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Coincidence and fixed point results in ordered G-cone metric spaces

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ABSTRACT

We prove some coincidence and fixed point theorems for mappings satisfying contractive conditions under φ-maps in partially ordered G-cone metric spaces.

1. Introduction

In 2006, Z. Mustafa in collaboration with B. Sims introduced a new notion of generalized metric space called G-metric space [1]. In this generalization to every triplet of elements in the space, a non-negative real number is assigned. Analysis of the structure of these spaces was done in some detail in [1]. Fixed point theory in such spaces was initiated in [2] and studied further in [3,4]. In particular, the Banach contraction mapping principle was established in these works. Subsequently, several authors proved fixed point results in these spaces (see, e.g., [5–10]).

The notion of a cone metric space (under various names) is very old. Metric spaces, in which the metric takes values in an ordered space, were first introduced in 1934 by Kurepa [11]. Huang–Zhang's definition [12] of a cone metric space can be seen, e.g., in Chung's papers [13,14]. Chung named such spaces "cone-valued metric spaces". In these papers Chung also introduced the notions of convergence and completeness in cone metric spaces (over a solid Banach space). See also [15], the well-known monograph of Colatz [16], and the well-known survey paper of Zabrejko [17].

Several authors obtained further fixed point results in such spaces (see, e.g., [18–21] and a review of these results in [22]). Recently, Beg et al. [23] introduced G-cone metric spaces which are generalization of G-metric spaces and cone metric spaces. They proved some fixed point theorems under certain contractive conditions. Shatanawi [10] worked on fixed points for φ-maps in G-metric spaces which are extended to G-cone metric spaces for a pair of maps by Ozturk and Basarir [24].

Fixed point theory has also developed rapidly in metric spaces endowed with a partial ordering (see details in [25–32] and references therein). Fixed point problems have also been considered in partially ordered cone metric spaces [33] and partially ordered G-metric spaces [34].

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In this paper, we study common fixed point theorems for mappings satisfying contractive conditions related to a nondecreasing φ-map [19,20] in partially ordered G-cone metric spaces. Our results are ordered G-cone version extension of work presented by Shatanawi [10] and Ozturk and Basarir [24]. It is worth mentioning that we do not use normality of the cone to obtain the results. On the way, we correct some formulations of results from [23].

2. Preliminaries

To ease understanding of the material incorporated in this paper we recall some basic definitions and results. For details on the following notions we refer to [10,12,22,24] and references therein.

The following concept (usually cited as taken from [12]) can also be seen in many earlier papers (see, e.g., [35–39] and historical notes in the beginning of Section 3 of Proinov [40]).

Let B be a real Banach space and P be a subset of B. By θ we denote the zero element of B and by $\operatorname{int} P$ the interior of P. The subset P is called an order cone if:

(i) P is closed, nonempty and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given an order cone $P \subset B$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ if $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$.

There exist two kinds of cones, normal and nonnormal ones. The order cone P is normal if

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0 \tag{2.1}$$

or equivalently, if there is a number $M > 0$ such that for all $x, y \in B$,

$$\theta \leq x \leq y \Rightarrow \|x\| \leq M\|y\|. \tag{2.2}$$

The least positive number M satisfying (2.2) is called the normal constant of P. From (2.1) one can conclude that P is nonnormal if and only if there exist sequences $x_n, y_n \in P$ such that

$$\theta \leq x_n \leq x_n + y_n, \quad \lim_{n\to\infty} (x_n + y_n) = \theta, \quad \text{but} \lim_{n\to\infty} x_n \neq \theta.$$

Definition 2.1 (*[23]*). Let X be a nonempty set, B be a real Banach space and $P \subset B$ be an order cone. Suppose a mapping $G : X \times X \times X \to B$ satisfies

(G1) $G(x, y, z) = \theta$ if $x = y = z$;
(G2) $\theta < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized cone metric on X and X is called a generalized cone metric space or, shortly, a G-cone metric space.

It is obvious that the concept of a G-cone metric space is more general than that of a G-metric space or a cone metric space. If $B = \mathbb{R}$ and $P = [0, +\infty)$ then a G-cone metric space becomes a G-metric space.

Example 2.2. Let $X = [0, +\infty)$, $d(x, y) = |x - y|$, $g(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, $B = \mathbb{R}^2$, $P = \{(x, y) \mid x \geq 0, y \geq 0\}$ and let $G : X \times X \times X \to P$ be defined by $G(x, y, z) = \{g(x, y, z), \alpha g(x, y, z)\}$ where $\alpha > 0$ is fixed. Then (X, G) is a G-cone metric space over the normal cone P.

Example 2.3. Let $B = C^1_{\mathbb{R}}[0, 1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty$ and $P = \{u \in B : u(t) \geq 0 \text{ for } t \in [0, 1]\}$. It is well known (see, e.g., [41]) that the cone P is not normal. Let $X = [0, +\infty)$, $d(x, y) = |x - y|$, $g(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, for $x, y, x \in X$, and let $G : X \times X \times X \to P$ be defined by $G(x, y, z) = g(x, y, z)u$ where $u \in P$ is fixed. Then (X, G) is a G-cone metric space over a nonnormal cone.

The following remark will be useful in the sequel.

Remark 2.4. For elements u, v, w of an order cone P, the following hold:

(1) if $u \leq v$ and $v \ll w$, then $u \ll w$;
(2) if $u \ll v$ and $v \leq w$, then $u \ll w$;
(3) if $\theta \leq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$.

Throughout the paper we assume that B is a real Banach space and P is a cone in B with $\operatorname{int} P \neq \emptyset$ (such cones are called solid). In this way, we uniquely determine the limit of a sequence. Normality of the cone is not assumed unless otherwise stated.

Definition 2.5 (*[23]*)**.** Let (X, G) be a G-cone metric space.

(1) A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $c \in B$ with $\theta \ll c$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $G(x_n, x_m, x) \ll c$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $c \in B$ with $\theta \ll c$ there is a positive integer N such that $G(x_n, x_m, x_\ell) \ll c$, for all $n, m, \ell \geq N$.
(3) (X, G) is said to be complete if every Cauchy sequence in X is convergent in X.

The following assertion was stated (without proof) in [23], claiming that it holds for arbitrary cones. In fact it is valid only if the underlying cone P is normal.

Lemma 2.6. *Let X be a G-cone metric space over a normal cone, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then the following are equivalent:*

(1) *$\{x_n\}$ is convergent to x;*
(2) *$G(x_n, x_n, x) \to \theta$ as $n \to \infty$;*
(3) *$G(x_n, x, x) \to \theta$ as $n \to \infty$;*
(4) *$G(x_m, x_n, x) \to \theta$ as $m, n \to \infty$.*

Remark 2.7. The respective assertion when the cone is nonnormal can be proved for the so-called c-sequences. Namely, a sequence $\{a_n\}$ in B is called a c-sequence if for each $c \in \operatorname{int} P$ there exists $N \in \mathbb{N}$ such that $a_n \ll c$ holds whenever $n > N$. Note that $a_n \to \theta$ when $n \to \infty$ implies that $\{a_n\}$ is a c-sequence, but the converse is true only if the cone P is normal.

It was proved in [1] that every G-metric space is topologically equivalent to a metric space. In a similar way, one can prove that each G-cone metric space is topologically equivalent to a cone metric space. Namely, the base of such topology τ_G is given by the family of G-balls of the form

$$B_G(x_0, c) = \{ y \in X : G(x_0, y, y) \ll c \}$$

for $x_0 \in X$ and $c \in \operatorname{int} P$. A sequence in X G-converges in X if and only if it τ_G-converges.

If G is a G-cone metric, then a cone metric defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x)$$

satisfies that

$$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 2G(x, y, y).$$

We conclude that G-cone metric and cone metric d_G give rise to the same topology, and so, among other things, they have the same convergent sequences. In particular, this topology is Hausdorff and hence the limit of a sequence is unique.

The following assertion about the topological structure of G-cone metric space was stated in [23]. However, the proof given there uses normality of the cone and in fact cannot be done without this assumption. We will give here an alternative proof.

Lemma 2.8. *Let (X, G) be a G-cone metric space over a normal cone P. If $\{x_m\}$, $\{y_n\}$, and $\{z_\ell\}$ are sequences in X such that $x_m \to x$, $y_n \to y$ and $z_\ell \to z$, then $G(x_m, y_n, z_\ell) \to G(x, y, z)$ as $m, n, \ell \to \infty$.*

Proof. Let $e \in \operatorname{int} P$ and let ε be a fixed positive real number. Then, similarly as in [23], it can be proved that

$$-\varepsilon e < -\frac{\varepsilon}{2} e \leq G(x_m, y_n, z_\ell) - G(x, y, z) \leq \frac{\varepsilon}{2} e < \varepsilon e. \tag{2.3}$$

Let q_e be the Minkowski functional of the order interval $[-e, e]$, which is an absolutely convex neighbourhood of θ in B. Since the cone P is solid and normal, q_e is a norm in B, equivalent to the given norm (for details see [21]). Relation (2.3) implies that

$$q_e(G(x_m, y_n, z_\ell) - G(x, y, z)) < \varepsilon$$

and so $\|G(x_m, y_n, z_l) - G(x, y, z)\| \to 0$ when $m, n, \ell \to \infty$. Hence,

$$G(x_m, y_n, z_\ell) - G(x, y, z) \to \theta \quad \text{when } m, n, \ell \to \infty. \quad \square$$

Definition 2.9. Let X be a nonempty set. Then $(X, G, \preceq)$ is called an ordered G-cone metric space if:

(i) (X, G) is a G-cone metric space,
(ii) $(X, \preceq)$ is a partially ordered set.

Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

In [30], Nashine and Samet introduced the following concept.

Let X be a non-empty set and let $R : X \to X$ be a given mapping. For every $x \in X$, we denote by $R^{-1}(x)$ the subset of X defined by $R^{-1}(x) := \{u \in X : Ru = x\}$.

Definition 2.10. Let $(X, \preceq)$ be a partially ordered set and let $T, S, R : X \to X$ be given mappings such that $TX \subseteq RX$ and $SX \subseteq RX$. We say that S and T are weakly increasing with respect to R if for all $x \in X$, we have:

$$Tx \preceq Sy, \quad \forall y \in R^{-1}(Tx) \quad \text{and} \quad Sx \preceq Ty, \quad \forall y \in R^{-1}(Sx).$$

If $T = S$, we say that T is weakly increasing with respect to R.

Remark 2.11. If $R : X \to X$ is the identity mapping ($Rx = x$ for all $x \in X$), then S and T are weakly increasing with respect to R if and only if S and T are weakly increasing mappings in the sense of [42], i.e., $Tx \preceq S(Tx)$ and $Sx \preceq T(Sx)$ hold for each $x \in X$.

Definition 2.12. Let $(X, \preceq)$ be an ordered G-cone metric space. We say that X is regular if the following condition holds: if $\{z_n\}$ is a non-decreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to \infty$, then $z_n \preceq z$ for all $n \in \mathbb{N}$.

3. Main results

To formulate the results, we give the definition of a φ-map.

Definition 3.1 (*[19,20]*)**.** Let P be an order cone. A nondecreasing function $\varphi : P \to P$ is called a φ-map if:

(i) $\varphi(\theta) = \theta$ and $\theta < \varphi(\omega) < \omega$ for $\omega \in P \setminus \{\theta\}$,
(ii) $\omega \in \operatorname{int} P$ implies $\omega - \varphi(\omega) \in \operatorname{int} P$,
(iii) if $\omega \in P \setminus \{\theta\}$ and $c \in \operatorname{int} P$, then there exists $n_0 \in \mathbb{N}$ such that $\varphi^n(\omega) \ll c$ for each $n \geq n_0$.

Example 3.2 (*[19]*)**. (i)** If P is an arbitrary cone in a Banach space B and $\lambda \in (0, 1)$, then $\varphi : P \to P$, defined by $\varphi(\omega) = \lambda\omega$ for $\omega \in P$, is a φ-map.

(ii) Let $\psi : [0, +\infty) \to [0, +\infty)$ be any real-valued φ-map. Let P be a cone in a Banach space B and $\lambda \in (0, 1)$ be fixed. Then the function $\varphi_\lambda : P \to P$ defined by $\varphi_\lambda(\omega) = \psi(\lambda)\omega$, is a φ-map. Examples of this kind are of particular interest in the case when the cone P is nonnormal. For example, one can take $B = C^1_{\mathbb{R}}[0, 1]$, $P = \{x \in B : x(t) \geq 0, t \in [0, 1]\}$ (see Example 2.3) and $\psi(\lambda) = \frac{\lambda}{1+\lambda}$, $\lambda \in (0, 1)$.

Our first result is the following.

Theorem 3.3. *Let $(X, \preceq)$ be a partially ordered set, P be an order cone and let G be a G-cone metric on X. Let $T, R : X \to X$ be two mappings such that*

$$G(Tx, Ty, Tz) \leq \varphi(G(Rx, Ry, Rz)) \tag{3.1}$$

for all $x, y, z \in X$ with $Rx \succeq Ry \succeq Rz$, where φ is a φ-map. We suppose the following:

(i) *T is weakly increasing with respect to R;*
(ii) *RX is a complete subspace of X;*
(iii) *X is regular.*

Then T and R have a coincidence point.

Proof. Let x_0 be an arbitrary point in X. Since $TX \subseteq RX$ (by Definition 2.10), we can construct a sequence $\{x_n\}$ in X defined by

$$Rx_{n+1} = Tx_n, \quad \forall n \in \mathbb{N}_0.$$

Now, since $x_1 \in R^{-1}(Tx_0)$ and $x_2 \in R^{-1}(Tx_1)$, using that T is weakly increasing with respect to R, we obtain that

$$Rx_1 = Tx_0 \preceq Tx_1 = Rx_2 \preceq Tx_2 = Rx_3.$$

Continuing this process, we get that

$$Rx_1 \preceq Rx_2 \preceq Rx_3 \preceq \cdots \preceq Rx_n \preceq Rx_{n+1} \preceq \cdots.$$

We will prove that $\{Rx_n\}$ is a Cauchy sequence in $(R(X), G)$. We distinguish two cases.

First case. There exists $n \in \mathbb{N}$ such that $Rx_n = Rx_{n+1}$. Using the considered contractive condition, we get $Tx_n = Tx_{n+1}$, thats is, $Rx_{n+1} = Rx_{n+2}$. So, for every $m \geq n$, we have $Rx_m = Rx_n$. This implies that $\{Rx_n\}$ is a Cauchy sequence.

Second case. The successive terms of $\{Rx_n\}$ are different. From (3.1), we have

$$\begin{aligned} G(Rx_n, Rx_{n+1}, Rx_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\le \varphi(G(Rx_{n-1}, Rx_n, Rx_n)) \\ &\le \varphi^2(G(Rx_{n-2}, Rx_{n-1}, Rx_{n-1})) \\ &\vdots \\ &\le \varphi^n(G(Rx_0, Rx_1, Rx_1)). \end{aligned}$$

Fix $c, \theta \ll c$. According to property (iii) of function φ, there is $n_0 \in \mathbb{N}$ such that $\varphi^n(G(Rx_0, Rx_1, Rx_1)) \ll c$ for $n \ge n_0$. Using Remark 2.4(1), we get that $G(Rx_n, Rx_{n+1}, Rx_{n+1}) \ll c$ for $n \ge n_0$. In a similar way, there is $N_1 \in \mathbb{N}$ such that

$$G(Rx_m, Rx_{m+1}, Rx_{m+1}) < c - \varphi(c) \quad \text{for all } m \ge N_1. \tag{3.2}$$

We claim that

$$G(Rx_n, Rx_m, Rx_m) \ll c \quad \forall\, m > n \ge N_1 \tag{3.3}$$

and prove it by induction on m. The inequality (3.3) holds for $m = n + 1$ by using (3.2) and the fact that $c - \varphi(c) < c$. Assume that (3.3) holds for $m = k$. For $m = k + 1$, we have (using Remark 2.4)

$$\begin{aligned} G(Rx_n, Rx_{k+1}, Rx_{k+1}) &\le G(Rx_n, Rx_{n+1}, Rx_{n+1}) + G(Rx_{n+1}, Rx_{k+1}, Rx_{k+1}) \\ &\ll c - \varphi(c) + \varphi(G(Rx_n, Rx_k, Rx_k)) \\ &\ll c - \varphi(c) + \varphi(c) = c. \end{aligned}$$

By induction on m, we conclude that (3.3) holds for all $m > n \ge N_1$. Now axiom (G5) of G-metric (see also Remark 2.7) implies that

$$G(x_m, x_n, x_\ell) \le G(x_m, x_n, x_n) + G(x_n, x_n, x_\ell) \ll 2c$$

holds for $m, n, \ell \ge N_1$. Hence $\{Rx_n\}$ is a G-Cauchy sequence in (RX, G) which is complete by assumption. Then, there exist $u = Rv, z \in X$ such that

$$\lim_{n\to\infty} Rx_n = u = Rz. \tag{3.4}$$

Since $\{Rx_n\}$ is a non-decreasing sequence and X is regular, it follows from (3.4) that $Rx_n \preceq Rz$ for all $n \in \mathbb{N}$. Assume $Rx_n \ne Rz$. Fix $c, \theta \ll c$, and, using Remark 2.7, choose a natural number n such that $G(Rx_n, Rx_n, Rz) \ll \frac{c}{2}$ and $G(Rx_{n+1}, Rz, Rz) \ll \frac{c}{2}$. Hence, we can apply the considered contractive condition to obtain

$$\begin{aligned} G(Tz, Rz, Rz) &\le G(Tz, Tx_n, Tx_n) + G(Tx_n, Rz, Rz) \\ &\le \varphi(G(Rx_n, Rx_n, Rz)) + G(Rx_{n+1}, Rz, Rz) \quad (\text{by } (3.1)) \\ &< G(Rx_n, Rx_n, Rz) + G(Rx_{n+1}, Rz, Rz) \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Since $c \in \operatorname{int} P$ is arbitrary, it follows by Remark 2.4(3) that $G(Tz, Rz, Rz) = \theta$ which by axiom (G2) implies that $Tz = Rz$. Then z is a coincidence point for the mappings T and R. □

Example 3.4. Let (X, G) be the G-cone metric space introduced in Example 2.3, but with the reverse order:

$$x \preceq y \;\Leftrightarrow\; x \ge y.$$

Consider mappings $T : X \times X \to X$ and $R : X \times X \to X$ given by $Tx = 2x$ and $Rx = 3x$, and a φ-map given by $\varphi(\omega) = \frac{1}{2}\omega$, $\omega \in P$. Then all the conditions of Theorem 3.3 are satisfied. In particular, condition (3.1) reduces to

$$2(|x-y| + |y-z| + |z-x|)u \ge \frac{1}{2} \cdot 3(|x-y| + |y-z| + |z-x|)u,$$

and holds for all $x, y, z \in [0, +\infty)$. Also, T is weakly increasing with respect to R since $Ry = Tx$ implies $3y = 2x$, i.e., $y = \frac{2}{3}x$, which in turn implies $Tx = 2x \ge 2y = Ty$, i.e., $Tx \preceq Ty$. Obviously, 0 is a coincidence point of T and R.

The following result is an immediate consequence of Theorem 3.3.

Corollary 3.5. *Let $(X, \preceq)$ be a partially ordered set, P be an order cone and suppose that G is a G-cone metric on X. Let $T, R : X \to X$ be nondecreasing mappings such that for some $k \in [0, 1)$*

$$G(Tx, Ty, Tz) \le k\, G(Rx, Ry, Rz)$$

holds for all $x, y, z \in X$ with $x \succeq y \succeq z$. We suppose the following:

(i) *T is weakly increasing with respect to R;*
(ii) *RX is a complete subspace of X;*
(iii) *X is regular.*

Then T and R have a coincidence point.

Proof. The result follows from Theorem 3.3 taking $\varphi(\omega) = k\omega$. □

If $R : X \to X$ is the identity mapping, we get the following fixed point result.

Corollary 3.6. *Let $(X, \preceq)$ be a partially ordered set, P be an order cone and suppose there is a metric G on X such that (X, G) is a complete G-cone metric space. Let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z))$$

holds for all $x, y, z \in X$ with $x \succeq y \succeq z$ where φ is a φ-map. We suppose the following:

(i) *$Tx \preceq T(Tx)$ for all $x \in X$;*
(ii) *X is regular.*

Then T has a fixed point.

Now, our second result is the following generalization of Theorem 3.3.

Theorem 3.7. *Let $(X, \preceq)$ be a partially ordered set, P be an order cone and suppose there is a G-cone metric G on X such that (X, G) is a complete G-cone metric space. Let $T, R : X \to X$ be nondecreasing mappings such that for all $x, y, z \in X$ with $Rx \succeq Ry \succeq Rz$ there exists*

$$\Theta(x, y, z) \in \{G(Rx, Ry, Rz), G(Rx, Tx, Tx), G(Ry, Ty, Ty), G(Tx, Ry, Rz)\}$$

such that

$$G(Tx, Ty, Tz) \leq \varphi(\Theta(x, y, z)),$$

where φ is a φ-map. We suppose the following:

(i) *T is weakly increasing with respect to R,*
(ii) *X is regular.*

Then T and R have a coincidence point.

Proof. Let x_0 be an arbitrary point in X. Since $TX \subseteq RX$ (by Definition 2.10), we can construct a sequence $\{x_n\}$ in X defined by:

$$Rx_{n+1} = Tx_n, \quad \forall\, n \in \mathbb{N}.$$

Now, since $x_1 \in R^{-1}(Tx_0)$ and $x_2 \in R^{-1}(Tx_1)$, using that T is weakly increasing with respect to R, we obtain that

$$Rx_1 = Tx_0 \preceq Tx_1 = Rx_2 \preceq Tx_2 = Rx_3.$$

Continuing this process, we get that

$$Rx_1 \preceq Rx_2 \preceq Rx_3 \preceq \cdots \preceq Rx_n \preceq Rx_{n+1} \preceq \cdots.$$

If there exists $n_0 \in \{1, 2, \ldots\}$ such that $\Theta(x_{n_0}, x_{n_0-1}, x_{n_0-1}) = \theta$ then it is clear that $Rx_{n_0-1} = Rx_{n_0} = Tx_{n_0-1}$ and so we are finished. Now we can suppose

$$\Theta(x_n, x_{n-1}, x_{n-1}) > \theta$$

for all $n \geq 1$.

Assume $Rx_n \neq Rx_{n-1}$, for each $n \in \mathbb{N}$. Thus for $n \in \mathbb{N}$, we have

$$G(Rx_n, Rx_{n+1}, Rx_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \leq \varphi(\Theta(x_{n-1}, x_n, x_n))$$

where

$$\begin{aligned}\Theta(x_{n-1}, x_n, x_n) &\in \{G(Rx_{n-1}, Rx_n, Rx_n), G(Rx_{n-1}, Tx_{n-1}, Tx_{n-1}), G(Rx_n, Tx_n, Tx_n), G(Tx_{n-1}, Rx_n, Rx_n)\}\\ &= \{G(Rx_{n-1}, Rx_n, Rx_n), G(Rx_{n-1}, Rx_n, Rx_n), G(Rx_n, Rx_{n+1}, Rx_{n+1}), G(Rx_n, Rx_n, Rx_n)\}\\ &= \{G(Rx_{n-1}, Rx_n, Rx_n), G(Rx_n, Rx_{n+1}, Rx_{n+1}), \theta\}.\end{aligned}$$

• If $\Theta(x_{n-1}, x_n, x_n) = G(Rx_n, Rx_{n+1}, Rx_{n+1})$, then

$$G(Rx_n, Rx_{n+1}, Rx_{n+1}) \preceq \varphi(G(Rx_n, Rx_{n+1}, Rx_{n+1}))$$

and by the property of φ we have

$$G(Rx_n, Rx_{n+1}, Rx_{n+1}) < G(Rx_n, Rx_{n+1}, Rx_{n+1})$$

which is impossible.

• If $\Theta(x_{n-1}, x_n, x_n) = \theta$, then

$$G(Rx_n, Rx_{n+1}, Rx_{n+1}) \preceq \varphi(\theta) < \theta$$

which is a contradiction. Therefore, $\Theta(x_{n-1}, x_n, x_n) = G(Rx_{n-1}, Rx_n, Rx_n)$, and then

$$G(Rx_n, Rx_{n+1}, Rx_{n+1}) \preceq \varphi(G(Rx_{n-1}, Rx_n, Rx_n)).$$

Thus for $n \in N$, we have

$$\begin{aligned} G(Rx_n, Rx_{n+1}, Rx_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\preceq \varphi(G(Rx_{n-1}, Rx_n, Rx_n)) \\ &\preceq \varphi^2(G(Rx_{n-2}, Rx_{n-1}, Rx_{n-1})) \\ &\vdots \\ &\preceq \varphi^n(G(Rx_0, Rx_1, Rx_1)). \end{aligned}$$

By an argument similar to that in the proof of Theorem 3.3, one can show that $\{Rx_n\}$ is a Cauchy sequence. Since X is G-complete, Rx_n is convergent to $u \in X$. Now we show that $Ru = Tu$.

Since $\{Rx_n\}$ is a nondecreasing sequence and $Rx_n \to u$, by regularity of X we have $Rx_n \preceq u$ for all n. If $Rx_n = u$ for some n, then, by construction, $Rx_{n+1} = u$ and u is a fixed point. So we assume that $Rx_n \neq u$. Then, for $n \in \mathbb{N}$, we have

$$\begin{aligned} G(Ru, Ru, Tu) &\preceq G(Ru, Ru, Rx_n) + G(Rx_n, Rx_n, Tu) \\ &= G(Ru, Ru, Rx_n) + G(Tx_{n-1}, Tx_{n-1}, Tu) \\ &\preceq G(Ru, Ru, Rx_n) + \varphi(\Theta(x_{n-1}, x_{n-1}, u)) \end{aligned}$$

where

$$\begin{aligned} \Theta(x_{n-1}, x_{n-1}, u) &\in \{G(Rx_{n-1}, Rx_{n-1}, Ru), G(Rx_{n-1}, Tx_{n-1}, Tx_{n-1}), G(Rx_{n-1}, Tx_{n-1}, Tx_{n-1}), G(Tx_{n-1}, Rx_{n-1}, Ru)\} \\ &= \{G(Rx_{n-1}, Rx_{n-1}, Ru), G(Rx_{n-1}, Rx_n, Rx_n), G(Rx_n, Rx_{n-1}, Ru)\}. \end{aligned}$$

Fix $c, \theta \ll c$. Choose a natural number N_1 such that $G(Ru, Ru, Rx_n) \ll \frac{c}{2}$ and $G(Rx_{n-1}, Rx_{n-1}, Ru) \ll \frac{c}{2}$, for all $n \geq N_1$. We investigate these situations as follows:

Case 1. If $\Theta(x_{n-1}, x_{n-1}, u) = G(Rx_{n-1}, Rx_{n-1}, Ru)$, then we have

$$\begin{aligned} G(Ru, Ru, Tu) &\preceq G(Ru, Ru, Rx_n) + \varphi(G(Rx_{n-1}, Rx_{n-1}, Ru)) \\ &< G(Ru, Ru, Rx_n) + G(Rx_{n-1}, Rx_{n-1}, Ru) \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Case 2. If $\Theta(x_{n-1}, x_{n-1}, u) = G(Rx_{n-1}, Rx_n, Rx_n)$, then we have

$$\begin{aligned} G(Ru, Ru, Tu) &\preceq G(Ru, Ru, Rx_n) + \varphi(G(Rx_{n-1}, Rx_n, Rx_n)) \\ &< G(Ru, Ru, Rx_n) + G(Rx_{n-1}, Rx_n, Rx_n) \ll c. \end{aligned}$$

Case 3. If $\Theta(x_{n-1}, x_{n-1}, u) = G(Rx_n, Rx_{n-1}, Ru)$, then we have

$$\begin{aligned} G(Ru, Ru, Tu) &\preceq G(Ru, Ru, Rx_n) + \varphi(G(Rx_n, Rx_{n-1}, Ru)) \\ &< G(Ru, Ru, Rx_n) + G(Rx_n, Rx_{n-1}, Ru) \\ &\preceq G(Ru, Ru, Rx_n) + G(Rx_n, Rx_{n-1}, Rx_{n-1}) + G(Rx_{n-1}, Rx_{n-1}, Ru) \\ &\ll c \end{aligned}$$

whenever $n \in \mathbb{N}$. Thus in all cases $G(Ru, Ru, Tu) \ll c$ for arbitrary $c \in \operatorname{int} P$. By Remark 2.4(3), it follows that $G(Ru, Ru, Tu) = \theta$ which implies that $Tu = Ru$. Then u is a coincidence point for the mappings T and R. □

The following result is an immediate consequence of Theorem 3.7.

Corollary 3.8. *Let $(X, \preceq)$ be a partially ordered set, P be an order cone and suppose there is a G-cone metric G on X such that (X, G) is a complete G-cone metric space. Let $T, R : X \to X$ be nondecreasing mappings such that for some $k \in [0, 1)$, and for all $x, y, z \in X$ with $Rx \succeq Ry \succeq Rz$, there exists*

$$\Theta(x, y, z) \in \{G(Rx, Ry, Rz), G(Rx, Tx, Tx), G(Ry, Ty, Ty), G(Tx, Ry, Rz)\}$$

such that

$$G(Tx, Ty, Tz) \leq k\,\Theta(x, y, z).$$

We suppose the following:

(i) *T is weakly increasing with respect to R,*
(ii) *X is regular.*

Then T and R have a coincidence point.

If $R : X \to X$ is the identity mapping, we get easily the following fixed point result from Theorem 3.7.

Corollary 3.9. *Let $(X, \preceq)$ be a partially ordered set, P be an order cone and suppose there is a G-cone metric G on X such that (X, G) is a complete G-cone metric space. Let $T : X \to X$ be a nondecreasing mapping such that*

$$G(Tx, Ty, Tz) \leq \varphi(\Theta(x, y, z))$$

where

$$\Theta(x, y, z) \in \{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(Tx, y, z)\}$$

for all $x, y, z \in X$ with $x \succeq y \succeq z$, and φ is a φ-map. We suppose the following:

(i) *$Tx \preceq T(Tx)$ for all $x \in X$;*
(ii) *X is regular.*

Then T has a fixed point.

In the following result we present a sufficient condition for the uniqueness of the point of coincidence.

Theorem 3.10. *Under assumptions of Theorem 3.7 suppose that X is a totally ordered set. Then the point of coincidence of R and T is unique. If, additionally, R and T are weakly compatible, then they have a unique common fixed point.*

Proof. Suppose that T and R have two points of coincidence,

$$Tu = Ru \quad \text{and} \quad Tw = Rw, \qquad Ru \neq Rw.$$

As X is totally ordered set and $u, w \in X$, suppose that $u \prec w$. Applying the contractive condition we have that for some

$$\begin{aligned}\Theta(u, u, w) &\in \{G(Ru, Ru, Rw), G(Ru, Tu, Tu), G(Ru, Tu, Tu), G(Tu, Ru, Rw)\}\\ &= \{\theta, G(Ru, Ru, Rw)\},\end{aligned}$$

$G(Ru, Ru, Rw) = G(Tu, Tu, Tw) \leq \varphi(\Theta(u, u, w))$ holds. In both possible cases, using property of φ-function, a contradiction is obtained. Thus $Ru = Rw$. Hence T and R have a unique point of coincidence $Tu = Ru$.

The final assertion follows from a classical result of G. Jungck. □

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