Analysis of a preemptive priority retrial queue with negative customers, starting failure and at most J vacations

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Abstract: An analysis of single server preemptive priority retrial queue with at most J vacations where two types of customers called (priority customers and ordinary customers) are considered in this paper. The priority customers do not have queue and they have higher priority to receive their services over ordinary customers. If negative customer is arriving during the service time of any positive customer (priority customer or ordinary customer), it will remove the positive customer from the service. If the interrupted customer is an ordinary customer, he may join the orbit and the priority customer will leave the system. As soon as the system is empty, the server takes at most J vacations. The probability generating functions for the system/orbit size in steady state is obtained using supplementary variable method. Some important system measures and the stochastic decomposition are discussed. Numerical examples are presented to picturise the effect of parameters on system performance measures.

Keywords: stochastic processes; preemptive priority queue; J vacations; starting failure; negative customer; supplementary variable technique.

Reference to this paper should be made as follows: Yuvarani, S. and Saravanarajan, M.C. (2017) 'Analysis of a preemptive priority retrial queue with negative customers, starting failure and at most J vacations', *Int. J. Knowledge Management in Tourism and Hospitality*, Vol. 1, No. 1, pp.76–109.

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1 Introduction

Queueing theory is recognised as an important research area due to many applications in different areas like communication networks, transportation networks, operation systems and production lines. Gomez-Corral (2006) is given the general models of queueing in various aspects. Retrial queues are defined as the repeated trials such that the new customer who found the server busy upon arrival is requested to leave the service area and join a trial group, called orbit. After some time (retrial time) the customer in the orbit can repeat their request for service according to FCFS. Any arbitrary customer in the orbit who repeats the request for service is independent of the other customers in the orbit. Upon the arrival of a customer, if the server is busy or under repair or on vacation, the customer will join the orbit and try his luck again for some time later. This kind of (retrial) queue plays a superior role in computer networks, telecommunication, telephone systems communication protocols, retail shopping queues, etc.

In the earlier years, retrial queues with two varieties of customers have been widely studied by many of the researchers, Artalejo et al. (2001), Wang (2008) and Liu and Gao (2011). The high priority customers are formed in the queue or not in the queue and served according to the discipline of preemptive or non-preemptive. If blocked pool of customers, low priority customers (called as ordinary customers) leave the system and join the retrial group to retry its service after some time when the server is free. Mostly, in many systems, an arriving higher priority customer will push out the lower priority customers from the service to the queue or into the orbit. According to the above concepts, Choi and Chang (1999) first studied a priority retrial queue, in which priority customers do not preempt the ordinary customers and are queued in FCFS discipline. Krishnakumar et al. (2002) deliberated a single server retrial queueing system with preemptive resume and two-phase service. Liu and Wu (2009) considered a Markovian arrival process of queues with preemptive resume, negative customers, and multiple vacations which present the importance of a preemptive resume in practical situations. In recent times, Wu and Lian (2013) studied a single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. For a comprehensive analysis of priority queueing models, the reader may refer Liu and Gao (2011), Senthilkumar et al. (2013). Peng (2016) have discussed the discrete-time Geo/G/1 retrial queueing system with preemptive resume. Gao (2015) have analysed about the preemptive priority queues with general times. Yuvarani and Saravanarajan, (2015) have discussed the preemptive priority retrial queue with single vacation and server breakdown.

The concept of the positive and negative customer is widely discussed by many researchers recently, due to their several extensive applications such as communications networks, server breakdown, and manufacturing system. The positive customer who comes to the system will get the service in the normal manner. The negative customer arrives at the system only when the positive customer is in service. This type of negative customer do not form any queue and they do not get any type of service. After some repair the negative customer will vanish from the system and reduce one positive customer may in service, then the positive may leave the system or join the queue again for another service. Negative customers have been considered as a virus, inhibitor signals, system or server disaster in communication, and operation mistakes. Wang and Zhang (2009) discussed retrial queues with negative customers and unreliable server. Some authors like Gao (2014), Yang et al. (2013) and Wu and Lian (2013) discussed different types of queueing models dealing with the presence of negative customers.

Dimitriou (2013) have considered a preemptive resume priority retrial queue with State dependent arrivals, unreliable server and negative customers. Kirupa and Chandrika (2015) have analysed the concepts of batch arrival queue with negative customer, feedback and multi-optional service.

A vacation queueing model is considered as an extension of the standard queueing system in which server may not be available for a period of time due to many reasons, like being checked for maintenance, any damage occurred to the server or simply it is taking a break. The period of time taken to come out of the server absence is considered as a server vacation. The server vacation models are essential for the systems in which server wants to utilise the idle time for many purposes. Various authors have analysed queueing models of server vacations with many combinations. A literature survey on queueing systems with server vacations is done by Doshi (1986). Most vacation models deal with the exhaustive policy that is the system must be empty when the server starts a vacation. Chang and Ke (2009) examined a batch arrival model with J vacations in which, if the orbit is empty, the server takes at most J vacations repeatedly until at least one customer in the orbit upon returning from vacation. Jain and Bhagat, (2014) developed the concepts of bulk retrial queues with delayed repairs and modified vacation policy. Recently, Dimitriou (2013), Ke and Chang (2009), Rajadurai et al. (2015a, 2015b) and Wang and Zhang (2009) have analysed about the retrial queue with J vacations. Yang et al. (2016) have discussed the concepts of unreliable retrial queue with general repeated attempts and J vacations.

In most of the literature survey related to queueing theory, it is assumed that the server is available always in the system. The server is assumed to be liable in most of the cases and always available for the customers to be served. Sometimes we come across the cases where the server may breakdown and resume its service after repair. In an instance, in manufacturing systems, the machine may break down due to Mechanical or job-related problems. Ke and Chang (2009) discussed a batch arrival retrial queueing system with two phases of service using the concept of Bernoulli vacation and server breakdown due to starting failure. Ke and Chang (2009) has analysed about the M[x]/ (G1, G2)/1 retrial queue with general attempts under Bernoulli vacation schedules and starting failures. Sumitha and Chandrika (2012), have discussed the concept of a single server with single vacation and starting failure and orbital search. Rajadurai et al. (2014) have analysed the repairable $M^{[x]}$/(G1, G2)/1 retrial G-queue with balking and starting failures under at most J vacations. Yang et al. have discussed the concepts of retrial queue with unreliable server and J vacations. Retrial queues are mostly used in real-time systems, manufacturing system, operating systems and simulations. The application of server vacation system can be found in production systems, designing of local area networks, data communication systems. Queueing systems with breakdown are very common in manufacturing systems and computer priority networks.

To the authors best of Knowledge, there are many works existing in the model of retrial queueing system with negative customer, at most J vacation and starting failure, but there is no work published in the queueing survey with the combination of preemptive priority retrial queueing system with negative customer, at most J vacation and starting failure by using the supplementary variable technique. Mathematical results and concepts of queues dicussed in this model provide a wide application in the telecommunications, computer processing and production and manufacturing system.

rest of this paper is structured as follows. The detailed mathematical description of the model and practical applications is given in Section 2. The stability condition of the model is analysed in this Section 3. In Section 4, the steady state joint distribution of the server state and the number of customers in the orbit/system are obtained. Some system performance measures are discussed in Section 5. The stochastic decomposition is shown good for our model in Section 6. Important special cases are derived in Section 7. In Section 8, the effects of various parameters on the system performance are discussed numerically. In Section 9, Conclusion and summary of the paper are discussed.

2 Description of the model and its practical applications

In this subdivision, we consider a preemptive priority retrial queue with two types of customers, J vacations, and negative customer where the server is subject to starting failure repair. The detailed description of model is given as follows:

- There are two types of customers like priority customers and ordinary customers, arrives at the system. Priority customers have higher priorities over ordinary customers in service time of the busy server. We assume that both priority customers and ordinary customers arrive at the system according to two independent Poisson processes with rates λ and δ respectively.
- Newly arriving (priority or ordinary) customer will commence its service immediately if the server is free. If a priority customer is in service, then the newly arriving priority customer will depart the system directly without service. While the server is busy with working an ordinary customer, the arriving priority customer will disturb the service of the ordinary customer. Then the server begins priority customer's service immediately. We imagine that when an ordinary customer is preempted by a priority customer, the ordinary customer who was just being served before waits in the service area for the remaining service to complete.
- If the server is being busy or on vacation, the arriving customers will join the pool of blocked customers called an orbit with FCFS discipline. Here, only one customer at the head of the orbit queue can access to the server. Also retrial times have an arbitrary distribution $R(t)$ with corresponding Laplace Stieltjes-transform (LST)$R^*(\vartheta)$.
- If the server is free, any arriving customer is allowed to receive their service and the startup time of server could be negligible. Moreover, the server may have a starting failure with a probability $\bar{\alpha} = 1 - \alpha,$ and then the server has to be repaired immediately. The customer is in service must leave the service area. The probability of successful commencement of service for a new customer is. Here, the repair time of the failure server is of random length H with distribution function, LST $H^*(\vartheta)$ and finite k^{th} moment $h^{(k)}(k = 1, 2)$.
- The negative customers always arrive only at the service time of the positive customers from outside the system according to a Poisson rate δ. Once the negative customer comes into the system it will remove the positive customer (priority customer or ordinary customer). The interrupted ordinary customer either enters into the orbit with probability θ $(0 \leq \theta \leq 1)$ or leave the system forever with the probability $(1 - \theta)$ and the priority customer will leave the system.

- Once the negative customer comes into the system, it will remove the positive customer (priority customer or ordinary customer). The interrupted ordinary customer either enters into the orbit with probability θ ($0 \leq \theta \leq 1$) or leave the system forever with the probability $(1 - \theta)$ and the priority customer will leave the system.
- Whenever the orbit is empty, the server leaves for a vacation of random length V. When the server returns from a vacation and if there is no customer in the orbit then it leaves again for another vacation with the same length. This pattern continues until it returns from a vacation to find at least one customer in the orbit or it has already taken J vacations. At the end of the J^{th} vacation, if the orbit is empty then the server remains idle for new arrivals in the system. At a vacation completion epoch, if the orbit is empty, the server waits for the customers in the orbit or for a new arrival. During the vacation period, the service time follows a general random variable V with distribution function and LST and finite k^{th} moment $v^{(k)}$ ($k = 1, 2$).
- As soon as the breakdown occurs the server is sent for repair during that time it stops providing service to the customers till service channel is got repaired. The customer just being served before server breakdown will be waiting on the server, to complete th remaining service. The repair time (denoted by G) of the server is generally distributed with d.f $G(y)$, LST $G^*(\vartheta)$, and finite k^{th} moment $g^{(k)}$ ($k = 1,2$).
- In the normal busy period, there is a single server which provides regular service. The service time of priority customers follows a general distribution and denoted by the random variable S_p with distribution function $S_p(t)$, having LST $S_p^*(\vartheta)$ and the first and second moments are $\beta_p^{(1)}$ and $\beta_p^{(2)}$. The service time of ordinary customers follows a general distribution and denoted by the random variable S_b with distribution function $S_b(t)$ having LST $S_b^*(\vartheta)$ and the first and second moments are $\beta_b^{(1)}$ and $\beta_b^{(2)}$.
- Various stochastic processes involved in the system are assumed to be independent of each other.
- Throughout the rest of the paper, we denote by $\bar{F}(x) = 1 - F(x)$ the tail of distribution function $F(x)$. We also denote $F^*(s) = \int_0^\infty e^{-sx} dF(x)$, the Laplace-Stieltjes transform $F(x)$ of $\tilde{F}(s) = \int_0^\infty e^{-sx} F(x) dx$, and to be the Laplace transform of $F(x)$ and we assume the notation $\bar{F}^*(s) = \frac{1 - F^*(s)}{s}$.

2.1 Practical justification of the model

The suggested model has potential application in the field of telecommunications, computer processing, production and manufacturing system, inventory control system and operating systems. We consider a telecommunications system for an example. In telecommunications, call centres play a vital role. The customers contact the call centres through the agent or a customer service representative over the telephone (the regular server). In addition to contacting over the phone (priority customer), the customer can contact the call centre through the internet via e-mail, fax or live chat sessions (ordinary customer). If the customer service representative is idle then he can attend the voice call or e-mail immediately. Suppose, at the time of voice calling, if the agent is busy with other calls then the arriving voice call will lose its service. If the agent is busy with the e-mail or message, the arriving voice call has a preemptive priority over the e-mail service and the preempted service (e-mail) will wait to complete its service. If the arriving e-mail or message found the customer care representative is busy with the voice call then they are temporarily stored in retrial buffer called orbit and they will be served after some time (retrial time) according to FCFS. When the agent finds no voice calls or mail services, it will perform a sequence of maintenance jobs such as virus scan or server maintenance (J vacation). Sometimes, the server will have a starting problem due to a network problem or electronic failure (starting failure), then the voice call or mail services have to wait till the server get repaired. Meanwhile, the working server may receive a flow of virus (negative customer), which will stop the service of the system and the voice call or mail services which are in service will lose their service and it will force to leave the system.

3 Stability condition

In this section, we will carry out the discussion of the stability condition of the system by using embedded Markov chain technique. Let $\{t_n; n = 1, 2, \ldots\}$ be the sequence of epochs of the regular service completion times for priority customers, ordinary customers, a vacation period completion occurs or starting failure repair period ends. Then the state of the queueing system can be described by the bivariate Markov process $\{C(t), N(t); t > 0\}$ where $C(t)$ denotes the server state $(0,1,2,3,4, \ldots, J + 4,5)$ depending on the server is free, busy on priority customers, busy on preemptive priority customers, busy on ordinary customers, on first vacation, on jth vacation, on repair due to starting failure. $N(t)$ denotes the number of ordinary customers in the orbit.

In addition, let $R^0(t), S_p^0(t), S_b^0(t), V_j^0(t)$ and $H^0(t)$ be the elapsed retrial time, elapsed service time of the priority customer, elapsed service time of the ordinary customer, elapsed vacation time of any customer and elapsed repair time on starting failure of any customer respectively at time t. Further, we introduce the random variable,

$$C(t)=\begin{cases}0, & \text{if the server is idle at time } t\\ 1, & \text{if the server is busy with a priority customer without preempting}\\ & \text{an ordinary customer and in regular service period at time } t,\\ 2, & \text{if the server is busy with a priority customer with preempting}\\ & \text{an ordinary customer and in regular service period at time } t,\\ 3, & \text{if the server is busy with an ordinary customer}\\ & \text{and in regular service period at time } t,\\ 4, & \text{if the server is on first vacation at time } t,\\ \vdots & \\ j+4 & \text{if the server is on } j^{th} \text{ vacation at time } t,\\ 5, & \text{if the server is on repair due to starting failure at time } t\end{cases}$$

If $C(t) = 0$ and $N(t) > 0$, then $R^0(t)$ represent the elapsed retrial time. If $C(t) = 1$ and $N(t) \geq 0$ then $S_p^0(t)$ corresponding to the elapsed service time of the priority customer being served in the regular busy period. If $C(t) = 2$ and $N(t) \geq 0$ then $S_p^0(t)$ corresponding to the elapsed service time of the preemptive priority customer and $S_b^0(t)$ corresponding to the elapsed service time of the interrupted ordinary customer being served in the regular busy period. If $C(t) = 3$ and $N(t) \geq 0$ then 0( ) b S t corresponding to the elapsed service time of the ordinary customer being served in the regular busy period. If $C(t) = 4$ and $S_b^0(t)$ then corresponding to the elapsed first vacation time of any customer. If $C(t) = j + 4$ and $N(t) \geq 0$ then $V_j^0(t)$ corresponding to the elapsed j^{th} vacation time of any customer. If $C(t) = 5$ and $N(t) \geq 0$ then $H^0(t)$ corresponding to the elapsed time of server being repaired.

Let $\{tn;\ n = 1, 2 \ldots\}$ be then the sequence of epochs at which is either a vacation period ends, service completion occurs or a repair period ends. Then the sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. It follows from Appendix that $\{Zn;\ n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda + \delta)$ then the system will be stable where

$$\rho=\begin{bmatrix}\bar{\alpha}\left[R^*(\lambda+\delta)+\lambda R^*(\lambda+\delta)\right]+\alpha\delta\bar{R}^*(\lambda+\delta)\left[\delta_p^*(\beta)+\dfrac{\left(1-S_p^*(\beta)\right)}{\tau}\right]\\ +\alpha\left[R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right]\left[\delta_b^*(\tau)+\dfrac{\left(1-S_b^*(\tau)\right)}{\tau}\right]\end{bmatrix}$$

4 Steady state analysis of the system

In this section, we develop the steady state difference-differential equations for the retrial queueing system by treating the elapsed retrial times, the elapsed service times, the elapsed vacation time and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states, the PGF for a number of customers in the system and orbit.

In steady state, we assume that $R(0) = 0$, $R(\infty) = 1$, $S_p(0) = 0$, $S_p(\infty) = 1$, $S_b(0) = 0$, $S_b(\infty) = 1$, $V_j(0) = 0$, $V_j(\infty) = 1$, $G(0) = 0$, $G(\infty) = 1$ are continuous at $x = 0$. So that the functions $a(x)$, $\mu_p(x)$, $\mu_b(x)$, $\gamma(x)$ and $\xi(x)$ are the conditional completion rates (hazard rate) for retrial, service of a priority customer and ordinary customer, vacation completion rate, repair completion rate of a customer respectively.

$$\text{i.e., } a(x)dx = \frac{dR(x)}{1-R(x)};\ \mu_p(x)dx = \frac{dS_p(x)}{1-S_p(x)};\ \mu_b(x)dx = \frac{dS_b(x)}{1-S_b(x)};$$

$$\gamma(x)dx\frac{dV(x)}{1-V(x)};\ \eta(x)dx = \frac{dH(x)}{1-H(x)}$$

For the process, we define the limiting probabilities $P_0(t) = P\{X(t) = 0, N(t) = 0\}$ and the probability densities

$$P_n(x,t)dx = P\left\{C(t)=0,\ N(t)=n,\ x < R^0(t) \le x+dx\right\}, \quad \text{for } t \ge 0,\ x \ge 0 \text{ and } n \ge 1.$$

$$\Pi_{1,n}(x,t)dx = P\left\{C(t)=1,\ N(t)=n,\ x < S_p^0(t) \le x+dx\right\}, \quad \text{for } t \ge 0,\ x \ge 0 \text{ and } n \ge 0.$$

$$\Pi_{2,n}(x,y,t)dx = P\left\{C(t)=2,\ N(t)=n,\ x < S_p^0(t) \le x+dx,\ y < S_b^0(t) \le y+dy\right\},$$
$$\text{for } t \ge 0,\ x \ge 0 \text{ and } y \ge 0,\ n \ge 0.$$

$$\Pi_{3,n}(x,t)dx = P\left\{C(t)=3,\ N(t)=n,\ x < S_b^0(t) \le x+dx\right\}, \quad \text{for } t \ge 0,\ x \ge 0 \text{ and } n \ge 0.$$

$$\Omega_{j,n}(x,t)dx = P\left\{C(t)=j+3,\ N(t)=n,\ x < V_j^0(t) \le x+dx\right\},$$
$$\text{for } (1 \le j \le J),\ t \ge 0,\ x \ge 0 \text{ and } n \ge 0.$$

$$Q_n(x,t)dx = P\left\{C(t)=5,\ N(t)=n,\ x < H^0(t) \le x+dx\right\}, \quad \text{for } t \ge 0,\ x \ge 0 \text{ and } n \ge 0.$$

We assume that the stability condition is fulfilled in the sequel and so that we can set $P_0 = \lim_{t\to\infty} P_0(t);$ and limiting densities for $t \ge 0$, $x \ge 0$ and $n \ge 1$.

$$P_n(x) = \lim_{t\to\infty} P_n(x,t); \quad \Pi_{1,n}(x) = \lim_{t\to\infty} \Pi_{1,n}(x,t); \quad \Pi_{2,n}(x) = \lim_{t\to\infty} \Pi_{2,n}(x,y,t);$$
$$\Pi_{3,n}(x) = \lim_{t\to\infty} \Pi_{3,n}(x,t); \quad \Omega_{j,n}(x) = \lim_{t\to\infty} \Omega_{j,n}(x,t); \quad Q_n(x) = \lim_{t\to\infty} Q_n(x,t);$$

4.1 The steady state equations

By using the method of supplementary variable technique, we formulate the system of governing equations of this model as follows:

$$(\lambda+\delta)P_0 = \int_0^\infty \Omega_{J,0}(x)\gamma(x)dx \tag{4.1}$$

$$\frac{dP_n(x)}{dx} + \left(\lambda+\delta+a(x)\right)P_n(x) = 0,\ n \ge 1 \tag{4.2}$$

$$\frac{d\Pi_{1,0}(x)}{dx} + \left(\lambda + \beta + \mu_p(x)\right)\Pi_{1,0}(x) = 0, n = 0, \tag{4.3}$$

$$\frac{d\Pi_{1,n}(x)}{dx} + \left(\lambda + \beta + \mu_p(x)\right)\Pi_{1,n}(x) = \lambda\Pi_{1,n-1}(x), n \geq 1 \tag{4.4}$$

$$\frac{\partial\Pi_{2,0}(x, y)}{\partial x} + \left(\lambda + \beta + \mu_p(x)\right)\Pi_{2,0}(x, y) = 0, n = 0 \tag{4.5}$$

$$\frac{\partial\Pi_{2,n}(x, y)}{\partial x} + \left(\lambda + \beta + \mu_p(x)\right)\Pi_{2,n}(x, y) = \lambda\Pi_{2,n-1}(x, y), n \geq 1, \tag{4.6}$$

$$\frac{d\Pi_{3,0}(x)}{dx} + \left(\lambda + \delta + \beta + \mu_b(x)\right)\Pi_{3,0}(x) = \int_0^\infty \Pi_{2,0}(y, x)\mu_p(y)dy, n = 0 \tag{4.7}$$

$$\begin{aligned}\frac{d\Pi_{3,n}(x)}{dx} + \left(\lambda + \delta + \beta + \mu_b(x)\right)\Pi_{3,n}(x) &= \lambda\Pi_{3,n-1}(x) \\ &+ \int_0^\infty \Pi_{2,n}(y, x)\mu_p(y)dy, n \geq 1\end{aligned} \tag{4.8}$$

$$\frac{d\Omega_{j,0}(x)}{dx} + \left(\lambda + \gamma(x)\right)\Omega_{j,0}(x) = 0, n = 0 \tag{4.9}$$

$$\frac{d\Omega_{j,n}(x)}{dx} + \left(\lambda + \gamma(x)\right)\Omega_{j,n}(x) = \lambda\Omega_{j,n-1}(x), n \geq 1 \tag{4.10}$$

$$\frac{\partial Q_0(x)}{\partial x} + \left(\lambda + \xi(x)\right)Q_0(x) = 0, n = 0 \tag{4.11}$$

$$\frac{\partial Q_n(x)}{\partial x} + \left(\lambda + \xi(u)\right)Q_n(x) = \lambda Q_{n-1}(x), n \geq 1 \tag{4.12}$$

To solve the equations (4.2) to (4.12), the stead state boundary conditions at $x = 0$ and $y = 0$ are followed,

$$\begin{aligned}P_n(0) = \sum_{j=1}^{J}\int_0^\infty \Omega_{j,n}(x)\gamma(x)dx + \int_0^\infty Q_n(x)\eta(x)dx + \beta\theta\left(\int_0^\infty \Pi_{3,n-1}(x)dx\right) \\ + \beta(1-\theta)\left(\int_0^\infty \Pi_{3,n}(x)dx\right) + \beta\int_0^\infty \Pi_{1,n}(x)\mu_p(y)dy + \int_0^\infty \Pi_{3,n}(x)\mu_b(x)dx, \\ n \geq 1\end{aligned} \tag{4.13}$$

$$\Pi_{1,0}(0) = \alpha\left(\delta p_0\right), \quad n = 0 \tag{4.14}$$

$$\Pi_{1,n}(0) = \alpha\left(\delta\int_0^\infty P_n(x)dx\right), n \geq 1 \tag{4.15}$$

$$\Pi_{2,n}(0, x) = \delta \Pi_{3,n}(x),\ n \geq 0 \tag{4.16}$$

$$\Pi_{3,0}(0) = \alpha \left(\int_0^\infty P_1(x) a(x) dx + \lambda P_0 \right),\ n = 0 \tag{4.17}$$

$$\Pi_{3,n}(0) = \alpha \left(\int_0^\infty P_{n+1}(x) a(x) dx + \lambda \int_0^\infty P_n(x) dx \right),\ n \geq 1 \tag{4.18}$$

$$\Omega_{1,n}(0) = \begin{bmatrix} \int_0^\infty \Pi_{1,0}(y) \mu_p(y) dy + \int_0^\infty \Pi_{3,0}(x) \mu_b(x) dx + (1-\theta)\beta \int_0^\infty \Pi_{3,0}(x) dx \\ + \beta \int_0^\infty \Pi_{1,0}(x) dx + \int_0^\infty Q_0(x) \eta(x) dx \end{bmatrix} \tag{4.19}$$

$$\Omega_{j,n}(0) = \begin{cases} \int_0^\infty \Omega_{j-1,n}(x) \gamma(x) dx, & n = 0,\ j = 2, \ldots J \\ 0, & n \geq 1 \end{cases} \tag{4.20}$$

$$Q_0(0) = \bar{\alpha} \delta P_0, \quad n = 0 \tag{4.21}$$

$$Q_n(0) = \bar{\alpha} \left(\lambda \int_0^\infty P_{n-1}(x) a(x) dx + \int_0^\infty P_n(x) dx + (\lambda + \delta) P_0 \right), \quad n \geq 1 \tag{4.22}$$

The normalising condition is

$$P_0 + \sum_{n=1}^{\infty} \int_0^\infty P_n(x) dx + \sum_{n=0}^{\infty} \begin{pmatrix} \int_0^\infty \Pi_{1,n}(x) dx + \int_0^\infty \int_0^\infty \Pi_{2,n}(x, y) dx dy + \int_0^\infty \Pi_{3,n}(x) dx \\ + \int_0^\infty Q_n(x) dx + \sum_{j=1}^{J} \int_0^\infty \Omega_{j,n}(x) dx \end{pmatrix} = 1 \tag{4.23}$$

4.2 The steady state solution

The steady state solution of the retrial queueing model is obtained by using the PGF technique. To solve the above equations, we define the generating functions for $|z| \leq 1$ as follows:

$$P(x, z) = \sum_{n=1}^{\infty} P_n(x) z^n; \quad P(0, z) = \sum_{n=1}^{\infty} P_n(0) z^n; \quad \Pi_1(x, z) = \sum_{n=0}^{\infty} \Pi_{1,n}(x) z^n;$$

$$\Pi_1(0, z) = \sum_{n=0}^{\infty} \Pi_{1,n}(0) z^n; \quad \Pi_2(x, y, z) = \sum_{n=0}^{\infty} \Pi_{2,n}(x, y) z^n;$$

$$\Pi_2(x, 0, z) = \sum_{n=0}^{\infty} \Pi_{2,n}(x, 0) z^n; \quad \Pi_3(x, z) = \sum_{n=0}^{\infty} \Pi_{3,n}(x) z^n;$$

$$\Pi_3(0, z) = \sum_{n=0}^{\infty} \Pi_{3,n}(0) z^n; \quad \Omega_j(x, z) = \sum_{n=0}^{\infty} \Omega_{j,n}(x) z^n; \Omega_j(0, z) = \sum_{n=0}^{\infty} \Omega_{j,n}(0) z^n;$$

$$Q(x, z) = \sum_{n=0}^{\infty} Q_n(x) z^n; Q(0, z) = \sum_{n=0}^{\infty} Q_n(0) z^n;$$

On multiplying the equations (4.2) to (4.12) by z^n and summing over n, (n = 0, 1, 2…) and Solving the partial differential equations, we get

$$P(x, z) = P(0, z)\left[1 - R(x)\right] e^{-(\lambda+\delta)x} \tag{4.24}$$

$$\Pi_1(x, z) = \Pi_1(0, z)\left[1 - S_p(x)\right] e^{-A_p(z)x}, \tag{4.25}$$

$$\Pi_2(x, y, z) = \Pi_2(0, y, z)\left[1 - S_p(x)\right] e^{-A_p(z)x}, \tag{4.26}$$

$$\Pi_3(x, z) = \Pi_3(0, z)\left[1 - S_b(x)\right] e^{-A_b(z)x}, \tag{4.27}$$

$$\Omega_j(x, z) = \Omega_j(0, z)\left[1 - V(x)\right] e^{-b(z)x}, \quad \text{for } (j = 1, 2, \ldots J) \tag{4.28}$$

$$Q(x, z) = Q(0, z)\left[1 - H(x)\right] e^{-b(z)x}, \tag{4.29}$$

where $A_p(z) = \lambda(1 - z) + \beta$, $A_b(z) = (A_p(z) + \delta(1 - S_p^*(A_p(z))))$ and $b(z) = (\lambda(1 - z))$.

From (4.10) we obtain,

$$\Omega_{j,0}(x) = \Omega_{j,0}(0)\left[1 - V(x)\right] e^{-\lambda x}, \quad \text{for } (j = 1, 2, \ldots J) \tag{4.30}$$

Multiplying with equation (4.30) by $\gamma(x)$ on both sides for $j = J$ and integrating with respect to x from 0 to ∞, then from (4.1) we have:

$$\Omega_{J,0}(0) = \frac{(\lambda + \delta) p_0}{V^*(\lambda)} \tag{4.31}$$

From equation (4.30) and solving (4.20), (4.30) over the range $j = J - 1, J - 2, \ldots, 1$, after some simplifications, we will have:

$$\Omega_{j,0}(0) = \frac{(\lambda + \delta) p_0}{\left(V^*(\lambda)\right)^{J-j+1}}, \quad j = 1, 2, \ldots, J - 1. \tag{4.32}$$

from (4.20), (4.31) and (4.32), we obtain

$$\Omega_J(0, z) = \frac{(\lambda + \delta) p_0}{\left(V^*(\lambda)\right)^{J-j+1}}, \quad j = 1, 2, \ldots, J - 1 \tag{4.33}$$

Integrating the equation (3.33) from 0 to ∞ and using (4.31) and (4.32) again, we finally obtain

$$\Omega_{j,0}(0, z) = \frac{(\lambda+\delta)p_0\left(1-V^*(\lambda)\right)}{\left(V^*(\lambda)\right)^{J-j+1}}, \quad j = 1, 2, \ldots J. \tag{4.34}$$

Note that $\Omega_{j,0}$ represents the steady-state probability that no customer appears while the server is on the j^{th} vacation.

Let us define as the probability that no customer appears in the system while the server is on vacation. Then,

$$\Omega_0 = \frac{(\lambda+\delta)p_0\left(1-\left(V^*(\lambda)\right)^J\right)}{\left(V^*(\lambda)\right)^J\left(1-V^*(\lambda)\right)} \tag{4.35}$$

From the equations (4.13) to (4.22), we can obtain

$$\begin{aligned} P(0, z) = {} & \int_0^\infty \Pi_1(x, z)\mu_p(x)dx + \int_0^\infty \Pi_3(x, z)\mu_b(x)dx + \sum_{j=1}^{J}\int_0^\infty \Omega_j(x, z)\gamma(x)dx \\ & + \int_0^\infty Q(x, z)\eta(x)dx + \beta(1-\theta+\theta z)\int_0^\infty \Pi_3(x, z)dx \\ & + \beta\int_0^\infty \Pi_1(x, z)dx - \left(\sum_{j=1}^{J}\Omega_{j,0}(0) + (\lambda+\delta)P_0\right) \end{aligned} \tag{4.36}$$

$$\Pi_1(0, z) = \alpha\left(\delta\int_0^\infty P(x, z)dx + \delta P_0\right), \tag{4.37}$$

$$\Pi_2(0, x, z) = \delta\Pi_3(x, z) \tag{4.38}$$

$$\Pi_3(0, z) = \alpha\left(\frac{1}{z}\int_0^\infty P(x, z)a(x)dx + \lambda\int_0^\infty P(x, z)dx + \lambda P_0\right), \tag{4.39}$$

$$Q(0, z) = \overline{\alpha}\left(\int_0^\infty P(x, z)a(x)dx + \lambda z\int_0^\infty P(x, z)dx + (\lambda+\delta)P_0\right) \tag{4.40}$$

Inserting the equation (4.24) in (4.37), we get

$$\Pi_1(0, z) = \alpha\left(\delta P(0, z)\overline{R}^*(\lambda+\delta) + \delta P_0\right), \tag{4.41}$$

where $\overline{R}^*(\lambda+\delta) = \left(\dfrac{1-R^*(\lambda+\delta)}{\lambda+\delta}\right)$.

Inserting equation (4.24) in (4.38) and make some manipulation, finally we get,

$$\Pi_3(0, z) = \frac{\alpha P(0, z)}{z}\left(R^*(\lambda+\delta) + \lambda z\overline{R}^*(\lambda+\delta)\right) + \alpha\lambda P_0 \tag{4.42}$$

Inserting the equation (4.24) in (4.39), we get

$$Q(0,z) = \bar{\alpha} P(0, z)\left(R^*(\lambda+\delta) + \lambda z \bar{R}^*(\lambda+\delta)\right) + \bar{\alpha}(\lambda+\delta)P_0 \tag{4.43}$$

Using (4.24) to (4.29) in (4.36) and make some manipulation, we get

$$P(0, z) = \begin{cases} \Pi_1(0, z)S_p^*\left(A_p(z)\right) + \Pi_3(0, z)S_b^*\left(A_b(z)\right) \\ \quad -\sum_{j=1}^{J} \Omega(0, z)V^*\left(b(z)\right) + \beta(1-\theta+\theta z)\Pi_3(0, z)\dfrac{\left(1-S_b^*\left(A_b(z)\right)\right)}{\left(A_b(z)\right)} \\ Q(0,z)H^*(b(z)) + \beta\Pi_1(0, z)\dfrac{\left(1-S_p^*\left(A_p(z)\right)\right)}{\left(A_p(z)\right)} \\ \quad -(\lambda+\delta)p_0\dfrac{\left(1-\left(V^*(\lambda)\right)^J\right)}{\left(V^*(\lambda)\right)^J} - (\lambda+\delta)p_0 \end{cases} \tag{4.44}$$

Using the equation (4.41) to (4.43) and (4.33) in (4.44), we get

$$P(0, z) = \frac{Nr(z)}{Dr(z)} \tag{4.45}$$

$$Nr(z) = zP_0\begin{Bmatrix} \left[\left((\lambda+\delta)\left[N(z)-1\right]\right) + \bar{\alpha}(\lambda+\delta)H^*\left(b(z)\right) + \alpha\lambda S_b^*\left(A_b(z)\right)\right. \\ \left. +\alpha\delta S_p^*\left(A_p(z)\right)\right]A_b(z)A_p(z) + \alpha\lambda\beta(1-q-\theta z) \\ \left(1-S_b^*\left(A_b(z)\right)\right)A_p(z) + \alpha\delta\beta\left(1-S_p^*\left(A_p(z)\right)\right)A_b(z) \end{Bmatrix}$$

$$Dr(z) = zA_b(z)A_p(z) - \begin{Bmatrix} \begin{bmatrix} \bar{\alpha}z\left(R^*(\lambda+\delta) + \lambda z\bar{R}^*(\lambda+\delta)\right)H^*\left(b(z)\right) \\ +\alpha\left(R^*(\lambda+\delta) + \lambda z\bar{R}^*(\lambda+\delta)\right)S_b^*\left(A_b(z)\right) \\ +z\alpha\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right) \end{bmatrix} A_b(z)A_p(z) \\ +\alpha\beta(1-\theta+\theta z)\left(1-S_b^*\left(A_b(z)\right)\right) \\ \left(R^*(\lambda+\delta) + \lambda z\bar{R}^*(\lambda+\delta)\right)A_p(z) \\ +\alpha\delta\beta z\left(1-S_p^*\left(A_p(z)\right)\bar{R}^*(\lambda+\delta)A_b(z)\right) \end{Bmatrix}$$

Using the equation (4.45) in (4.41), we get

$$\Pi_1(0, z) = \alpha\delta p_0\begin{Bmatrix} z\left(\bar{R}^*(\lambda+\delta)\left[N(z)-1\right]-1\right) \\ +\bar{\alpha}z\left[(\lambda+\delta-\lambda z)\bar{R}^*(\lambda+\delta) - R^*(\lambda+\delta)\right] \\ H^*\left(b(z)\right) - \alpha R^*(\lambda+\delta)S_b^*\left(A_b(z)\right) \\ -\alpha\beta R^*(\lambda+\delta)(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)A_p(z) \end{Bmatrix} \Big/ Dr(z) \tag{4.46}$$

Using the equation (4.45) in (4.42), we get

$$\Pi_3(0,z)=\alpha P_0\left\{\begin{array}{l}\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\\ \left(\begin{array}{l}\left((\lambda+\delta)\left[N(z)-1\right]\right)\\ +\bar{\alpha}(\lambda+\delta-\lambda z)H^*\left(b(z)\right)\end{array}\right)\\ +\lambda z+\alpha\delta R^*(\lambda+\delta)S_p^*\left(A_p(z)\right)\end{array}\right\}A_p(z)A_b(z)\Big/Dr(z) \tag{4.47}$$

Using the equation (4.39), (4.45) in (4.38), we get

$$\Pi_2(0,x,z)=\frac{\alpha\delta P_0\left[1-S_b^*(x)\right]e^{-A_b(z)x}\left\{\begin{array}{l}\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\\ \left((\lambda+\delta)\left[N(z)-1\right]\right)+\alpha(\lambda+\delta-\lambda z)\\ \left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\\ H^*\left(b(z)\right)+\lambda z\\ \alpha\delta R^*(\lambda+\delta)S_p^*\left(A_p(z)\right)\end{array}\right\}A_p(z)A_b(z)}{Dr(z)} \tag{4.48}$$

Using the equation (4.45) in (4.43) we get

$$Q(0,z)=\bar{\alpha}P_0\left\{\begin{array}{l}z\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)(\lambda+\delta)\\ \left[N(z)-1\right]A_p(z)A_b(z)\\ +\alpha\delta zA_b(z)\left(R^*(\lambda+\delta)+\left(\lambda z-(\lambda+\delta)\right)\bar{R}^*(\lambda+\delta)\right)\\ \left[S_p^*\left(A_p(z)\right)A_p(z)+\beta\left(1-S_p^*\left(A_p(z)\right)\right)\right]\\ +\alpha\left(\lambda z-(\lambda+\delta)\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\\ A_p(z)\left(\beta(1-\theta-\theta z)\right)\\ \left(1-S_b^*\left(A_b(z)\right)\right)+S_b^*\left(A_b(z)\right)A_b(z)\end{array}\right\}\Big/Dr(z) \tag{4.49}$$

Using the equations (4.45) to (4.51) in (4.24) to (4.29), then we get the results for the following PGFs $P(x, z)$, $\Pi_1(x, z)$, $\Pi_2(x, y, z)$, $\Pi_3(x, z)$, $\Omega j(x, z)$ and $Q(x, z)$. Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Theorem 4.1. Under the stability condition $\rho < R^*(\lambda + \delta)$, the marginal probability distributions of the number of customers in the orbit when server being idle, busy serving priority customers without preempting an ordinary customer, busy serving priority customers with preempting an ordinary customer, busy serving ordinary customers, on vacation and repair due to starting failure is given by

$$P(z)=\frac{Nr(z)}{Dr(z)} \tag{4.50}$$

$$Nr(z)=zP_0\bar{R}^*(\lambda+\delta)\left\{\begin{array}{l}\left[\begin{array}{l}\left((\lambda+\delta)\left[N(z)-1\right]\right)+\bar{\alpha}(\lambda+\delta)H^*\left(b(z)\right)\\ +\alpha\lambda S_b^*\left(A_b(z)\right)+\alpha\delta S_p^*\left(A_p(z)\right)\end{array}\right]A_b(z)A_p(z)\\ +\alpha\lambda\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)A_p(z)\right)\\ +\alpha\delta\beta\left(1-S_p^*\left(A_p(z)\right)A_b z\right)\end{array}\right\}$$

$$Dr(z)=zA_b(z)A_p(z)-\left\{\begin{array}{l}\left[\begin{array}{l}\bar{\alpha}z\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)H^*\left(b(z)\right)\\ +\alpha\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)S_b^*\left(A_b(z)\right)\\ +z\alpha\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right)\end{array}\right]A_b(z)A_p(z)\\ +\alpha\beta(1-\theta+\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\\ \quad\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)A_p(z)\\ +\alpha\delta\beta z\left(1-S_p^*\left(A_p(z)\right)\bar{R}^*(\lambda+\delta)A_b(z)\right)\end{array}\right\}$$

$$\Pi_1(z)=\left.\begin{array}{l}\alpha\delta P_0\left[1-S_p^*\left(A_p(z)\right)\right]\\ \left\{\begin{array}{l}\left(\begin{array}{l}z\left(\bar{R}^*(\lambda+\delta)\left[N(z)-1\right]-1\right)\\ \bar{\alpha}z\left[(\lambda+\delta-\lambda z)\bar{R}^*(\lambda+\delta)-R^*(\lambda+\delta)\right]\\ H^*\left(b(z)\right)-\alpha R^*(\lambda+\delta)S_b^*\left(A_b(z)\right)\end{array}\right)A_b(z)\\ -\alpha\beta R^*(\lambda+\delta)(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\end{array}\right\}\\ A_p(z)A_b(z)\end{array}\right/ Dr(z) \tag{4.51}$$

$$\Pi_2(z)=\left.\begin{array}{l}\alpha\delta P_0\left[1-S_b^*\left(A_b(z)\right)\right]\left[1-S_p^*\left(A_p(z)\right)\right]\\ \left\{\begin{array}{l}\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\left((\lambda+\delta)\left[N(z)-1\right]\right)\\ +\bar{\alpha}(\lambda+\delta-\lambda z)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\\ H^*\left(b(z)\right)+\lambda z+\alpha\delta R^*(\lambda+\delta)S_p^*\left(A_p(z)\right)\end{array}\right\}\end{array}\right/ Dr(z) \tag{4.52}$$

$$\Pi_3(z)=\left.\begin{array}{l}\alpha P_0\left[1-S_b^*\left(A_b(z)\right)\right]\\ \left\{\begin{array}{l}\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\\ \left(\left((\lambda+\delta)\left[N(z)-1\right]\right)+\bar{\alpha}(\lambda+\delta-\lambda z)H^*\left(b(z)\right)\right)\\ +\lambda z+\alpha\delta R^*(\lambda+\delta)S_p^*\left(A_p(z)\right)\end{array}\right\}A_p(z)\end{array}\right/ Dr(z) \tag{4.53}$$

$$\Omega(z)=\frac{(\lambda+\delta)P_0\left[1-V^*\left(b(z)\right)\right]}{\left[V^*(\lambda)\right]^{J-j+1}b(z)},\quad j=1,2,\ldots,J \tag{4.54}$$

$$\frac{\bar{\alpha}P_0\left[1-H^*\left(b(z)\right)\right]}{b(z)}$$

$$Q(z)=\left.\left\{\begin{array}{l} z\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)(\lambda+\delta)\left[N(z)-1\right] \\ A_p(z)A_b(z)+\alpha\delta zA_b(z) \\ \left(R^*(\lambda+\delta)+\left(\lambda z-(\lambda+\delta)\right)\bar{R}^*(\lambda+\delta)\right) \\ \left[S_p^*\left(A_p(z)\right)A_p(z)\right]+\alpha\delta zA_b(z) \\ \left(R^*(\lambda+\delta)+\left(\lambda z-(\lambda+\delta)\right)\bar{R}^*(\lambda+\delta)\right) \\ \beta\left(1-S_p^*\left(A_p(z)\right)\right)+\alpha\left(\lambda z-(\lambda+\delta)\right) \\ \left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)A_p(z) \\ \left(\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\right) \\ +\alpha\left(\lambda z-(\lambda\delta)\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right) \\ A_p(z)S_b^*\left(A_b(z)\right)A_b(z) \end{array}\right\}\right/ Dr(z) \tag{4.55}$$

where

$$P_0=\frac{R^*(\lambda+\delta)-\rho}{\omega} \tag{4.56}$$

$$\omega=\left\{\begin{array}{l} N^{(1)}(1)(\lambda+\delta)\left[\frac{1}{\lambda}\left(R^*(\lambda+\delta)+\lambda R^*(\lambda+\delta)\right)2\bar{\alpha}h^{(1)}\right]+R^*(\lambda+\delta)(\lambda+\delta)[\bar{\alpha}-1] \\ +\alpha\left[\delta S_p^*(\beta)+\lambda S_b^*(\tau)\right]+\frac{\alpha\lambda\beta}{\tau}\left[1-S_b^*(\tau)+\alpha\delta\left[1-S_p^*(\beta)\right]\right]+\frac{\alpha\delta}{\tau}\left[1-S_p^*(\beta)\right] \\ \left\{\begin{array}{l} 1-R^*(\lambda+\delta)+\bar{\alpha}\left[\left((\lambda+\delta)-\lambda z\right)\bar{R}^*(\lambda+\delta)-R^*(\lambda+\delta)\right] \\ -\alpha R^*(\lambda+\delta)S_b^*(\tau)-\frac{\alpha\beta R^*(\lambda+\delta)}{\tau}\left[1-S_b^*(\tau)\right] \end{array}\right\} \\ +\frac{\alpha S_b^*(\tau)A_b(1)}{\beta\tau}\left[\begin{array}{l} \left(R^*(\lambda+\delta)+\lambda^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right) \\ \left[\bar{\alpha}\delta-(\lambda+\delta)\right]+\lambda+\alpha\delta P^*(\lambda+\delta) \end{array}\right]D_p^*(\beta) \\ +\frac{2\alpha h^{(1)}}{\beta\tau}\left[\begin{array}{l} \left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)(\lambda+\delta)\left[\beta A_b'(1)-\tau(\beta+\lambda)\right] \\ +\beta\tau\lambda(\lambda+\delta)\bar{R}^*(\lambda+\delta)+\alpha\delta\left(R^*(\lambda+\delta)-\delta\bar{R}^*(\lambda+\delta)\right) \\ \left[\lambda\tau S_p^*(\beta)-\beta A_b'(1)\right]+\left[\beta\left(1-S_b^*(\tau)\right)+\tau\beta S_b^*(\tau)\right] \\ \left[\lambda\alpha\delta\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)-\alpha\delta\tau R^*(\lambda+\delta)\right] \\ -\beta\alpha\delta\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right) \\ \left[\beta\theta\left(1-S_b^*(\tau)\right)+\beta_v^{(1)}A_b^{(1)}(\beta+\tau)+S_b^*(\tau)A_b'(1)\right] \end{array}\right] \end{array}\right\}$$

$$\rho = \begin{Bmatrix} \bar{\alpha}\left[R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right] \\ +\alpha\delta\bar{R}^*(\lambda+\delta)\left[S_p^*(\beta)+\dfrac{\left(1-S_p^*(\beta)\right)}{\tau}\right] \\ -\alpha\left[R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right]\left[S_p^*(\tau)+\beta\dfrac{\left(1-S_p^*(\tau)\right)}{\tau}\right] \end{Bmatrix}; b(z)=\lambda(1-z);$$

$$A_p(z)=\left(\lambda(1-z)+\alpha(1-G^*\left(b(z)\right)\right) \quad \text{and}$$
$$A_b(z)=\left(\lambda(1-z)+\alpha\left(1-G^*\left(b(z)\right)\right)+\delta\left(1-S_p^*\left(A_p(z)\right)\right)\right)$$

Proof. Integrating the equation (4.24) to (4.29) with respect to x, we define the PGFs as,

$$P(z)=\int_0^\infty P(x,z)dx, \quad \Pi_1(z)=\int_0^\infty \Pi_1(x,z)dx, \quad \Pi_3(z)=\int_0^\infty \Pi_3(x,z)dx,$$

$$\Omega_j(z)=\int_0^\infty \Omega_j(x,z)dx \quad \text{and} \quad Q(z)=\int_0^\infty Q(x,z)dx.$$

Integrating the above equation (4.26) by x and y, define the partial generating functions as $\Pi_2(x,z)=\int_0^\infty \Pi_2(x,y,z)dy$, and $\Pi_2(z)=\int_0^\infty \Pi_2(x,z)dx$. Using the normalised condition, we can be determined the probability that the server is idle (P0). Thus, by setting $z = 1$ in (4.52) to (4.57) and applying L-Hospital's rule whenever necessary and we get

$$P_0+P(z)+\Pi_1(z)+\Pi_2(z)+\Pi_3(z)+\Omega_j(z)+Q(z)=1.$$

Corollary 4.1.: if the system satisfies the stability condition $\rho < R^*(\lambda+\delta)$ the PGF of number of customers in the system at stationary point of time is

$$K_s(z)=\frac{Nr_s(z)}{Dr_s(z)}=P_0+P(z)+z\left(\Pi_1(z)+\Pi_2(z)+\Pi_3(z)+Q(z)\right)+\Omega_j(z) \tag{4.57}$$

where

$$Dr_s(z)=zA_b(z)A_p(z)-\begin{Bmatrix} \begin{bmatrix} \left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\left[\bar{\alpha}zH^*\left(b(z)\right)\right. \\ \left.+\alpha S_b^*\left(A_b(z)\right)\right]+z\alpha\delta\bar{R}^*(\lambda+\delta)\left[S_p^*\left(A_p(z)\right)\right. \\ \left.+\beta\left(1-S_p^*\left(A_p(z)\right)\right)/A_p(z)\right] \end{bmatrix} A_p(z)A_b(z) \\ +\alpha\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right) \\ \left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)A_p(z) \end{Bmatrix}$$

$$
Nr_s(z)=\frac{P_0}{b(z)}\left\{\begin{array}{l}
Dr_s(z)+(\lambda+\delta)N(z)Dr_s(z)+z\overline{R}^*(\lambda+\delta)\\
\left\{\begin{array}{l}
\left[\begin{array}{l}\left((\lambda+\delta)\left[N(z-1)\right]\right)+\overline{\alpha}(\lambda+\delta)H^*\left(b(z)\right)\\
+\alpha\lambda S_b^*\left(A_b(z)\right)\left(A_b(z)\right)+\alpha\delta S_p^*(z)\end{array}\right]A_b(z)A_p(z)\\
+\alpha\lambda\beta(1-\theta-\theta z)\left(\begin{array}{l}1-S_b^*\left(A_b(z)\right)A_p(z)\\
+\alpha\delta\beta\left(1-S_p^*\left(A_p(z)\right)A_b(z)\right)\end{array}\right)
\end{array}\right\}\\
+z\alpha\delta\left[1-S_p^*\left(A_p(z)\right)\right]\\
\left\{\left(\begin{array}{l}
z\left(\overline{R}^*(\lambda+\delta)\left[N(z)-1\right]-1\right)\\
+\overline{\alpha}z\left[(\lambda+\delta-\lambda z)\overline{R}^*(\lambda+\delta)-R^*(\lambda+\delta)\right]\\
H^*\left(b(z)\right)-\alpha R^*(\lambda+\delta)S_b^*\left(A_b(z)\right)
\end{array}\right)\right\}\\
+z\alpha\left[1-S_b^*\left(A_b(z)\right)\right]\\
\left\{\begin{array}{l}
\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right)\left(\begin{array}{l}\left((\lambda+\delta)\left[N(z)-1\right]\right)\\
+\overline{\alpha}(\lambda+\delta-\lambda z)H^*\left(b(z)\right)\end{array}\right)\\
+\lambda z+\alpha\delta R^*(\lambda+\delta)S_p^*\left(A_p(z)\right)
\end{array}\right\}\\
+z\overline{\alpha}\left[1-H^*\left(b(z)\right)\right]\\
\left\{\begin{array}{l}
z\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right)(\lambda+\delta)\left[N(z)-1\right]A_p(z)A_b(z)\\
+\alpha\delta zA_b(z)\left(R^*(\lambda+\delta)+\left(\lambda z-(\lambda+\delta)\right)\overline{R}^*(\lambda+\delta)\right)\\
\left[S_p^*\left(A_p(z)\right)A_p(z)\right]+\alpha\delta zA_b(z)\\
\left(R^*(\lambda+\delta)+\left(\lambda z-(\lambda+\delta)\right)\overline{R}^*(\lambda+\delta)\right)\beta\left(1-S_p^*\left(A_p(z)\right)\right)\\
+\alpha\left(\lambda z-(\lambda+\delta)\right)\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right)A_p(z)\\
\left(\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\right)+\alpha\left(\lambda z-(\lambda+\delta)\right)\\
\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right)A_p(z)S_b^*\left(A_b(z)A_b(z)\right)
\end{array}\right\}
\end{array}\right\}
$$

$$
K_o(z)=\frac{Nr_o(z)}{Dt_s(z)}=P_0+P(z)+\Pi_1(z)+\Pi_2(z)+\Pi_3(z)+Q(z)+\Omega_j(z) \tag{4.58}
$$

$$Nr_o(z)=\frac{P_0}{b(z)}\left\{\begin{array}{l} Dr_0(z)+(\lambda+\delta)N(z)Dr_0(z)+z\overline{R}^*(\lambda+\delta) \\ \left\{\begin{array}{l}\left[\begin{array}{l}\left((\lambda+\delta)\left[N(z)-1\right]\right)+\overline{\alpha}(\lambda+\delta)H^*\left(v(z)\right) \\ +\alpha\lambda S_b^*\left(A_b(z)\right)+\alpha\delta S_p^*\left(A_p(z)\right)\end{array}\right]A_b(z)A_p(z) \\ +\alpha\lambda\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)A_p(z)\right) \\ +\alpha\delta\beta\left(1-S_p^*\left(A_p(z)\right)A_b(z)\right)\end{array}\right\} \\ +\alpha\delta\left[1-S_p^*\left(A_p(z)\right)\right] \\ \left\{\begin{array}{l}\left(\begin{array}{l} z\left(\overline{R}^*(\lambda+\delta)\left[N(z)-1\right]-1\right) \\ +\overline{\alpha}z\left[(\lambda+\delta-\lambda z)\overline{R}^*(\lambda+\delta)-R^*(\lambda+\delta)\right]H^*\left(b(z)\right) \\ -\alpha R^*(\lambda+\delta)S_b^*\left(A_b(z)\right)\end{array}\right)A_b(z) \\ +\alpha\beta R^*(\lambda+\delta)(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\end{array}\right\} \\ +\alpha\left[1-S_b^*\left(A_b(z)\right)\right] \\ \left\{\begin{array}{l}\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right) \\ \left(\left((\lambda+\delta)\left[N(z)-1\right]\right)+\overline{\alpha}(\lambda+\delta-\lambda z)H^*\left(b(z)\right)\right) \\ +\lambda z+\alpha\delta R^*(\lambda+\delta)S_p^*\left(A_p(z)\right)\end{array}\right\}A_b(z) \\ \overline{\alpha}\left[1-H^*\left(b(z)\right)\right] \\ \left\{\begin{array}{l} z\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right)(\lambda+\delta)\left[N(z)-1\right]A_p(z)A_b(z) \\ +\alpha\delta zA_b(z)\left(R^*(\lambda+\delta)+\left(\lambda z-(\lambda+\delta)\right)\overline{R}^*(\lambda+\delta)\right) \\ \left[S_p^*\left(A_p(z)\right)A_p(z)\right]+\alpha\delta zA_b(z) \\ \left(R^*(\lambda+\delta)+\left(\lambda z-(\lambda+\delta)\right)\overline{R}^*(\lambda+\delta)\right)\beta\left(1-S_p^*\left(A_p(z)\right)\right) \\ +\alpha\left(\lambda z-(\lambda+\delta)\right)\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right)A_p(z) \\ \left(\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\right) \\ +\alpha\left(\lambda z-(\lambda+\delta)\right)\left(R^*(\lambda+\delta)+\lambda z\overline{R}^*(\lambda+\delta)\right) \\ A_p(z)S_b^*\left(A_b(z)\right)A_b(z)\end{array}\right\}\end{array}\right\}$$

where P_0 is given in equation (4.62).

5 System performance measures

In this section, we derive some system probabilities, a mean number of customers in the orbit/system, mean busy period and the busy cycle of the model.

5.1 System state probabilities

From equations (4.52) to (4.57), by setting $z \to 1$ and applying L-Hospital's rule whenever necessary, then we get the following results.

1 the steady-state probability that the server is idle during the retrial is given by,

$$P = P(1) = \frac{R^*(\lambda+\delta)}{\omega}\begin{bmatrix}(\lambda+\delta)[\overline{\alpha}-1] + \alpha\left(\delta S_p^*(\beta) + \lambda S_b^*(\tau)\right) \\ +\frac{\alpha\lambda\beta}{\tau}\left[1 - S_b^*(\tau)\right] + \alpha\delta\left[1 - S_p^*(\beta)\right]\end{bmatrix}$$

2 the steady-state probability that the server is busy serving priority customers without preempting an ordinary customer is given by,

$$\Pi_1 = \Pi_1(1) = \frac{\alpha\delta\left[1 - S_p^*(\beta)\right]}{\beta\omega}\begin{bmatrix}\left(\overline{\alpha}\left[(\lambda+\delta) - \lambda z\right]\overline{R}^*(\lambda+\delta) - R^*(\lambda+\delta)\right) \\ -R^*(\lambda+\delta)\left[1 - \alpha S_b^*(\tau)\right] - \frac{\alpha\beta R^*(\lambda+\delta)}{\tau}\left[1 - S_b^*(\tau)\right]\end{bmatrix}$$

3 the steady-state probability that the server is busy serving priority customers with preempting an ordinary customer is given by,

$$\Pi_2 = \Pi_2(1) = \frac{\alpha\delta\left[1 - S_p^*(\beta)\right]\left[1 - S_b^*(\tau)\right]}{\beta\tau\omega}\begin{bmatrix}\left[\overline{\alpha}\delta - (\lambda+\delta)\right]\left(R^*(\lambda+\delta) + \lambda\overline{R}(\lambda+\delta)\right) \\ +\lambda + \alpha\delta R^*(\lambda+\delta)S_p^*(\beta)\end{bmatrix}$$

4 the steady-state probability that the server is busy serving ordinary customers is given by,

$$\Pi_3 = \Pi_3(1) = \frac{\alpha\left[1 - S_b^*(\tau)\right]}{\omega\tau}\begin{bmatrix}\left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)\left[\overline{\alpha}\delta - (\lambda+\delta)\right] \\ +\lambda + \alpha\delta R^*(\lambda+\delta)S_p^*(\beta)\end{bmatrix}$$

5 The steady-state probability that the server is on vacation, is given by

$$\Omega = \Omega(1) = \frac{(\lambda+\delta)N^{(1)}[1-\rho]}{\lambda\omega}$$

6 The steady-state probability that the server is on starting failure is given by,

$$Q = Q(1) = \frac{2\overline{\alpha}h^{(1)}}{\beta\tau\omega}\begin{bmatrix}\left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)(\lambda+\delta)\left[\beta A_b'(1) - \lambda\tau - \beta\tau\left[N^{(1)}(1) - 1\right]\right] \\ +\beta\tau\lambda(\lambda+\delta)\overline{R}^*(\lambda+\delta) - \alpha\delta\left(R^*(\lambda+\delta) - \delta\overline{R}^*(\lambda+\delta)\right) \\ \left[\lambda\tau S_p^*(\beta) - \beta A_b'(1)\right] + \left[\beta\left[1 - S_p^*(\beta)\right] + \tau S_p^*(\beta)\right] \\ \left(\alpha\lambda\delta\left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right) - \alpha\delta\tau R^*(\lambda+\delta)\right) \\ +\beta\alpha\delta\left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right) \\ \left[\beta\theta\left[1 - S_p^*(\beta)\right] + \beta_b^{(1)}A_b'(1)(\beta+\tau) + S_b^*(\tau)A_b'(1)\right]\end{bmatrix}$$

5.2 Mean system size and orbit size

If the system is in steady state condition, the expected number of customers in the orbit (L_q) is obtained by differentiating (4.64) with respect to z and evaluating at $z = 1$.

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z) = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_q''(1) = 2\left\{\begin{array}{l}
N'(1)(\lambda+\delta)\left\{\begin{array}{l}
\left[\beta\left(\tau + A_b(1)\right) - \tau\lambda\right] + \left(R^*(\lambda+\delta) + \lambda\bar{R}^*(\lambda+\delta)\right) \\
\left[\begin{array}{l}\bar{\alpha}\left(\beta A_b'(1) - \lambda\tau\right) + 2\beta\tau - \alpha S_b^*(\tau) \\ +\beta\tau\alpha A_b'(1)\beta_b^{(1)} - \alpha\beta S_b^*(\tau)A_b'(1) + \left[1S_b^*(\tau)\right]\beta^2\theta\tau\alpha\end{array}\right] \\
\alpha\bar{R}^*(\lambda+\delta)\left\{\begin{array}{l}\delta S_p^*(\beta) - \beta\lambda\tau S_b^*(\tau) - \beta\tau\delta S_p^*(\beta) - \beta\tau\delta\beta_p^{(1)} \\ +\lambda\beta^2\left(1 - S_b^*(\tau)\right) + \bar{\alpha}\lambda h^{(1)}\bar{R}^*(\lambda+\delta)\end{array}\right\}
\end{array}\right\} \\
-\lambda\left\{\begin{array}{l}
\left[\beta\left(\tau A_b'(1)\right) - \tau\lambda\right] - \bar{\alpha}\left(R^*(\lambda+\delta) - \delta\bar{R}^*(\lambda+\delta)\right) \\
\left[\left[\beta\left(\tau + A_b'(1)\right) - \tau\lambda\right] - \lambda\beta\tau h^{(1)}\right] - \bar{\alpha}\beta\lambda\bar{R}^*(\lambda+\delta) \\
-\alpha R^*(\lambda+\delta)\left[(\beta-\lambda)\tau S_b^*(\tau) + \left[1 - S_b^*(\tau)\right]\left[\beta\theta\tau - \lambda\right]\right]
\end{array}\right\} \\
-\alpha\lambda\left[\begin{array}{l}
\lambda S_p^*(\beta)\left\{\begin{array}{l}\tau - \tau\bar{R}(\lambda+\delta) + \bar{\alpha}\tau\left(R^*(\lambda+\delta) + \delta\bar{R}^*(\lambda+\delta)\right) \\ -\alpha R^*(\lambda+\delta)\left[\tau S_b^*(\tau) + \beta\left[1 - S_b^*(\tau)\right]\right]\end{array}\right\} \\
+\left[1 - S_p^*(\beta)\right]\left\{\begin{array}{l}
\left(A_b'(1) + \tau\right)\left[1 - R^*(\lambda+\delta)\right] + \tau R^*(\lambda+\delta)N^*(1) \\
+\bar{\alpha}\left[\lambda\tau\bar{R}^*(\lambda+\delta) + \left(R^*(\lambda+\delta) + \delta\bar{R}^*(\lambda+\delta)\right)\right] \\
\left[\tau - \lambda\tau h^{(1)} + A_b'(1)\right] + \alpha\bar{R}^*(\lambda+\delta) \\
\left[\begin{array}{l}\lambda\tau\beta_p^{(1)} - S_p^*(\beta)A_b'(1) \\ +\beta\theta\left[1 - S_b^*(\tau)\right] + \beta\beta_b^{(1)}A_b'(1)\end{array}\right]
\end{array}\right\} \\
+\beta_b^{(1)}A_b'(1)\left\{\begin{array}{l}\left(R^*(\lambda+\delta) + \lambda\bar{R}^*(\lambda+\delta)\right)\left(\bar{\alpha}\delta - (\lambda+\delta)\right) \\ +\lambda + \alpha\delta\bar{R}^*(\lambda+\delta)S_p^*\beta\end{array}\right\} \\
\left[1 - S_b^*(\lambda)\right] \\
\left\{\begin{array}{l}\lambda\bar{R}^*(\lambda+\delta)\left(\bar{\alpha}\delta - (\lambda+\delta)\right) + \left(R^*(\lambda+\delta) + \lambda\bar{R}^*(\lambda+\delta)\right) \\ \left[(\lambda+\delta)N'(1) - \lambda\alpha\left[1 + \delta h^{(1)}\right] + \lambda\left[1 - \alpha\delta R^*(\lambda+\delta)\beta_b^{(1)}\right]\right]\end{array}\right\}
\end{array}\right] \\
+\bar{\alpha}\lambda h^{(1)}\left[\begin{array}{l}
\lambda R^*(\lambda+\delta)\left[\beta\tau\left(\bar{\alpha}\delta - (\lambda+\delta)\right)\right] + \left(R^*(\lambda+\delta) + \lambda\bar{R}^*(\lambda+\delta)\right) \\
\left\{\begin{array}{l}(\lambda+\delta)\left[\lambda\tau - \beta\left(\tau + A_b'(1)\right)\right] - N'(1)\beta\tau \\ \alpha\lambda\left[\beta\left(\left[1 - S_b^*(\tau)\right](1-\delta) + \beta\tau S_b^*(\tau)\right)\right]\alpha\delta\left(\beta^2\theta - \beta\lambda\right) \\ -\beta^2\beta_b^{(1)}A_b'(1) + \beta\tau\beta_b^{(1)}A_b'(1) + S_b^*(\tau)\left[\beta A_b'(1) - \lambda\tau\right]\end{array}\right\}
\end{array}\right] \\
+\alpha\delta\left[\begin{array}{l}
\beta\tau\left(R^*(\lambda+\delta) - (\lambda+\delta)\bar{R}^*(\lambda+\delta)\right) \\
+\left(\beta A_b'(1) - \alpha\lambda\delta\tau\beta_b^{(1)}\right)\left(R^*(\lambda+\delta) - \delta\bar{R}^*(\lambda+\delta)\right)
\end{array}\right]
\end{array}\right\}$$

$$
Nr_q'''(1) = 3\left\{
\begin{array}{l}
(\lambda+\delta)N''(1)\left[\left[\left\{\begin{array}{l}1-\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\\ \quad\left[\alpha\lambda S_b^*(\tau)+\bar{\alpha}\right]+\bar{R}^*(\lambda+\delta)\alpha\delta S_p^*(\beta)\end{array}\right\}\left[\beta A_b'(1)-\lambda\tau\right]\right]\right] \\
+(\lambda+\delta)N''(1)\beta\tau \\
\quad\left\{\begin{array}{l}
1-\alpha\lambda\bar{R}^*(\lambda+\delta)S_b^*(\tau)-\alpha\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\beta_b^{(1)}A_b'(1) \\
+\alpha\delta P^*(\lambda+\delta)\left[S_p^*(\beta)-\beta_p^{(1)}\right]+\bar{\alpha}\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right) \\
+\alpha\beta^2\tau\left[1-S_b^*(\tau)\right]\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right) \\
+\alpha\beta\left\{\begin{array}{l}\lambda\bar{R}^*(\lambda+\delta)\beta\left[1-S_b^*(\tau)\right]+\left(R^*(\lambda+\delta)+\lambda R^*(\lambda+\delta)\right)\\ \left(\left[1-S_b^*(\tau)\right]+\beta_b^{(1)}A_b'(1)\right)\end{array}\right\}
\end{array}\right\} \\
(\lambda+\delta)N'(1)term1+\lambda term2-\alpha\lambda\left[1-S_p^*(\beta)\right]term3 \\
\quad+\alpha\lambda\left[\lambda h^{(2)}term4+h^{(1)}term5\right] \\
-2\alpha\delta\lambda\beta_p^{(1)}\left[\begin{array}{l}
\left(1-R^*(\lambda+\delta)\right)\left[\tau+A_b'(1)\right]+R^*(\lambda+\delta)\tau N'(1) \\
+\bar{\alpha}\left(R^*(\lambda+\delta)+\delta\bar{R}^*(\lambda+\delta)\right)\tau\left(1-\lambda h^{(1)}\right)+A_b'(1) \\
+\alpha\left\{\begin{array}{l}\lambda\tau R^*(\lambda+\delta)-R^*(\lambda+\delta)\\ \left(\begin{array}{l}S_p^*(\beta)\beta_p^{(1)}-\lambda\tau\beta_p^{(1)}-\beta R^*(\lambda+\delta)\\ \left(\theta\left[1-S_b^*(\tau)\right]+\beta_b^{(1)}A_b'(1)\right)\end{array}\right)\end{array}\right\}
\end{array}\right] \\
+\alpha\lambda^2\beta_p^{(2)}\left[\begin{array}{l}\tau\left(1-R^*(\lambda+\delta)\right)-\alpha R^*(\lambda+\delta)\left[\tau S_p^*(\beta)+\left[1-S_b^*(\tau)\right]\right]\\ \quad+\bar{\alpha}\tau\left[\delta\bar{R}^*(\lambda+\delta)+R^*(\lambda+\delta)\right]\end{array}\right] \\
2\alpha\lambda\beta_b^{(1)}A_b'(1)\left[\begin{array}{l}\lambda+\lambda\bar{R}^*(\lambda+\delta)\left(\delta\bar{\alpha}-(\lambda+\delta)-\alpha\delta\beta_p^{(1)}\right)\\ +\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\\ \left((\lambda+\delta)N'(1)-\lambda\bar{\alpha}\left(1+\delta h^{(1)}\right)\right)\end{array}\right] \\
+\alpha\lambda\beta_b^{(2)}\left(A_b'(1)\right)^2\left\{\begin{array}{l}\lambda+\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\left[\delta\bar{\alpha}-(\lambda+\delta)\right]\\ +\alpha\delta R^*(\lambda+\delta)S_p^*(\beta)\end{array}\right\} \\
+\lambda\alpha\left[1-S_b^*(\tau)\right]\left[\begin{array}{l}\lambda\bar{R}^*(\lambda+\delta)\left\{2(\lambda+\delta)N'(1)+\alpha\delta\lambda S_p^*(\beta)-\delta\lambda\bar{\alpha}h^{(1)}\right\}\\ +\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\\ \left[(\lambda+\delta)N''(1)+2\lambda^2\bar{\alpha}h^{(1)}+\alpha\lambda\beta_p^{(1)}\right]\end{array}\right]
\end{array}\right\}
$$

$$
term1=\left\{\begin{array}{l}
\left[\begin{array}{l}1-\left(R^{*}(\lambda+\delta)+\lambda\bar{R}^{*}(\lambda+\delta)\right)\left[\alpha\lambda S_b^{*}(\tau)+\bar{\alpha}\right]\\ +\bar{R}^{*}(\lambda+\delta)\alpha\delta S_p^{*}(\beta)\end{array}\right]\left[\beta A_b''(1)-2\lambda A_b'(1)\right]\\
+2\left[\beta A_b'(1)-\lambda\tau\right]\\
\quad\left\{\begin{array}{l}1-\alpha\lambda\bar{R}^{*}(\lambda+\delta)S_b^{*}(\tau)-\alpha\left(R^{*}(\lambda+\delta)+\lambda\bar{R}^{*}(\lambda+\delta)\right)\beta_b^{(1)}A_b'(1)\\ +\alpha\delta R^{*}(\lambda+\delta)\left[S_p^{*}(\beta)-\beta_p^{(1)}\right]+\bar{\alpha}\left(R^{*}(\lambda+\delta)+\lambda\bar{R}^{*}(\lambda+\delta)\right)\\ +\bar{\alpha}h^{(1)}\left[R^{*}(\lambda+\delta)+2\lambda\bar{R}^{*}(\lambda+\delta)\right]\end{array}\right\}\\
-\beta\tau\left(\begin{array}{l}\bar{\alpha}^{*}(\lambda+\delta)\left[\left[\lambda\beta_b^{(1)}+\lambda\beta_b^{(1)}A_b'(1)\right]+\left[\delta\beta_p^{(1)}A_p'(1)+\delta\beta_b^{(2)}\left(A_b'(1)\right)^2\right]\right]\\ +\lambda\bar{\alpha}\bar{R}^{*}(\lambda+\delta)\left(1+h^{(1)}\right)+\left(R^{*}(\lambda+\delta)+\lambda\bar{R}^{*}(\lambda+\delta)\right)\\ \left(\left[\alpha\beta_b^{(1)}A_b''(1)+\alpha\beta_b^{(2)}\left(A_b'(1)\right)^2\right]-\bar{\alpha}(1-\lambda)h^{(1)}+\lambda h^{(2)}\right)\end{array}\right)\\
+2\alpha\beta\theta\left[\begin{array}{l}\beta\lambda\bar{R}^{*}(\lambda+\delta)\left[1-S_b^{*}(\tau)\right]-\left(R^{*}(\lambda+\delta)+\lambda\bar{R}^{*}(\lambda+\delta)\right)\\ \left[\lambda\left(1-S_b^{*}(\tau)\right)-2\beta\beta_b^{(1)}A_b'(1)\right]\end{array}\right]\\
\alpha\beta\left\{\begin{array}{l}\left(R^{*}(\lambda+\delta)+\lambda\bar{R}^{*}(\lambda+\delta)\right)\left(-\beta_b^{(1)}\left[A_b''(1)+2\lambda A_b'(1)\right]-\beta_b^{(2)}\left(A_b'(1)\right)^2\right)\\ +\lambda\bar{R}^{*}(\lambda+\delta)\left[2\beta\beta_b^{(1)}A_b'(1)+2\left[1-S_b^{*}(\tau)\right]\beta_b^{(1)}\right]+\delta\bar{R}^{*}(\lambda+\delta)\\ \left[-2\lambda\beta\beta_p^{(1)}+\left[1-S_p^{*}(\beta)\right]\left(2A_b'(1)+A_b''(1)\right)-\beta\lambda\left(\lambda\beta_b^{(2)}+\beta_p^{(1)}\right)\right]\end{array}\right\}
\end{array}\right\}
$$

$$
term2=\left[\begin{array}{l}
\left\{2\beta A_b'(1)-2\lambda\tau-2\lambda A_b'(1)+2\beta A_b''(1)\right\}\\
-\bar{\alpha}\left\{\begin{array}{l}\left[R^{*}(\lambda+\delta)-\delta\bar{R}^{*}(\lambda+\delta)\right]\left(\begin{array}{l}-2\lambda\beta h^{(1)}\left(\tau+A_b'(1)\right)-2\lambda\tau\\ +3\beta A_b'(1)+\lambda^2\tau h^{(1)}(2+\beta)\end{array}\right)\\ \lambda\bar{R}^{*}(\lambda+\delta)\left\{\beta\tau\left(1-\lambda h^{(1)}\right)-\lambda\tau+A_b'(1)\right\}\end{array}\right\}\\
-\alpha R^{*}(\lambda+\delta)\left[\begin{array}{l}\beta\tau\left(A_b'(1)\right)^2\beta_b^{(2)}-2\lambda\tau A_b'(1)\beta_b^{(1)}-2\lambda\left(A_b'(1)\right)^2\beta_b^{(1)}\\ -A_b'(1)S_b^{*}(\tau)(2\lambda-\beta)+\beta\theta\\ \left\{-2\beta A_b'(1)\beta_b^{(1)}(1+2\lambda)-2\lambda\left[1-S_b^{*}(\tau)\right]-\beta^2\left(A_b'(1)\right)^2\beta_b^{(1)}\right\}\end{array}\right]
\end{array}\right]
$$

$$
term3=\left[\begin{array}{l}
R^{*}(\lambda+\delta)\left\{-\left(A_b'(1)+A_b''(1)\right)2\left(A_b'(1)+\tau\right)N'(1)+\tau N'(1)\right\}+2A_b'(1)+A_b''(1)\\
-\bar{\alpha}\left\{2\lambda\bar{R}^{*}(\lambda+\delta)\left[\tau\left(1-h^{(1)}\right)+A_b'(1)\right]+\left[R^{*}(\lambda+\delta)+\delta\bar{R}^{*}(\lambda+\delta)\right]\right\}\\
\left(\left(-2\lambda\tau+\tau\lambda^2+A_b'(1)-2\lambda A_b'(1)\right)h^{(1)}+A_b''(1)\right)\\
-\alpha R^{*}(\lambda+\delta)\left[\begin{array}{l}\tau\lambda^2\beta_p^{(2)}-2\lambda A_b'(1)\beta_p^{(1)}+A_b''(1)\left(S_p^{*}(\beta)+\beta\beta_b^{(1)}\right)\\ -2\beta A_b'(1)\beta_b^{(1)}+\beta\beta_b^{(2)}\left(A_b'(1)\right)^2\end{array}\right]
\end{array}\right]
$$

$$term4=\begin{bmatrix} \lambda\bar{R}^*(\lambda+\delta)\left\{-\beta\tau(\lambda+\delta)+\alpha\delta\beta\tau\right\}+\alpha\delta\left[R^*(\lambda+\delta)+\delta\bar{R}^*(\lambda+\delta)\right] \\ \begin{bmatrix} -\beta\tau(\lambda+\delta)\left[N'(1)-1\right]-(\lambda+\delta)\left[\beta A_b'(1)-\lambda\tau\right]+\alpha\lambda\beta\tau \\ +\alpha\delta\left[\beta^2\theta\left(1-S_b^*(\tau)\right)-\beta A_b'(1)\beta_b^{(1)}(\beta-\tau)\right] \\ -\lambda\beta+S_b^*(\tau)\left[\beta A_b'(1)-\lambda(\tau+\beta)\right] \end{bmatrix} \\ +\alpha\delta\begin{Bmatrix} \left[R^*(\lambda+\delta)-\delta\bar{R}^*(\lambda+\delta)\right]\left(\beta\delta A_b'(1)+\alpha\delta\beta\tau S_p^*(\beta)\right) \\ +\beta\tau\left[R^*(\lambda+\delta)-(\lambda+\delta)\bar{R}^*(\lambda+\delta)\right] \end{Bmatrix} \end{bmatrix}$$

$$term5=\begin{bmatrix} 2\lambda\bar{R}^*(\lambda+\delta)\begin{Bmatrix} \beta\tau(\lambda+\delta)\left[N'(1)-1\right]-(\lambda+\delta)\left[\beta A_b'(1)-\lambda\tau\right] \\ +\alpha\lambda\beta\left[\beta-S_b^*(\tau)(\beta-\tau)\right]+\alpha\delta\beta^2\theta\left(1-S_b^*(\tau)\right) \\ -\beta A_b'(1)\beta_b^{(1)}(\beta+\tau)-\lambda\beta\left(1-S_b^*(\tau)\right)\left[\beta A_b'(1)-\lambda\tau\right] \end{Bmatrix} \\ +\left[R^*(\lambda+\delta)+\delta\bar{R}^*(\lambda+\delta)\right] \\ \begin{bmatrix} \beta\tau N'(1)-\left[\beta A_b'(1)-\lambda\tau\right]+(\lambda+\delta)\begin{Bmatrix} \beta\tau N''(1)+2N'(1)\left[\beta A_b'(1)-\lambda\tau\right] \\ +2\lambda A_b'(1)-\beta A_b''(1) \end{Bmatrix} \\ +2\alpha\lambda\begin{Bmatrix} \beta^2\theta-\lambda\beta\left(1-S_b^*(\tau)\right)-\beta^2 A_b'(1)\beta_b^{(1)} \\ +\beta\tau A_b'(1)\beta_b^{(1)}S_b^*(\tau)\left[\beta A_b'(1)-\lambda\tau\right] \end{Bmatrix} \\ +\alpha\lambda\begin{bmatrix} -\beta^2\theta A_b'(1)\beta_b^{(1)}-\beta\theta\left[\beta A_b'(1)S_b^*(\tau)+\lambda\left[1-S_b^*(\tau)\right]\right] \\ -\lambda\beta\theta\left(1-S_b^*(\tau)\right)+2\lambda\beta A_b'(1)\beta_b^{(1)}-(\beta+\tau)\beta\left(A_b'(1)\right)^2\beta_b^{(2)} \\ +\beta\tau A_b''(1)\beta_b^{(1)}+\beta\left(A_b'(1)\right)^2\beta_b^{(1)}-\lambda\tau A_b'(1)\beta_b^{(1)}+\beta A_b'(1)-\lambda\tau \\ +S_b^*(\tau)\left[-2\lambda A_b'(1)+A_b''(1)-\lambda\tau\right] \end{bmatrix} \end{bmatrix} \\ +2\alpha\delta\begin{Bmatrix} \left[R^*(\lambda+\delta)-\delta\bar{R}^*(\lambda+\delta)\right]\left(\beta-\lambda\tau S_p^*(\beta)\right) \\ +2\delta\left[R^*(\lambda+\delta)-(\lambda+\delta)\bar{R}^*(\lambda+\delta)\right] \\ \left(-S_b^*(\tau)3\lambda A_b'(1)+\beta A_b''(1)+2\tau\lambda^2\beta_p^{(1)}\right) \end{Bmatrix} \end{bmatrix}$$

$$Dr_q''(1)=-2\lambda\begin{bmatrix} (\beta-\tau)+\beta A_b'(1)-\alpha\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right) \\ \left\{\beta_b^1\beta\tau(1)-S_b^*(\tau)\left(\tau\lambda-\beta A_b'(1)\right)+\beta\theta^2\left[\beta_b^1 A_b'(1)-1-S_b^*(\tau)\right]\right\} \\ +\alpha R^*(\lambda+\delta)\left\{\beta\lambda\tau S_b^*(\tau)-\beta^2\lambda\left(1-S_b^*(\tau)\right)\right\}+\alpha\delta\bar{R}^*(\lambda+\delta) \\ \begin{Bmatrix} S_p^*(\beta)\beta\tau+\left[1-S_p^*(\beta)\right]\left(\beta\tau+A_b'(1)\right) \\ -\lambda\tau(1-\beta)\beta_p^{(1)}+\left(\beta A_b'(1)-\lambda\tau\right)S_p^*(\beta) \end{Bmatrix} \\ \bar{\alpha}\left(R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right)\left\{\beta\left(1-A_b'(1)\right)-\lambda\tau\left(1-\beta h^{(1)}\right)\right\} \\ +\bar{\alpha}\lambda\beta\tau R^*(\lambda+\delta) \end{bmatrix}$$

$$Dr_1'''(1) = -3\lambda \left[\begin{array}{l} (\beta-\lambda)A_b'(1) - \lambda\tau + \beta A_b''(1) - 2\lambda\alpha\overline{R}^*(\lambda+\delta)\left\{ \begin{array}{l} \beta\tau\beta_b^{(1)}A_b'(1) - \lambda\tau S_b^*(\tau) \\ +\beta S_b^*(\tau)A_b'(1) \end{array} \right\} \\ +2\left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right) \left[\begin{array}{l} \beta_b^2\left(A_b'(1)\right)^2 \beta\tau + A_b'(1)\beta_b^1\left(\beta\tau - 2\tau\lambda\right) \\ +(\beta-\lambda)\beta_b^1\left(A_b'(1)\right)^2 - 2\lambda S_b^*(\tau)A_b'(1) \\ +\beta A_b''(1)S_b^*(\tau) - 2\beta\lambda\theta\left(1 - S_b^*(\tau)\right) \\ -2\theta\beta^2\beta_b^{(1)}A_b'(1) \end{array} \right] \\ +\alpha\delta\overline{R}^*(\lambda+\delta) \left(\begin{array}{l} -\lambda\tau\left(\beta\beta_b^1 - S_p^*(\beta)\right) + 2S_p^*(\beta)\left(\begin{array}{l} \lambda^2\tau + \beta A_b'(1) \\ -\lambda\tau + \beta A_b'(1) \end{array} \right) \\ +\beta\left(1 - S_p^*(\beta)\right)\left(A_b'(1) + A_b''(1)\right) \\ +\beta\beta_b^{(1)}A_b'(1)(\lambda + 2\tau) + \lambda^2\tau\beta A_b''(1) \end{array} \right) \\ +\overline{\alpha}\lambda\overline{R}^*(\lambda+\delta)\left\{\beta\tau\left(1 - \lambda h^{(1)}\right) + \beta A_b'(1) - \lambda\tau\right\} \\ +\overline{\alpha}\overline{R}^*(\lambda+\delta)\left(\begin{array}{l} 2A_b'(1)(\beta-\lambda) - 2\lambda h^{(1)}\beta\tau - 2\lambda\tau + 2\lambda^2 h^{(1)}\tau \\ -2\lambda\beta A_b'(1)h^{(1)} - 2\lambda A_b'(1) + \beta A_b''(1) \end{array} \right) \end{array} \right]$$

where $A'(1) = -\lambda[(1-\delta\beta_p^{(1)})]$; $A''(1) = \lambda^2\delta\beta_p^{(2)}$.

1 the excepted number of customers in the system (L_s) is obtained by differentiating (4.50) with respect to z and evaluating at

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z) = P_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

where

$$Nr_s'''(1) = Nr_q'''(1) - 6\left[\begin{array}{l} \alpha\lambda^2\delta\beta_p^{(1)}\left\{ \begin{array}{l} \tau\left(1 - \overline{R}^*(\lambda+\delta)\right) + \overline{\alpha}\tau\left(R^*(\lambda+\delta) + \delta\overline{R}^*(\lambda+\delta)\right) \\ -\alpha\overline{R}^*(\lambda+\delta)\left[\tau S_p^*(\beta) - \beta\left[1 - S_p^*(\beta)\right]\right] \end{array} \right\} \\ +\alpha\lambda\left[1 - S_p^*(\beta)\left\{ \begin{array}{l} R^*(\lambda+\delta)\left[N'(1) - 1 + A_b'(1)\right] \\ +\overline{\alpha}\left[\begin{array}{l} \overline{\alpha}\lambda\theta\tau + \left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right) \\ \left[A_b'(1) - \lambda\tau h^{(1)}\right] \end{array} \right] \end{array} \right\} \right] \\ +\alpha\lambda\left\{\beta_b^{(1)}A_b'(1)\right\}\left[\begin{array}{l} \left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right)\left((\lambda+\delta) + \overline{\alpha}\delta\right) \\ +\lambda + \alpha\delta R^*(\lambda+\delta)S_b^*(\tau) \end{array} \right] \\ +\alpha\lambda\left[1 - S_b^*(\tau)\right]\left\{ \begin{array}{l} \lambda\overline{R}^*(\lambda+\delta)\left\{\left(-(\lambda+\delta) + \overline{\alpha}\delta\right)\right\} \\ +\left(R^*(\lambda+\delta) + \lambda\overline{R}^*(\lambda+\delta)\right) \\ \left[(\lambda+\delta)N'(1) - \lambda\alpha\delta\overline{R}^*(\lambda+\delta)\beta_p^{(1)}\right] \end{array} \right\} \end{array} \right]$$

2 the average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) can be found by using the Little's formula

$$W_s = \frac{L_s}{\lambda} \quad \text{and} \quad W_q = \frac{L_q}{\lambda}.$$

6 Stochastic decomposition

Stochastic decomposition has been discussed among M/G/1 type queueing models with server vacations by Fuhrman and Cooper (1985). In this result we analyses that the number of customers in the system in a steady state at a random point in time is distributed as the sum of two independent random variables, one of which is the number of customers in the corresponding standard queueing system (in steady state) at random point in time, the other random variable may have different probabilistic interpretations in specific cases depending on how the vacations are scheduled. Let $K(z)$ be the stationary size distribution of M/G/1 retrial queueing system with negative customers, starting failure and at most J vacation convolution of two independent random variables $\chi(z)$ and $\varphi(z)$.

The mathematical version of the stochastic decomposition law is $K(z) = \chi(z)$. $\varphi(z)$ and is expressed in the form

1 the system size distribution of M/G/1 queueing system with two type of customers, Negative customers, starting failure and at most J vacations [represented in the first term of $K(z)$]

2 the conditional distribution of the number of customers in the vacation system at a random point in time given the server is idle [represented in the second term of $K(z)$].

The number of arrivals in the vacation system at a random point in time given that the server is on vacation or idle. In fact, the second term can also obtain through the vacation definition of our system, i.e.

$$\varphi(z) = \left(P_0 + P(z) + \Omega(z)\right) / \left(P_0 + P(1) + \Omega(1)\right)$$

$$\varphi(z) = \frac{\lambda\omega\left\{Dr(z)\left[1 + \dfrac{(\lambda+\delta)P_0\left[1 - V^*\left(b(z)\right)\right]}{\left[V^*(\lambda)\right]^{J-j+1} b(z)}\right] + Nr(z)\right\}}{Dr(z)\left\{\begin{array}{l} \lambda\omega + \lambda R^*(\lambda+\delta)\left[\begin{array}{l}(\lambda+\delta)[\bar{\alpha}-1] \\ +\alpha\left(\delta S_p^*(\beta) + \lambda S_b^*(\tau)\right) \\ +\dfrac{\alpha\lambda\beta}{\tau}\left[1 - S_b^*(\tau)\right] \\ +\alpha\delta\left[1 - S_p^*(\beta)\right]\end{array}\right] \\ +(\lambda+\delta)\mathbf{N}^{(1)}(1)[1-\rho] \end{array}\right\}}, \quad j = 1, 2, \ldots J$$

where

$$Nr(z)=zP_0\bar{R}^*(\lambda+\delta)\left\{\begin{array}{l}\left[\begin{array}{l}\left((\lambda+\delta)\left[N(z)-1\right]\right)+\bar{\alpha}(\lambda+\delta)H^*\left(b(z)\right)\\+\alpha\lambda S_b^*\left(A_b(z)\right)+\alpha\delta S_p^*\left(A_p(z)\right)\end{array}\right]A_b(z)A_p(z)\\+\alpha\lambda\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)A_p(z)\right)\\+\alpha\delta\beta\left(1-S_p^*\left(A_p(z)\right)A_b(z)\right)\end{array}\right\}$$

$$Dr(z)=zA_b(z)A_p(z)-\left\{\left[\begin{array}{l}\bar{\alpha}z\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\left\{\begin{array}{l}\bar{\alpha}zH^*\left(b(z)\right)\\+\alpha S_b^*\left(A_b(z)\right)\end{array}\right\}\\+z\alpha\delta\bar{R}^*(\lambda+\delta)\left(\begin{array}{l}S_p^*\left(A_p(z)\right)\\+\beta\left(1-S_p^*\left(A_p(z)\right)\right)/A_b(z)\end{array}\right)\end{array}\right]\right\}$$

The first term can be obtained through the without vacation definition of our system

$$\chi(z)=\frac{Nr_1(z)}{Dr(z)},$$

where

$$Nr_1(z)=\left\{\begin{array}{l}Dr(z)+z\bar{R}^*(\lambda+\delta)\left\{\begin{array}{l}\left[\begin{array}{l}+\bar{\alpha}(\lambda+\delta)H^*\left(b(z)\right)\\+\alpha\lambda S_b^*\left(A_b(z)\right)+\alpha\delta S_p^*\left(A_p(z)\right)\end{array}\right]A_b(z)A_p(z)\\+\alpha\lambda\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)A_p(z)\right)\\+\alpha\delta\beta\left(1-S_p^*\left(A_p(z)\right)A_b(z)\right)\end{array}\right\}\\+z\alpha\delta\left[1-S_p^*\left(A_p(z)\right)\right]\left\{\begin{array}{l}\left(\begin{array}{l}\bar{\alpha}z\left[\begin{array}{l}(\lambda+\delta-\lambda z)\bar{R}^*(\lambda+\delta)\\-R^*(\lambda+\delta)H^*\end{array}\right]\left(b(z)\right)\\-\alpha R^*(\lambda+\delta)S_b^*\left(A_b(z)\right)\end{array}\right)A_b(z)\\-\alpha\beta R^*(\lambda+\delta)(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\end{array}\right\}\\+z\alpha\left[1-S_b^*\left(A_b(z)\right)\right]\left\{\begin{array}{l}\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)\\\left(\bar{\alpha}(\lambda+\delta-\lambda z)H^*\left(b(z)\right)\right)\\+\lambda z+\alpha\delta R^*(\lambda+\delta)S_p^*\left(A_p(z)\right)\end{array}\right\}A_b(z)\\+z\bar{\alpha}\left[1-H^*\left(b(z)\right)\right]\left\{\begin{array}{l}\alpha\delta zA_b(z)\left(R^*(\lambda+\delta)+\left(\lambda-(\lambda+\delta)\right)\bar{R}^*(\lambda+\delta)\right)\\\left\{\left[S_p^*\left(A_p(z)\right)A_p(z)\right]+\beta\left(1-S_p^*\left(A_p(z)\right)\right)\right\}\\+\alpha\left(\lambda z-(\lambda+\delta)\right)\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)A_p(z)\\\left\{\begin{array}{l}\left(\beta(1-\theta-\theta z)\left(1-S_b^*\left(A_b(z)\right)\right)\right)\\+S_b^*\left(A_b(z)\right)A_b(z)\end{array}\right\}\end{array}\right\}\end{array}\right\}$$

From above stochastic decomposition law, we observe that $K(z) = \chi(z).\varphi(z)$ which confirm that the decomposition result of Fuhrman and Cooper (1985) also valid for this special vacation system.

7 Special cases

In this section, we analyse briefly some special cases of our model, which are consistent with the existing literature.

Case (i): no vacation, no negative customer, no orbital search and no starting failure

In this case, we put $Pr[V = 0] = 1$; $\alpha = \theta = 1$; $\beta = 0$; our model can be reduced to a preemptive priority retrial queueing system and $K_s(z)$ can be obtained as follows,

$$K_s(z) = \frac{P_0(z-1)R^*(\lambda+\delta)\left\{A_b(z)S_b^*\left(A_b(z)\right)+\delta\left[1-S_b^*\left(A_b(z)\right)\right]S_p^*\left(A_p(z)\right)\right\}A_b(z)}{\left\{\begin{bmatrix} z-\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)S_b^*\left(A_b(z)\right) \\ +z\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right) \end{bmatrix} A_b(z)A_b(z)\right\}}$$

This coincides with the result of Gao (2015).

Case (ii): no priority arrival, no negative customer and no starting failure

In this case, we put $\delta = \beta = 0$; $\alpha = 1$, our model can be reduced to a single server retrial queueing system with j vacation and $K_s(z)$ can be obtained as follows,

$$Ks(z) = \frac{P_0\left\{\begin{aligned} &(1+\lambda N(z))\left[z-\left(R^*(\lambda)+\lambda z\bar{R}^*(\lambda)\right)S_b^*\left(b(z)\right)\right] \\ &+z\bar{R}^*(\lambda)\left\{\left[\left(\lambda\left[N(z)-1\right]\right)+\lambda S_b^*\left(b(z)\right)\right]\right\}+z\left[1-S_b^*\left(b(z)\right)\right] \\ &\left\{\left(R^*(\lambda)+\lambda z\bar{R}^*(\lambda)\right)\left(\lambda\left[N(z)-1\right]\right)+\lambda z\right\} \end{aligned}\right\}}{\left\{\left[z-\left(R^*(\lambda)+\lambda z\bar{R}^*(\lambda)\right)S_b^*\left(b(z)\right)\right]b(z)\right\}}$$

Case (iii): no priority arrival, no starting failure, and no vacation

In this case, we put $Pr[V = 0] = 1$; $\delta = 0$; $\alpha = 1$, our model can be reduced to a single server retrial queueing system with starting failure, and negative customer and $K_s(z)$ can be obtained as follows,

$$Nr_s(z) = \frac{P_0(z-1)\left\{R^*(\lambda)\left\{\beta+\lambda\beta\left(1-S_b^*\left(A_b(z)\right)A_p(z)\right)\right\}+z\left[1-S_b^*\left(A_b(z)\right)\right]A_p(z)\right\}}{z-\left(R^*(\lambda)+\lambda z\bar{R}^*(\lambda)\left\{S_b^*\left(A_p(z)\right)\lambda(1-z)+\beta\right\}\right)}$$

Case (iv): no priority arrival, no vacation, no starting failure, and no negative customer

In this case, we put $Pr[V = 0] = 1$; $\delta = \beta = \theta = 0$; $\alpha = 1$, our model can be reduced to an M/G/1 retrial queue and $K_s(z)$ can be obtained as follows,

$$Ks(z) = \frac{P_0(1-z)\left[R^*(\lambda)-\lambda\right]S_b^*(\lambda-\lambda z)}{z-\left(R^*(\lambda)+\lambda z\bar{R}^*(\lambda)\right)S_b^*(\lambda-\lambda z)}$$

Case (v): no priority arrival, no retrial; no vacation, no negative customer and no starting failure

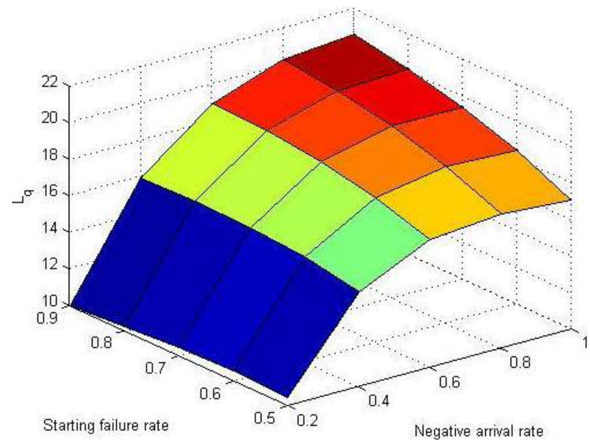
In this case, we put $R^*(\lambda) \to 1$; $Pr[V = 0] = 1$; $\delta = \beta = \theta = 0$; $\alpha = 1$, our model can be reduced to an M/G/1 queue and $K_s(z)$ can be obtained as follows

$$K_s(z) = \frac{[1-\lambda]S_b^*(\lambda-\lambda z)(z-1)}{\left\{z-S_b^*(\lambda-\lambda z)\right\}}$$

8 Numerical examples

We present some numerical examples in this section, to study the effect of various parameters in the system performance measures of our system where all retrial times, arrival times, service times, vacation times and repair times are exponential. We assume arbitrary values to the parameters such that the steady state condition is satisfied. MATLAB software has been used to illustrate the results numerically. Probability density functions for the exponential distribution is $f(x) = \upsilon e^{-\upsilon x}$, $x > 0$.

Figure 1 L_q versus starting failure and β (see online version for colours)



For the effect of the parameters λ, a, δ, α and γ on the system performancemeasures, three-dimensionalgraphs are illustrated in Figure 1 to Figure 4. Figure. 1 shows that the surface displays an upward trend as expected for increasing the value of starting failure rate and Negative arrival rate (β) against the mean orbit size (L_q). Figure 2 shows that the probability that server is idle (P_0) decreases with increasing values of the arrival rate (λ) and successful service rate (α). Figure 3 show that the mean orbit size (L_q) is decreasing for the increasing values of the retrial rate and vacation rate (γ). Figure 4 show that the probability that server is idle (P_0) is decreased for the increasing values of the vacation rate and successful service rate (α).

Figure 2 α and λ versus P_0 (see online version for colours)

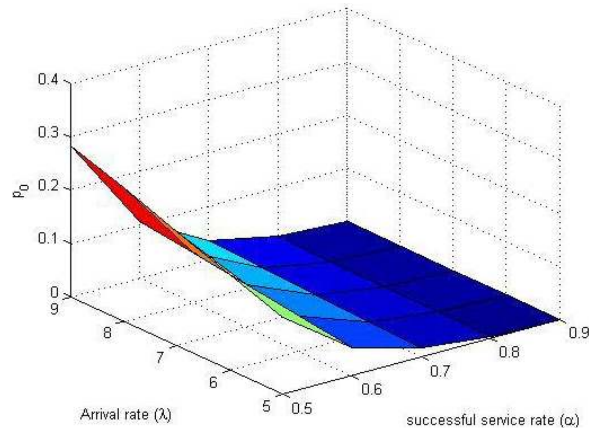
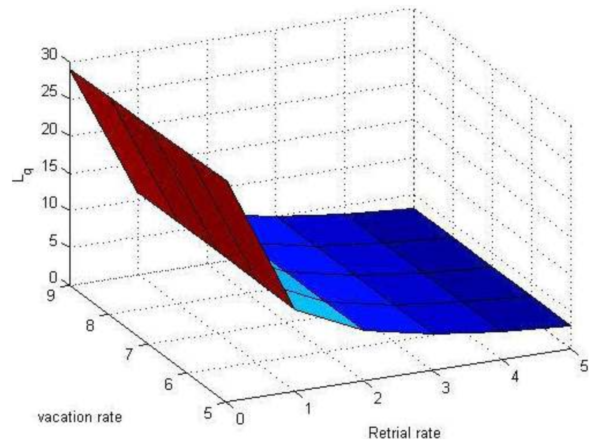
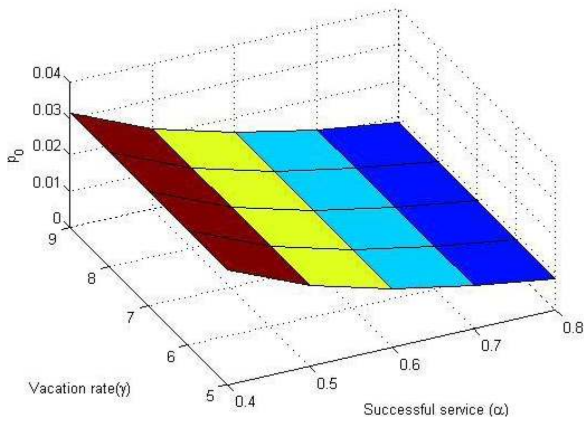


Figure 3 L_q versus retrial rate and γ (see online version for colours)



From the above numerical examples, we can find the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

Figure 4 P_0 versus α and γ (see online version for colours)



9 Conclusions

In this paper, we have examined a preemptive priority retrial queueing system with the negative customer, at most J vacations and repair due to starting failure. Using the method of supplementary variable technique, the PGFs for the numbers of customers in the system when it is free, busy with priority customer, busy with preemptive priority customer, busy with an ordinary customer, on vacations and under repair due to starting failure is found. Some important system performance measures and stochastic decomposition law are also discussed. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, the analytical results are validated with the help of numerical illustrations.

The present investigation includes features simultaneously such as,

- preemptive priority retrial queue
- negative customer
- at most J vacations
- repair due to starting failure.

Our suggested model and its results have a specific and potential application in the field of telephone consultation service and in the area of computer processing system. This work can be further extended in many directions by incorporating the concepts of

- batch arrival
- optional re-service
- orbital search
- working vacation
- immediate feedback.

Hopefully, this investigation will be of great help for the system managers to take decisions about the system size and other parameters in a perfect manner.

Acknowledgements

The authors are pleased to thank the anonymous referees, for their valuable comments and suggestions, which helped in improving the paper.

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Appendix

Sufficient condition of ergodicity

The embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if $\rho < R^* (\lambda + \delta)$ for our system to be stable, where

$$\rho = \begin{bmatrix} \bar{\alpha}\left[R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right]+\alpha\delta\bar{R}^*(\lambda+\delta)\left[S_p^*(\beta)+\dfrac{\left(1-S_p^*(\beta)\right)}{\tau}\right] \\ +\alpha\left[R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\right]\left[S_b^*(\tau)+\beta\dfrac{\left(1-S_b^*(\tau)\right)}{\tau}\right] \end{bmatrix}$$

Proof:

To prove the sufficient condition of ergodicity, it is very convenient to use Foster's criterion (see Pakes, 1969), which states that the chain $\{Z_n;\, n \in N\}$ is an irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j)$, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E[f(z_{n+1}) - f(z_n)/z_n = j]$ is finite for all $j \in N$ and $\psi_j \le -\varepsilon$ for all $j \in N$, except perhaps for a finite number j's. In our case, we consider the function $f(j) = j$. then we have

$$\psi_j = \begin{cases} \rho - 1, & \text{if } j = 0, \\ \rho - R^*(\lambda+\delta), & \text{if } j = 1, 2\ldots \end{cases}$$

Clearly, the inequality $\rho < R^*(\lambda + \delta)$ is sufficient condition for ergodicity.

To prove the necessary condition, As noted in Sennott et al. (1983), if the Markov chain $\{Z_n;\, n \ge 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \ge 0$ and there exits $j_0 \in N$ such that $\psi_j \ge 0$ for $j \ge j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one-step transition matrix of $\{Z_n;\, n \in N\}$. Then $\rho \ge R^*(\lambda + \delta)$ implies the non-ergodicity of the Markov chain.