An unreliable group arrival queue with *k* stages of service, retrial under variant vacation policy

An unreliable group arrival queue with *k* stages of service, retrial under variant vacation policy

J Radha, K Indhira and V M Chandrasekaran
Department of Mathematics, School of Advanced Sciences, VIT University, Vellore 632014, India

E-mail: vmcsn@vit.ac.in

Abstract. In this research work we considered repairable retrial queue with group arrival and the server utilize the variant vacations. A server gives service in k stages. Any arriving group of units finds the server free, one from the group entering the first stage of service and the rest are joining into the orbit. After completion of the i^{th} stage of service, the customer may have the option to choose $(i+1)^{\text{th}}$ stage of service with probability θ_i, with probability p_i may join into orbit as feedback customer or may leave the system with probability $q_i = \begin{cases} 1 - p_i - \theta_i, i = 1, 2, ...k-1 \\ 1 - p_i, i = k \end{cases}$. If the orbit is empty at the service completion of each stage service, the server takes modified vacation until at least one customer appears in the orbit on the server returns from a vacation. Busy server may get to breakdown and the service channel will fail for a short interval of time. By using the supplementary variable method, steady state probability generating function for system size, some system performance measures are discussed.

1. Introduction

Queues with number of unsatisfied attempts (retrial) have widely used to provide stochastic modeling of many problems arising in telecommunication and computer network. Artalejo[1], Artalejo et al. [2] and Choudhury [2] analyzed, the retrial policy in queueing systems.

The service has many stages in nature. Here a server gives the multi optional stages of service. Autors like, Bagyam et al. [3], Chen et al. [5] and Salehurad et al. [9] are surveyed the multi stage, two phase and multiphase service in queueing system. Service station breakdowns are very common in queueing systems. Keet al. [8] discussed about two phases of service batch retrial queueing pattern and delaying repair. Recently Wang et al. [12] discussed, the repairable queueing system.

In this work, server takes J number of vacations if the orbit has no units. Changet al. [4], Chen et al. [6] and Keet al. [7] are discussed the J vacations queueingmodels. Wanget al. [12] and Zhang et al. [13] discussed the vacations in queueing system.

2. Model description

2.1 Arrival process

Units arriving the system in batches with Poisson arrival rate λ. Let X_k, the number of units in the k^{th} batch, where k =1,2 ,3,… with common distribution $\Pr[X_k = n] = \chi_n$, n= 1,2,3… The PGF (probability generating function) of X *is* $X(z)$.The first and second moments are $E(X)$ and $E(X(X-1))$.

2.2 Retrial process:

If there is no space to wait, one from the arriving unit begins service (if the server is free) and rest is waiting in the orbit. If an arriving group finds the server either busy or on vacation or breakdown, then the group joins into an orbit. Herethe time interval between two continuous arrivals has the distribution $R(x)$ with Laplace-Stieltijes transform (LST) $R^*(s)$.

2.3 Service process

Here a server gives k stages of service. The First Stage Service (FSS) is followed by $i$stages of service. The service time $S_i$for$i=1,2,...k$ has a distribution (general) function $S_i(x)$ having LST $S_i^*(s)$ and first and second moments are $E(S_i)$ and $E(S_i^2)$, (i = 1,2,…k).

2.4 Feedback rule

After completion of i^{th} stage of service the customer may go to $(i+1)^{th}$ stage with probability θ_i or may join into the orbit as feedback customer with probability p_i or leaves the system with probability $q_i = 1-\theta_i - p_i$ for i = 1,2,…k-1. If the customer in the last k^{th}stage may join to the orbit with probability p_k or leaves the system with probability $q_k = 1 - p_k$. From this model, the service time or the time required by the customer to complete the service cycle is a random variable S is given by $S = \sum_{i=1}^{k} \Theta_{i-1} S_i$ having the LST $S^*(s) = \prod_{i=1}^{k} \Theta_{i-1} S_i^*(s)$ and the expected value is $E(S) = \sum_{i=1}^{k} \Theta_{i-1} E(S_i)$, where $\Theta_i = \theta_1\theta_2 ...\theta_i$ and $\Theta_0 = 1$.

2.5 Vacation process

If the orbit has no units to serve, then an idle server may do some other job or taking break is considered as vacation of length V which is random. In this work server is taking *1,2,...j* vacations in succession if the orbit is free. If the orbit has units then the server should be in the service station after the first stage of vacation ends, unless he continues the second stage of vacation. This vacation process continues up to last stage of vacation (free orbit) ends. Otherwise the server returns to the service station to give service without taking all the stages of vacation (busy orbit). Here the distribution function $V(x)$ and LST $V^*(s)$ with moments $E(V)$ and $E(V^2)$.

2.6 Breakdown and repair

The service station may down at any time with Poisson rate α_i where i=1,2,…k during service. The unit on service has to wait to complete the remaining service. The server continues the service for this unit after the repair process. The repair time G_i has the distributions function $G_i(y)$and LST $G_i^*(s)$for (i=1,2,…k).

The random variable at time *t*,

$$C(t)=\begin{cases}0, & \text{idle}\\ 1, & \text{busy on } i^{\text{th}} \text{ stage service}\\ 2, & \text{repair on } i^{\text{th}} \text{ stage}\\ 3, & \text{on first stage of vacation}\\ 4, & \text{on second stage of vacation}\\ . & \\ j+2, & \text{on j}^{\text{th}} \text{ vacation}\\ . & \\ J+2, & \text{J}^{\text{th}} \text{ vacation}\end{cases}$$

The Markov process $\{C(t), N(t); t \geq 0\}$ describes the system state, where $C(t)$ - the server state and $N(t)$ - the number in orbit at time *t*.Thendefine $B_i^* = S_1^* S_2^* ... S_i^*$ and $B_0^* = 1$.The first moment M_{1i} and second moment M_{2i} of B_i^* are given by

$$M_{1i} = \lim_{z \to 1} dB_i^*[A_i(z)] \big/ dz = \sum_{j=1}^{i} \lambda E(X) E(S_j)\left(1+\alpha_j E(G_j)\right)$$

$$M_{2i} = \lim_{z \to 1} d^2 B_i^*[A_i(z)] \big/ dz^2 = \sum_{j=1}^{i} \begin{bmatrix} \lambda E(X(X-1)) E(S_j)\left(1+\alpha_j E(G_j)\right) + \alpha_j \left(\lambda E(X)\right)^2 E(S_j) E(G_j^2) \\ + \left(\lambda E(X)\right)^2 E(S_j^2)\left(1+\alpha_j E(G_j)\right)^2 \end{bmatrix}$$

where

$$A_i(z) = \alpha_i \left(1 - G_i^*(b(z))\right) + b(z) \text{ and } b(z) = \left(1 - X(z)\right)\lambda$$

Let $\{t_n;\ n = 1,2,\ldots\}$ be the service period ending time or repair period ending time. In this system, $Z_n = \left\{C\left(t_n+\right), N\left(t_n+\right)\right\}$ forms an embedded Markov chain which is ergodic $\Leftrightarrow \rho < 1$, where $\rho = (E(X) - \overline{r})\left(1 - R^*(\lambda)\right) + \omega,$ and $\omega = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} + \sum_{i=1}^{k} p_i \Theta_{i-1} - \sum_{i=1}^{k-1} \Theta_i M_{1i}$.

3. Steady state distribution

For $\{N(t),\ t \geq 0\}$, we define the probability functions at time *t*,

$Q_0 = Pr$ (no customer when servers vacation time)

- $P_0(t)$ = Pr (the system is empty).
- $P_n(x,t)$ = Pr (customers undergoing retrial with elapsed retrial time is *x).*
- $\Pi_{i,n}(x,t), (1 \leq i \leq k)$ = Pr(customer undergoing service with elapsed service time *x).*
- $Q_{j,n}(x,t), (j = 1,2,...J)$ = Pr (customer on j^{th} vacation with elapsed vacation time*x).*
- $R_{i,n}(x,y,t), (1 \leq i \leq k)$ =Pr (customer undergoing service with elapsed times for service is *x* and repair is *y*on i^{th} stage).

3.1 Steady state equations

The following are the governing equations of our system for (i=1,2,..k)

$$\lambda P_0 = \int_0^\infty \gamma(x) Q_{j,0}(x)dx \tag{1}$$

$$\frac{dP_n(x)}{dx} = -P_n(x)[\lambda + a(x)],\ n = 1,2,... \tag{2}$$

$$\frac{d\Pi_{i,0}(x)}{dx} = -\Pi_{i,0}(x)[\lambda + \alpha_i + \mu_i(x)] + \int_0^\infty \xi_i(y) R_{i,0}(x,y)dy, n = 0 \tag{3}$$

$$\frac{d\Pi_{i,n}(x)}{dx} = -\Pi_{i,n}(x)[\lambda + \alpha_i + \mu_i(x)] + \lambda\sum_{k=1}^{n}\chi_k \Pi_{i,n-k}(x) + \int_0^\infty \xi_i(y) R_{i,n}(x,y)dy, n = 1,2,... \tag{4}$$

$$\frac{dQ_{j,0}(x)}{dx} = -Q_{j,0}(x)[\lambda + \gamma(x)] \text{ where } j = 1,2,...J \tag{5}$$

$$\frac{dQ_{j,n}(x)}{dx} = -Q_{j,n}(x)[\lambda + \gamma(x)] + \lambda\sum_{k=1}^{n}\chi_k Q_{j,n-k}(x), \text{ where } n = 1,2,... \text{ and } j = 1,2,...J \tag{6}$$

$$\frac{dR_{i,0}(x,y)}{dy} = -R_{i,0}(x,y)[\lambda + \xi_i(y)] \tag{7}$$

$$\frac{dR_{i,n}(x,y)}{dy} = -R_{i,n}(x,y)[\lambda + \xi_i(y)] + \lambda\sum_{k=1}^{n}\chi_k R_{i,n-k}(x,y), \text{ where } n = 1,2,... \tag{8}$$

The boundary conditions at $x = 0$ and $y = 0$ are

$$P_n(0) = \begin{pmatrix} \sum_{i=1}^{k-1} q_i \int_0^\infty \mu_i(x)\Pi_{i,n}(x)dx + (1-p_k)\int_0^\infty \mu_k(x)\Pi_{k,n}(x)dx \\ +\sum_{j=1}^{J}\int_0^\infty \gamma(x)Q_{j,n}(x)dx + \sum_{i=1}^{k} p_i \int_0^\infty \mu_i(x)\Pi_{i,n-1}(x)dx \end{pmatrix}, \text{ where } n = 1,2,... \tag{9}$$

$$\Pi_{i,0}(0) = \lambda\chi_1 P_0 + \int_0^\infty a(x)P_1(x)dx \tag{10}$$

$$\Pi_{1,n}(0) = \lambda\chi_{n+1}P_0 + \int_0^\infty a(x)P_{n+1}(x)dx + \lambda\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k+1}(x)dx, \text{ where } n = 1,2,... \tag{11}$$

$$\Pi_{i,n}(0) = \theta_{i-1}\int_0^\infty \mu_{i-1}(x)\Pi_{i-1,n}(x)dx, \text{ where } n \ge 1,\ (2 \le i \le k) \tag{12}$$

$$Q_{1,0}(0) = \sum_{i=1}^{k-1} q_i \int_0^\infty \mu_i(x)\Pi_{i,0}(x)dx + (1-p_k)\int_0^\infty \mu_k(x)\Pi_{k,0}(x)dx \tag{13}$$

$$Q_{j,n}(0) = \int_0^\infty \gamma(x)Q_{j-1,n}(x)dx, \text{ where } n = 0,\ j = 2,3,...J \tag{14}$$

$$R_{i,n}(x,0) = \alpha_i \Pi_{i,n}(x), \text{ where } n \ge 0 \tag{15}$$

The normalizing condition is

$$\left(P_0+\sum_{n=1}^{\infty}\int_0^{\infty}P_n(x)dx+\sum_{n=0}^{\infty}\sum_{i=1}^{k}\int_0^{\infty}\Pi_{i,n}(x)dx+\sum_{n=0}^{\infty}\sum_{i=1}^{k}\int_0^{\infty}\int_0^{\infty}R_{i,n}(x,y)dxdy+\sum_{j=1}^{J}\int_0^{\infty}Q_{j,n}(x)dx\right)=1 \tag{16}$$

3.2 Steady state solutions

The above equations are solved by using the method of generating functions. Multiplying Eqns.(2) to (15) by $\sum_{n=o}^{\infty} z^n$ then,

$$\frac{\partial P(x,z)}{\partial x}=-P(x,z)[\lambda+a(x)] \tag{17}$$

$$\frac{\partial \Pi_i(x,z)}{\partial x}=-\Pi_i(x,z)[\lambda(1-X(z))+\alpha_i+\mu_i(x)]+\int_0^{\infty}\xi_i(y)R_i(x,y,z)dy \tag{18}$$

$$\frac{\partial Q_j(x,z)}{\partial x}=-Q_j(x,z)[\lambda(1-X(z))+\gamma(x)] \tag{19}$$

$$\frac{dR_i(x,y,z)}{dy}+[\lambda(1-X(z))+\xi_i(y)]R_i(x,y,z)=0 \tag{20}$$

The boundary conditions at $x = 0$ and $y = 0$ are

$$P(0,z)=\sum_{i=1}^{k}\left\{\left(p_i z+q_i\right)\int_0^{\infty}\Pi_i(x,z)\mu_i(x)dx\right\}+\int_0^{\infty}Q_j(x,z)\gamma(x)dx-\lambda P_0-Q_{j,0}(0) \tag{21}$$

$$\Pi_1(0,z)=\frac{1}{z}\int_0^{\infty}a(x)P(x,z)dx+\frac{\lambda P_0 X(z)}{z}+\frac{X(z)\lambda}{z}\int_0^{\infty}P(x,z)dx \tag{22}$$

$$\Pi_i(0,z)=\theta_{i-1}\int_0^{\infty}\mu_{i-1}(x)\Pi_{i-1}(0,z)dx,(i=2,3,...k). \tag{23}$$

$$R_i(x,0,z)=\alpha_i\Pi_i(x,z) \tag{24}$$

Solving (17) to (20), it follows that for ($1 \le i \le k$)

$$P(x,z)=P(0,z)e^{-\lambda x}[1-R(x)] \tag{25}$$

$$\Pi_i(x,z)=\Pi_i(0,z)e^{-A_i(z)x}[1-S_i(x)] \tag{26}$$

$$Q_j(x,z)=Q_j(0,z)e^{-b(z)x}[1-V(x)] \tag{27}$$

$$R_i(x,y,z)=R_i(x,0,z)e^{-b(z)y}[1-G_i(y)] \tag{28}$$

From (5) we obtain,

$$Q_{j,0}(x)=Q_{j,0}(0)e^{-b\lambda x}[1-V(x)] \tag{29}$$

Multiplying equation (29) by $\gamma(x)$ for $j=J$ and integrating w.r to x from 0 to ∞ on both sidesthen from (1),

$$Q_{j,0}(0)=\frac{\lambda P_0}{V^*(\lambda)} \tag{30}$$

From Eq.(30) and solving (14) and(29), we have

$$Q_{j,0}(0)=\frac{\lambda P_0}{\left[V^*(\lambda)\right]^{J-j+1}},\ j=1,2,...J-1 \tag{31}$$

From (14),(30) and(31), we get

$$Q_j(0,z)=\frac{\lambda P_0}{\left[V^*(\lambda)\right]^{J-j+1}},\ j=1,2,...J \tag{32}$$

Solving Eqn. (29) then

$$Q_{j,0}(0,z)=\frac{P_0(1-V^*(\lambda))}{\left[V^*(\lambda)\right]^{J-j+1}},\ j=1,2,...J \tag{33}$$

$$Q_0=\frac{P_0\left(1-\left[V^*(\lambda)\right]^J\right)}{\left[V^*(\lambda)\right]^J},\ j=1,2,...J \tag{34}$$

Inserting (25) in (22), we obtain

$$\Pi_1(0,z)=\left[R^*(\lambda)\frac{P(0,z)}{z}+X(z)\frac{P(0,z)}{z}\left(1-R^*(\lambda)\right)\right]+\frac{\lambda X(z)}{z}P_0 \tag{35}$$

Inserting (34) in (23), we obtain

$$\Pi_i(0,z)=\Theta_{i-1}\Pi_1(0,z)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right),\ (i=2,3,...k) \tag{36}$$

Inserting (28) in (24), we obtain

$$R_i(x,0,z)=\Pi_i(0,z)e^{-A_i(z)x}[1-S_i(x)]\alpha_i \tag{37}$$

Using (26) and (32) in (21), finally,

$$P(0,z)=\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Pi_i(0,z)\left(S_i^*\left[A_i(z)\right]\right)\right\}+Q_j(0,z)V^*\left[b(z)\right]-\lambda P_0-\frac{\lambda P_0}{\left[V^*(\lambda)\right]^{J-j+1}} \tag{38}$$

Solving (32),(35),(36) and (38), we get

$$P(0,z)=\lambda P_0\times\left\{\frac{X(z)\left\{\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right\}+zN(z)-z}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\} \tag{39}$$

where $N(z)=\frac{\left(1-\left[V^*(\lambda)\right]^J\right)}{\left[V^*(\lambda)\right]^J\left(1-\left[V^*(\lambda)\right]\right)}\left(V^*(b(z))-1\right)$

Using (39) in (35), we get,

$$\Pi_1(0,z)=\lambda P_0\times\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\} \tag{40}$$

$$\Pi_i(0,z)=\lambda P_0\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\} \tag{41}$$

Using (41) in (37), we get

$$R_i(x,0,z)=\alpha_i\Theta_{i-1}\Pi_i(0,z)[1-S_i(x)]e^{-A_i(z)x} \tag{42}$$

Using (25) to (28), (39) to (42) and (32), then we get $P(x,z), \Pi_i(x,z), Q_j(x,z)$ and $R_i(x,y,z)$.

Theorem 3.1. Under $\rho < 1$, the stationary distributions of the numbers in the system when server being idle, busy during i[th] stage, on j[th] stage of vacation and repair on i[th] stage are given by

$$P(z)=\lambda P_0\left(1-R^*(\lambda)\right)\times\left\{\frac{X(z)\left\{\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right\}+z\left(N(z)-1\right)}{z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\} \tag{43}$$

$$\Pi_i(z)=b\lambda P_0\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)\left\{\frac{\Theta_{i-1}\left(\left(N(z)-1\right)\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]+X(z)\right)}{z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}}\right\} \tag{44}$$

$$Q_j(z)=\frac{P_0\left(1-V^*(b(z))\right)}{\left[V^*(\lambda)\right]^{J-j+1}(1-X(z))} \tag{45}$$

$$R_i(z)=\frac{\alpha_i\lambda P_0\left(1-S_i^*(A_i(z))\right)\left(1-G_i^*(b(z))\right)}{A_i(z)b(z)}\left\{\frac{\Theta_{i-1}\left(\left(N(z)-1\right)\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]+X(z)\right)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)}{\left(z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]\sum_{i=1}^{k}\left\{\left(p_iz+q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right)}\right\} \tag{46}$$

where $P_0=\dfrac{1}{\beta}\left\{1-E(X)\left(1-R^*(\lambda)\right)-\sum_{i=1}^{k}\Theta_{i-1}M_{1i}-\sum_{i=1}^{k}p_i\Theta_{i-1}+\sum_{i=1}^{k-1}\Theta_iM_{1i}\right\}$ (47)

$$\beta=\left(1+\frac{N^1(1)}{E(X)}\right)(1-\omega)-(1-\omega)\left(1-R^*(\lambda)\right)+\lambda\left\{\sum_{i=1}^{k}\Theta_{i-1}E(S_i)(N'(1)-R^*(\lambda))\left(1-\alpha_iE(G_i)\right)\right\}$$

Proof. The statement is obtained by using

$$P(z)=\int_0^\infty P(x,z)dx,\ \Pi_i(z)=\int_0^\infty \Pi_i(x,z)dx,\ Q_j(z)=\int_0^\infty Q_j(x,z)dx.\ R_i(x,z)=\int_0^\infty R_i(x,y,z)dy,\ R_i(z)=\int_0^\infty R_i(x,z)dx.$$

and $P_0+P(1)+\sum_{j=1}^{J}Q(1)+\sum_{i=1}^{k}\left(\Pi_i(1)+R_i(1)\right)=1.$

Theorem 3.2. Under $\rho < 1$, PGF of the system size and orbit size distribution at stationary point of time is

$$K(z)=\frac{Nr(z)}{Dr(z)} \tag{48}$$

$$Nr(z) = P_0 \left\{ \begin{array}{l} z\left\{\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)\left(\begin{array}{l}(N(z)-1)\\ \left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]+X(z)\end{array}\right)\right\} - N(z)\left(\begin{array}{l} z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right] \\ \sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\end{array}\right) \\ +\left[1-X(z)\right]\left(\begin{array}{l}\left[z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right]+ \\ \left(1-R^*(\lambda)\right)\left[\left\{\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}X(z)\left(B_i^*\left[A_i(z)\right]\right)\right\}\right\}+\left(zN(z)-z\right)\right]\end{array}\right)\end{array}\right\}$$

$$Dr(z) = \left[1-X(z)\right]\left(z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right),$$

$$H(z) = \frac{Nr(z)}{Dr(z)} \tag{49}$$

$$Nr(z) = P_0 \left\{ \begin{array}{l} \left\{\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)\left(\begin{array}{l}(N(z)-1)\\ \left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]+X(z)\end{array}\right)\right\} - N(z)\left(\begin{array}{l} z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right] \\ \sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\end{array}\right) \\ +\left[1-X(z)\right]\left(\begin{array}{l}\left[z-\left[X(z)-X(z)R^*(\lambda)+R^*(\lambda)\right]\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right]+ \\ \left(1-R^*(\lambda)\right)\left[\left\{\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}X(z)\left(B_i^*\left[A_i(z)\right]\right)\right\}\right\}+\left(zN(z)-z\right)\right]\end{array}\right)\end{array}\right\}$$

Proof. The statement is obtained by using $K(z) = P_0 + P(z) + \sum_{j=1}^{J} Q(1) + z\sum_{i=1}^{k}\left(\Pi_i(z)+R_i(z)\right)$ and

$$H(z) = P_0 + P(z) + \sum_{j=1}^{J} Q(1) + \sum_{i=1}^{k}\left(\Pi_i(z)+R_i(z)\right)$$

4. Performance measures

Here, the mean numbers in the orbit (L_q), the mean numbers in the system (L_s), the mean waiting time in the system (W_s) and in the queue (W_q) are required to analyze the model.

Theorem 4.1. If the system satisfies $\rho < 1$, then the following probabilities of the server state, that is the server is idle during the retrial, busy during i^{th} stage, on vacation, delaying repair during i^{th} stage and under repair on i^{th} stage respectively are obtained.

$$P = \frac{\left(1-R^*(\lambda)\right)}{\beta}\left(E(x)+N'(1)+\omega-1\right)$$

$$\Pi_i = \sum_{i=1}^{k}\Pi_i = \frac{1}{\beta}\sum_{i=1}^{k}\left\{\Theta_{i-1}\lambda E(S_i)\kappa\right\}$$

$$Q_j = \sum_{j=1}^{J} Q_j = \frac{1}{\beta}\left\{1-\omega-E(X)\left(1-R^*(\lambda)\right)\right\}\frac{N'(1)}{E(X)}$$

$$R_i = \sum_{i=1}^{k} R_i = \frac{1}{\beta}\sum_{i=1}^{k}\left\{\alpha_i\Theta_{i-1}\lambda\kappa E(S_i)E(G_i)\right\}$$

Proof**:** The statement followed by using

$$P=\lim_{z\to 1}P(z),\quad \sum_{i=1}^{k}\Pi_i=\lim_{z\to 1}\sum_{i=1}^{k}\Pi_i(z),\quad \sum_{j=1}^{J}Q_j=\lim_{z\to 1}\sum_{j=1}^{J}Q_j(z) \text{ and } \sum_{i=1}^{k}R_i=\lim_{z\to 1}\sum_{i=1}^{k}R_i(z).$$

Theorem 4.2. Let L_s, L_q, W_s and W_q be the average system size, average orbit size, average waiting time in the system and average waiting time in the orbit respectively, then under ρ < 1, $L_q = P_0\left[\dfrac{Nr_q'''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$,

$$Nr_q''(1)=-2\left\{\kappa\left\{\sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)E(S_i)\left(1+\alpha_iE(G_i)\right)\right\}+\left\{1-\omega-E(X)\left(1-R^*(\lambda)\right)\right\}\left(N'(1)+E(X)\right)+\left(1-R^*(\lambda)\right)E(X)\delta_1\right\}$$

$$Nr_q'''(1)=3\left(\begin{array}{l}-\sum_{i=1}^{k}\Theta_{i-1}\left[\begin{array}{l}\kappa\left(\begin{array}{l}\lambda E(X(X-1))E(S_i)\left(1+\alpha_iE(G_i)\right)+\alpha_i\left(\lambda E(X)\right)^2E(S_i)E(G_i^2)\\+\left(\lambda E(X)\right)^2E(S_i^2)\left(1+\alpha_iE(G_i)\right)^2+2\lambda E(X)E(S_i)\left(1+\alpha_iE(G_i)\right)M_{1i-1}\end{array}\right)\\+\lambda E(X)E(S_i)\left(1+\alpha_iE(G_i)\right)\\\left(\left(1-R^*(\lambda)\right)\left(\begin{array}{l}E(X(X-1))-\\2\left(1-R^*(\lambda)\right)E(X)N'(1)-N''(1)-E(X(X-1))\end{array}\right)\right)\end{array}\right]\\-N''(1)\left(1-\rho\right)+N'(1)\left[\tau+\left(1-R^*(\lambda)\right)(\delta_2-\delta_3)\right]-E(X(X-1))\left[1-\rho+\left(1-R^*(\lambda)\right)\delta_1\right]\end{array}\right)$$

$$Dr_q''(1)=-2E(X)\left(1-\rho\right)$$

$$Dr_q'''=3\left\{E(X)\left(\tau+2E(X)(1-R^*(\lambda))\omega+\left(1-R^*(\lambda)\right)\left(E(X(X-1))\right)\right)-E(X(X-1))(1-\rho)\right\}$$

$$\tau=\sum_{i=1}^{k}\Theta_{i-1}M_{2i}+2\sum_{i=1}^{k}p_i\Theta_{i-1}M_{1i}-\sum_{i=1}^{k-1}\Theta_iM_{2i},$$

$$\delta_1=E(X)+N'(1)+\omega-1,$$

$$\delta_2=\left(E(X(X-1))+2\omega E(X)\right),$$

$$\delta_3=\left(E(X(X-1))+2\omega E(X)+\tau+\mathrm{N}''(1)+2\,\mathrm{N}'(1)\right)$$

$$L_s = P_0\left[\frac{Nr_s'''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right],$$

where $Nr_s'''(1)=Nr_q'''(1)-6\sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)E(S_i)\left(1+\alpha_iE(G_i)\right)\left(N'(1)-R^*(\lambda)\right)$

$W_s=\dfrac{L_s}{\lambda E(X)}$ and $W_q=\dfrac{L_q}{\lambda E(X)}$

Proof. Under ρ < 1, L_q is obtained from

$$L_q=\frac{Nr(z)}{Dr(z)}=\lim_{z\to 1}\frac{d}{dz}H(z)=H'(1)=P_0\left[\frac{Nr_q'''(1)Dr_q''(1)-Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right].$$

And L_s is obtained from

$$L_s = \frac{Nr(z)}{Dr(z)} = \lim_{z\to 1}\frac{d}{dz}K(z) = K'(1) = P_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right].$$

5. Conclusion

Unreliable multi stage of vacation retrial queue and multi stages of service with batch arrival policy are meticulously studied. The PGF of the numbers in the system and orbit are found. The performance measures were obtained. L_s, L_q, W_s and W_q are obtained.

References

[1] Artalejo J R 1999 A classified bibliography of research on retrial queues(Progress in 1990-1999) Top 7 187- 211
[2] Artalejo J R and Gomez-Corral A 2008 Retrial queueing systems(*Springer)*
[3] Bagyam J E A. and Chandrika K U 2013 *Int. J. Sci.& Eng. Res.***4** 496-499
[4] Chang F M andKe J 2009*J ComputAppl Math***232**402-14
[5] Chen P, Choudhury G and Madan K C 2004 *Appl. Math. Computat.***149** 337-349
[6] Chen P, Zhu Y and Zhang Y 2009*IEEE* 978-1-4244-5540-9 26-30
[7] Ke J C, and Chang FM 2009 *ComputIndEng***57**433-43
[8] Ke, J C and Choudhury G 2012 *Appl. Math. Mod.***36** 255-269
[9] Salehurad M R and Badamchizadeh A 2009 *Cen. Eur. J. Oper. Res.* **17** 131-139
[10] Wang J and Li Q 2009 *J. of sys. Sci. Comp.* **22** 291-302
[11] Wang J and J Li 2008 *Qual. Tech. Quant. Manage* **5** 179-192
[12] Yang T and Templeton J G C 1987 *Queueing Systems* **2** 201-23
[13] Zhang M and Hou Z 2012 *J. Appl. Math.Comput.***39** 221–234