the truncation error is

$$\begin{aligned}|L^N(W_l^- - w_l^-)(x_i)| &\le |L_c^N(W_l^- - w_l^-)(x_i)| \\ &\le |\varepsilon w_l^{-(2)} + a(x_i)\mu w_l^{-(1)} - b(x_i)w_l^- \\ &- (\varepsilon\delta^2 + a(x_i)\mu D^0 - b(x_i))W_l^-| \\ &\le \left|Ch_1^2\varepsilon w_l^{-(4)}(x)\right| + \left|Ch_1^2\mu w_l^{-(3)}(x)\right| \\ &\le CN^{-2}(\varepsilon\tau_1^2\|w_l^-\|_4 + \mu\tau_1^2\|w_l^-\|_3) \\ |L^N(W_l^- - w_l^-)(x_i)| &\le C\frac{\mu^2}{\varepsilon}(N^{-1}\ln N)^2.\end{aligned}$$

Consider the barrier function [9],

$$\Psi(x_i) = C\left(N^{-2} + (N^{-1}\ln N)^2(\tau_1 - x_i)\frac{\mu}{\varepsilon}\right) \pm |(W_l^- - w_l^-)(x_i)|.$$

Here, $\Psi(0), \Psi(\tau_1)$ are non-negative and $L_c^N\Psi(x_i) < 0$. Applying Lemma 8, we have $\Psi(x_i) \ge 0$, hence

$$\begin{aligned}|(W_l^- - w_l^-)(x_i)| &\le C\left(N^{-2} + (N^{-1}\ln N)^2\tau_1\frac{\mu}{\varepsilon}\right) \\ &\le CN^{-2}\ln^3 N\end{aligned}$$

and similarly if $x_i \in (d, d+\tau_3)$, we obtain the same result.

If $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ for $x_i \in (0, \tau_1)$, we obtain

$$\begin{aligned}|L^N(W_l^- - w_l^-)(x_i)| &\le C(N^{-1}\ln N)^2\left(1+\frac{\mu}{\sqrt{\varepsilon}}\right)\\ &\le C(N^{-1}\ln N)^2\end{aligned}$$

and for $x_i \in (d, d+\tau_3)$, we get

$$|L^N(W_l^+ - w_l^+)(x_i)| \le C(N^{-1}\ln N)^2.$$

Combining all the above results, it is now possible to prove

$$\|(W_l - w_l)(x_i)\| \le \begin{cases} C(N^{-1}\ln N)^2, & \text{if } \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon},\\ CN^{-2}\ln^3 N, & \text{if } \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}.\end{cases} \quad \text{for } x_i \in \Omega^N_{\varepsilon/\mu}. \quad \square$$

Similarly, we can prove the right singular component in the following lemma.

Lemma 13. *Assume (30). The right singular component of the error satisfies the following estimate for* $x_i \in \Omega^N_{\varepsilon/\mu}$

$$\|(W_r - w_r)(x_i)\| \le \begin{cases} C(N^{-1}\ln N)^2, & \text{if } \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon},\\ C(N^{-1}\ln N)^2, & \text{if } \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}.\end{cases}$$

Lemma 14. *At the point of discontinuity* $x_{N/2} = d$*, the error* $e(d)$ *satisfies the following estimate.*

$$\left|L^N\left(Y_{N/2} - y_{N/2}\right)\right| \le \begin{cases} CN^{-1}\ln N, & \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon},\\ CN^{-1}\ln^2 N, & \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}.\end{cases}$$

Proof. Applying the corresponding arguments from [3,23], we can obtain

$$\begin{aligned}\left|L^N\left(Y_{N/2} - y_{N/2}\right)\right| &= \left|L^N Y_{N/2} - \hat{f}(d)\right|\\
&\le \left|\varepsilon\delta^2 y_{N/2} + \mu a(d)D^+ y_{N/2} - b(d)y_{N/2} - \hat{f}(d)\right|\\
&\le \left|\left(\hat{f} + by\right)(d)\right| + \left|\left(\varepsilon\delta^2 y + \mu a D^+ y\right)(d)\right|\\
&\le \frac{1}{h^+}\int_{t=d}^{d+h^+}\int_{s=d}^{t} \varepsilon y''(s)dsdt + \frac{1}{h^-}\int_{t=d-h^-}^{d}\int_{s=d}^{t}\varepsilon y''(s)dsdt\\
&\quad + \frac{1}{h^+}\int_{t=d}^{d+h^+}\int_{s=d}^{t}\mu a y'(s)dsdt + \frac{1}{h^-}\int_{t=d-h^-}^{d}\int_{s=d}^{t}\mu a y'(s)dsdt + \left[\left(\hat{f}+by\right)(d)\right]\\
&\le \frac{1}{h^+}\int_{t=d}^{d+h^+}\int_{s=d}^{t}\left[\varepsilon y''(s) + \mu a y'(s)\right]dsdt\\
&\quad + \frac{1}{h^-}\int_{t=d-h^-}^{d}\int_{s=d}^{t}\left[\varepsilon y''(s) + \mu a y'(s)\right]dsdt + \left[\left(\hat{f}+by\right)(d)\right]\\
&\le \frac{1}{h^+}\int_{t=d}^{d+h^+}\int_{s=d}^{t}\left[\left(\hat{f}+by\right)(s)\right]dsdt + \frac{1}{h^-}\int_{t=d-h^-}^{d}\int_{s=d}^{t}\left[\left(\hat{f}+by\right)(s)\right]dsdt + \left[\left(\hat{f}+by\right)(d)\right]\\
&\le \frac{1}{h^+}\int_{t=d}^{d+h^+}\int_{s=d}^{t}\int_{p=s}^{d+h^+} + \frac{1}{h^-}\int_{t=d-h^-}^{d}\int_{s=d}^{t}\int_{p=d-h^-}^{s}\left[\left(\hat{f}+by\right)'(p)dpdsdt\right]\\
&\quad + \frac{h^+ b(d+h^+)}{2}\int_{t=d}^{d+h^+} y'(t)dt + \frac{h^- b(d-h^-)}{2}\int_{t=d-h^-}^{d} y'(t)dt + O(h^4)\end{aligned} \tag{38}$$

when $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$, applying (12) and Lemma 3 on (38) we get

$$\left|L^N\left(Y_{N/2} - y_{N/2}\right)\right| \le CN^{-1}\ln N.$$

When $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$, applying (12), Lemma 3 and following the principles from [9,11], we prove

$$\left|L^N\left(Y_{N/2} - y_{N/2}\right)\right| \le CN^{-1}\ln^2 N. \quad \square$$

5. Error estimate

This section presents the main contribution of the article, namely, the theorem which conveys the ε–μ-uniform convergence error estimate $|e(x_i)| = \|Y(x_i) - y(x_i)\|$.

Theorem 2. *Let $y(x_i)$ be the solution of the continuous problem (1)–(2) and $Y(x_i)$ be the solution of the discrete problem (31)–(32). Then, for sufficiently large N, we have*

$$\|(Y-y)(x_i)\| \le \begin{cases} C(N^{-1}\ln N)^2, & \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}, \\ CN^{-2}(\ln N)^3, & \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}, \end{cases} \quad \forall\, x_i \in \overline{\Omega}^N_{\varepsilon/\mu}.$$

Proof. Define the mesh functions $t_1(x_j)$ and $t_2(x_j)$ to be

$$t_1(x_j) = \prod_{i=1}^{j}(1+\theta_1 h_i), \qquad t_2(x_j) = \prod_{i=j}^{N}(1+\theta_2 h_i)^{-1}.$$

Consider the following properties of these mesh functions

$$D^- t_1(x_i) = \frac{\theta_1}{1+\theta_1 h_i} t_1(x_i), \qquad D^+ t_1(x_i) = \theta_1 t_1(x_i),$$

$$D^- t_2(x_i) = -\theta_2 t_2(x_i), \qquad D^+ t_2(x_i) = -\frac{\theta_2}{1+\theta_2} t_2(x_i)$$

$$D^0 t_1(x_i) = t_1(x_i)\theta_1\left(1+\frac{\theta_1 h_i h_{i+1}}{2\bar{h}_i}\right), \qquad D^0 t_2(x_i) = -t_2(x_i)\theta_2\left(1+\frac{\theta_2 h_i h_{i+1}}{2\bar{h}_i}\right).$$

Define the three barrier functions $\psi_1(x_i)$, $\psi_2(x_i)$ and $\psi_3(x_i)$ as follows

$$\psi_1(x_i) = \begin{cases} \dfrac{x_i}{\tau_1}, & 0 \le x_i \le \tau_1, \\ 1, & \tau_1 \le x_i \le 1-\tau_4, \\ \dfrac{1-x_i}{\tau_4}, & 1-\tau_4 \le x_i \le 1, \end{cases}$$

$$\psi_2(x_i) = \begin{cases} \dfrac{t_1(x_i)}{t_1(d-\tau_2)}, & 0 \le x_i \le d-\tau_2, \\ 1, & d-\tau_2 \le x_i \le d+\tau_3, \\ \dfrac{t_2(x_i)}{t_2(d+\tau_3)}, & d+\tau_3 \le x_i \le 1, \end{cases}$$

$$\psi_3(x_i) = \begin{cases} \dfrac{t_1(x_i)}{t_1(d)}, & 0 \le x_i \le d, \\ \dfrac{t_2(x_i)}{t_2(d)}, & d \le x_i \le 1. \end{cases}$$

It is to be noted that, when $\tau_1 = \tau_2 = d/4$ use $t_1(x_i) = x_i$ (or) if, $\tau_3 = \tau_4 = (1-d)/4$ use $t_2(x_i) = 1-x_i$.

Case (i): $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$, θ_1, θ_2 are considered as in (12). From Lemmas 11–13 we get

$$\|(Y-y)(x_i)\| \le CN^{-2}\ln^2 N, \quad \forall\, x_i \in \overline{\Omega}^N_{\varepsilon/\mu}\setminus\{d\}. \tag{39}$$

From Lemma 14

$$|L^N(Y-y)(x_i)| \le CN^{-1}\ln N, \quad \text{for } x_i = d$$

by proper choice of C, define the following mesh function [23]

$$\zeta_1^{\pm}(x_i) = C(N^{-1}\ln N)^2\left[1+\sum_{j=1}^{3}\psi_j(x_i)\right] \pm e(x_i).$$

It is clear that $\zeta_1^{\pm}(0) \ge 0$, $\zeta_1^{\pm}(1) \ge 0$ and

$$L^N\zeta_1^{\pm}(x_i) < 0.$$

Table 1
Max. pointwise errors ($E^N_{\varepsilon/\mu}$) for $\mu = 10^{-4}$ and ε varies from 10^{-2} to 10^{-14} in Example 1.

	Number of mesh points N					
	128	256	512	1024	2048	4096
10^{-2}	1.29900E−04	2.24860E−05	3.45280E−06	4.82380E−07	6.38610E−08	8.21860E−09
10^{-4}	7.28060E−03	1.60150E−03	3.97790E−04	1.08900E−04	2.61190E−05	4.71860E−06
10^{-6}	4.25040E−03	1.81560E−03	7.61830E−04	3.05430E−04	1.05700E−04	3.57180E−05
10^{-8}	1.60760E−02	6.24470E−03	2.31990E−03	8.39840E−04	2.78980E−04	8.98980E−05
10^{-10}	1.39560E−02	4.81180E−03	1.55790E−03	4.82850E−04	1.43440E−04	4.28280E−05
10^{-12}	1.39080E−02	4.79850E−03	1.54500E−03	4.78180E−04	1.42500E−04	4.24190E−05
10^{-14}	1.39070E−02	4.79840E−03	1.54490E−03	4.78140E−04	1.42490E−04	4.24150E−05

Applying the discrete minimum principle to $\zeta_1^{\pm}(x_i)$, we get

$$|e(x_i)| \le CN^{-2}\ln^2 N. \tag{40}$$

Case (ii): $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$, θ_1, θ_2 are considered as in (15).
From Lemmas 11–13 we get

$$\|(Y-y)(x_i)\| \le CN^{-2}\ln^3 N, \quad \forall\, x_i \in \overline{\Omega}^N_{\varepsilon/\mu}\backslash\{d\}. \tag{41}$$

From Lemma 14

$$|L^N(Y-y)(x_i)| = CN^{-1}\ln^2 N, \quad \text{for } x_i = d$$

by proper choice of C, define the following mesh function

$$\zeta_2^{\pm}(x_i) = CN^{-2}\ln^3 N\left[1 + \sum_{j=1}^{3}\psi_j(x_i)\right] \pm e(x_i).$$

It is clear that $\zeta_2^{\pm}(0) \ge 0$, $\zeta_2^{\pm}(1) \ge 0$ and

$$L^N\zeta_2^{\pm}(x_i) < 0.$$

Applying the discrete minimum principle to $\zeta_2^{\pm}(x_i)$, we get

$$|e(x_i)| \le CN^{-2}\ln^3 N. \tag{42}$$

From (40) and (42), we get the required result. □

6. Numerical examples

To show the applicability and efficiency of the present method it is implemented to the following test problem. Consider the singularly perturbed two parameter BVPs with discontinuous source term.

Example 1.

$$\varepsilon y''(x) + \mu(1+x)^2 y'(x) - y(x) = f(x), \quad x \in \Omega^- \cup \Omega^+$$
$$y(0) = 0, \qquad y(1) = 0,$$

where,

$$f(x) = \begin{cases} 2x+1, & \text{for } 0 \le x \le 0.5 \\ -(3x+4), & \text{for } 0.5 < x \le 1. \end{cases}$$

To calculate the maximum pointwise error and the rate of convergence, we use the double mesh principle, which is followed in the literature [2,8,11]. The double mesh difference is defined by

$$E^N_{\varepsilon/\mu} = \max_{x_i \in \overline{\Omega}^N_{\varepsilon/\mu}} |Y^N(x_i) - Y^{2N}(x_i)|,$$

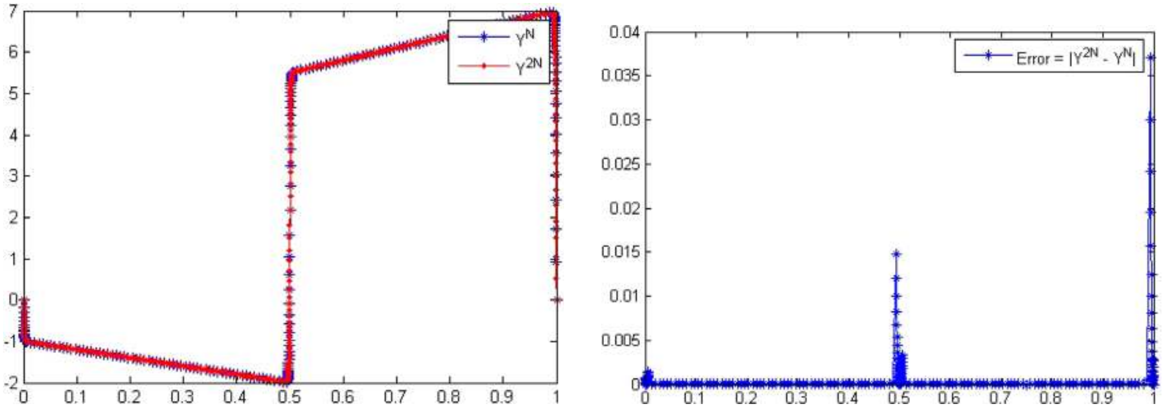
where $Y^N(x_i)$ and $Y^{2N}(x_i)$ denote respectively, the numerical solutions obtained using N and $2N$ mesh intervals. In addition, the parameter-robust orders of convergence are calculated from

$$R^N_{\varepsilon/\mu} = \log_2\left(\frac{E^N_{\varepsilon/\mu}}{E^{2N}_{\varepsilon/\mu}}\right).$$

The numerical solution and error plot for Example 1 are given in Figs. 1–2 and maximum pointwise error and order of convergence are given in Tables 1–2 respectively. The table highlights the error estimates obtained for Theorem 2.

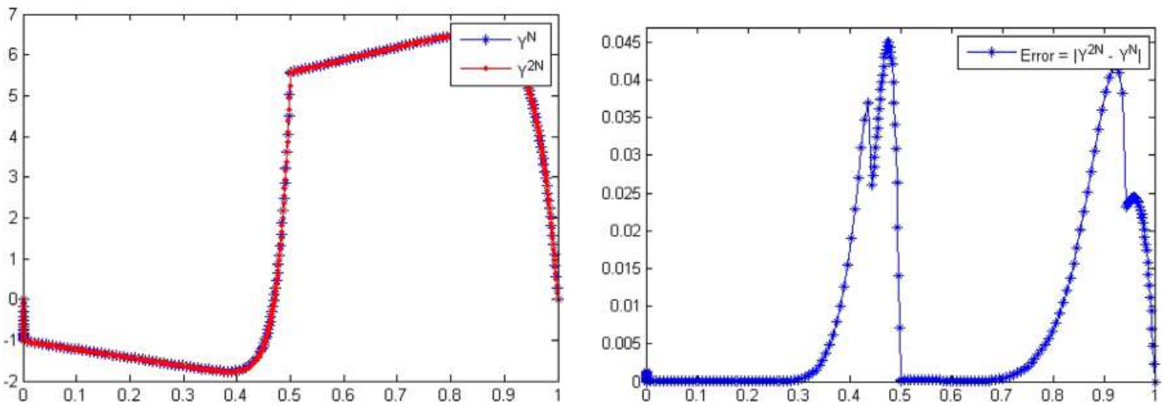
Table 2
Rate of convergence ($R^N_{\varepsilon/\mu}$) for $\mu = 10^{-4}$ and ε varies from 10^{-2} to 10^{-14} in Example 1.

	Number of mesh points N					
	128	256	512	1024	2048	4096
10^{-2}	2.530302481	2.703188370	2.839524772	2.917163043	2.957970581	2.978845275
10^{-4}	2.184633551	2.009344884	1.869003055	2.059832386	2.468668879	2.734399608
10^{-6}	1.227152225	1.252905388	1.318627327	1.530866397	1.565252171	1.692343833
10^{-8}	1.364204312	1.428569646	1.465876211	1.589952803	1.633800774	1.729889700
10^{-10}	1.536236919	1.626974048	1.689955649	1.751127684	1.743821181	1.818667052
10^{-12}	1.535259574	1.634976655	1.691981143	1.746591871	1.748179403	1.823360332
10^{-14}	1.535185905	1.635039971	1.692008449	1.746572429	1.748214207	1.823344653



(a) Numerical solutions. (b) Error $= |Y^N(x_j) - Y^{2N}(x_j)|$.

Fig. 1. The numerical solutions and the error for $\varepsilon = 10^{-6}$, $\mu = 10^{-4}$ and $N = 256$ for Example 1.



(a) Numerical solutions. (b) Error $= |Y^N(x_j) - Y^{2N}(x_j)|$.

Fig. 2. The numerical solutions and the error for $\varepsilon = 10^{-6}$, $\mu = 10^{-2}$ and $N = 256$ for Example 1.

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