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Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces

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ABSTRACT

Common fixed point theorems for $\mathcal{T}$-weakly isotone increasing mappings satisfying a generalized contractive type condition under a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$ and $\varphi(0) = 0$ in complete ordered partial metric spaces are proved. To illustrate our results and to distinguish them from the existing ones, we equip the paper with examples.

1. Introduction

In [1], Matthews introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, and obtained, among other results, a nice relationship between partial metric spaces and the so-called weightable quasi-metric spaces. He obtained the following Banach fixed point theorem in complete partial metric spaces.

Theorem 1.1 (*[1]*). *Let $\mathcal{T}$ be a mapping of a complete partial metric space $(\mathcal{X}, p)$ into itself such that there is a real number c with $0 \le c < 1$, satisfying for all $x, y \in \mathcal{X}$,*

$$p(\mathcal{T}x, \mathcal{T}y) \le cp(x, y).$$

Then $\mathcal{T}$ has a unique fixed point.

In partial metric spaces, the self-distance for any point need not be equal to zero. O'Neill [2] defined the concept of the dualistic partial metric, which is more general than the partial metric. In [3], Oltra and Valero gave a Banach fixed point theorem on complete dualistic partial metric spaces. Later, Valero [4] generalized the main theorem of [3] using a nonlinear contractive condition instead of a Banach contractive condition. Altun and Simsek [5], Ilić et al. [6], Oltra [7] and Romaguera [8] also studied fixed point theorems in partial metric spaces. Recently, Altun et al. [9] generalized Theorem 1.1 and obtained the following result (see also a correction in [10]):

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Theorem 1.2 *([9, Theorem 1], [10, Theorem 1]). Let $(\mathcal{X}, p)$ be a complete partial metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$p(\mathcal{T}x, \mathcal{T}y) \le \varphi\left(\max\left\{p(x, y), p(x, \mathcal{T}x), p(y, \mathcal{T}y), \frac{p(y, \mathcal{T}x) + p(x, \mathcal{T}y)}{2}\right\}\right) \tag{1.1}$$

for all $x, y \in \mathcal{X}$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function and $\sum_{n\ge 1} \varphi^n(t)$ is convergent for each $t > 0$. Then $\mathcal{T}$ has a unique fixed point.

On the other hand, many results have appeared recently related to fixed point theorems in complete metric spaces endowed with a partial ordering $\preceq$. Most of them are a hybrid of two fundamental principles: order iterative technique and various contractive conditions. Indeed, they deal with a monotone (either order-preserving or order-reversing) mapping $\mathcal{T}$ satisfying, with some restriction, a classical contractive condition, and are such that for some $x_0 \in \mathcal{X}$, either $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$ holds. The first result in this direction was given by Ran and Reurings [11, Theorem 2.1]. They proved an analogue of Banach's fixed point theorem in a metric space endowed with a partial order, and gave applications to matrix equations. In this way, they weakened the usual contraction condition but at the expense that the operator was monotone. Subsequently, Nieto and Rodríguez-López [12] extended the result of Ran and Reurings for nondecreasing mappings and applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, many fixed point problems have been considered in partially ordered sets (see [13–17] and references therein).

Ćirić [18] generalized Theorems 2.2 and 2.3 of Agarwal et al. [19] by introducing the concept of $\mathcal{S}$-monotone mapping and proved some common fixed point theorems for a pair of commuting mappings satisfying $\mathcal{S}$-nondecreasing generalized nonlinear contractions and some more conditions in partially ordered complete metric spaces. In [20], Nashine et al. extended Ćirić's result by using $\mathcal{T}$-weakly isotone increasing mappings and relaxing other conditions without taking into account any commutativity condition.

Altun and Erduran [21] used the idea of partial order and established fixed point theorems as a generalization of Theorem 1.2 to the frame of ordered partial metric spaces. One of their results is the following

Theorem 1.3 *([21, Theorem 2.2]). Let $(\mathcal{X}, p, \preceq)$ be a complete partially ordered partial metric space. Suppose $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a continuous and nondecreasing mapping (with respect to $\preceq$) such that (1.1) holds for all $x, y \in \mathcal{X}$ with $x \succeq y$, where $\varphi : [0, +\infty) \to [0, +\infty)$ satisfies all the conditions in Theorem 1.2. If there exists an $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$, then there exists $x \in \mathcal{X}$ such that $\mathcal{T}x = x$. Moreover, $p(x, x) = 0$.*

Aydi [22] and Samet et al. [23] also studied fixed point results on partially ordered partial metric spaces.

The aim of this paper is to give an extension of Theorem 1.3 for two mappings, using also weaker condition for the control function φ, that is, $\varphi(t) < t$ for all $t > 0$ in the place of condition $\sum_{n=1}^{\infty} \varphi^n(t) < 1$. We will do this by using the concept of $\mathcal{T}$-weakly isotone increasing mappings. In particular, we do not need any compatibility assumptions. Our results are the analogue of those of Nashine et al. [20] for ordered metric spaces; in particular, the continuity of $\mathcal{T}$ and $\mathcal{S}$ are both necessary. Examples are given to support the usability of our results and to show that they are proper extensions of the existing ones.

2. Preliminaries

The following definitions and details on partial metric spaces can be seen in [1,3–5,21,24].

Definition 2.1. A partial metric on a nonempty set $\mathcal{X}$ is a function $p : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ such that for all $x, y, z \in \mathcal{X}$:

(p_1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \le p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair $(\mathcal{X}, p)$ such that $\mathcal{X}$ is a nonempty set and p is a partial metric on $\mathcal{X}$.

It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [1,25].

Each partial metric p on $\mathcal{X}$ generates a T_0 topology τ_p on $\mathcal{X}$ which has as a base the family of open p-balls $\{\mathcal{B}_p(x, \varepsilon) : x \in \mathcal{X}, \varepsilon > 0\}$, where $\mathcal{B}_p(x, \varepsilon) = \{y \in \mathcal{X} : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in \mathcal{X}$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in $(\mathcal{X}, p)$ converges to a point $x \in \mathcal{X}$ (in τ_p) if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

Remark 2.2. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot, \cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$. For example, if $\mathcal{X} = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in \mathcal{X}$, then for $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x, x)$ for each $x \ge 1$ and so, e.g., $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

Definition 2.3. Let $(\mathcal{X}, p)$ be a partial metric space. Then:

(1) A sequence $\{x_n\}$ in $(\mathcal{X}, p)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).
(2) The space $(\mathcal{X}, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $\mathcal{X}$ converges, with respect to τ_p, to a point $x \in \mathcal{X}$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(3) A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be continuous at $x_0 \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{T}(\mathcal{B}_p(x_0, \delta)) \subset \mathcal{B}_p(\mathcal{T}x_0, \varepsilon)$.

It is easy to see that every τ_p-closed subset of a complete partial metric space is complete.

Lemma 2.4. *Let $(\mathcal{X}, p)$ be a partial metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a given mapping. Suppose that $\mathcal{T}$ is continuous at $x_0 \in \mathcal{X}$. Then, for each sequence $\{x_n\}$ in $\mathcal{X}$,*

$$x_n \to x_0 \quad \text{in } \tau_p \Rightarrow \mathcal{T}x_n \to \mathcal{T}x_0 \text{ in } \tau_p$$

holds.

If p is a partial metric on $\mathcal{X}$, then the function $p^s : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on $\mathcal{X}$.

Lemma 2.5. *Let $(\mathcal{X}, p)$ be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, p)$ if and only if it is a Cauchy sequence in the metric space $(\mathcal{X}, p^s)$.*
(b) *The space $(\mathcal{X}, p)$ is complete if and only if the metric space $(\mathcal{X}, p^s)$ is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 2.6. Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Then:

(a) elements $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds;
(b) a subset $\mathcal{K}$ of $\mathcal{X}$ is said to be well ordered if every two elements of $\mathcal{K}$ are comparable;
(c) a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called nondecreasing w.r.t. $\preceq$ if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$.

Definition 2.7. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, p, \preceq)$ is called an ordered (partial) metric space if:
(i) $(\mathcal{X}, p)$ is a (partial) metric space, and (ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

Definition 2.8. Let $(\mathcal{X}, p, \preceq)$ be an ordered partial metric space. We say that $\mathcal{X}$ is regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in $\mathcal{X}$ with respect to $\preceq$ such that $z_n \to z \in \mathcal{X}$ as $n \to \infty$, then $z_n \preceq z$ for all $n \in \mathbb{N}$.

3. Results

Definition 3.1. Let $(\mathcal{X}, \preceq)$ be a partially ordered set. A pair of mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be weakly increasing if $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preceq \mathcal{S}\mathcal{T}x$ for all $x \in \mathcal{X}$.

Note that two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [26].

In this section, we give a common fixed point theorem for a pair of maps satisfying $\mathcal{T}$-weakly isotone increasing property. For this we need the following definition, which is given in [20].

Definition 3.2. Let $(\mathcal{X}, \preceq)$ be a partially ordered set and let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be two mappings. The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-weakly isotone increasing if for all $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x \preceq \mathcal{S}\mathcal{T}\mathcal{S}x$.

Remark 3.3. If $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are weakly increasing, then $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing.

3.1. Auxiliary results

Assertions similar to the following lemma (see, e.g., [27]) were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 3.4. *Let $(\mathcal{X}, d)$ be a metric space and let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that*

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four sequences tend to ε when $k \to \infty$:

$$d(x_{2m(k)}, x_{2n(k)}), \qquad d(x_{2m(k)}, x_{2n(k)+1}), \qquad d(x_{2m(k)-1}, x_{2n(k)}), \qquad d(x_{2m(k)-1}, x_{2n(k)+1}).$$

As a corollary we obtain

Lemma 3.5. *Let $(\mathcal{X}, p)$ be a partial metric space and let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that*

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{3.1}$$

If $\{x_{2n}\}$ is not a Cauchy sequence in $(\mathcal{X}, p)$, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four sequences tend to ε when $k \to \infty$:

$$p(x_{2m(k)}, x_{2n(k)}), \qquad p(x_{2m(k)}, x_{2n(k)+1}), \qquad p(x_{2m(k)-1}, x_{2n(k)}), \qquad p(x_{2m(k)-1}, x_{2n(k)+1}). \tag{3.2}$$

Proof. Suppose that $\{x_n\}$ is a sequence in $(\mathcal{X}, p)$ satisfying (3.1) such that $\{x_{2n}\}$ is not Cauchy. According to Lemma 2.5, it is not a Cauchy sequence in the metric space $(\mathcal{X}, p^s)$, either. Applying Lemma 3.4 we get the sequences

$$p^s(x_{2m(k)}, x_{2n(k)}), \qquad p^s(x_{2m(k)}, x_{2n(k)+1}), \qquad p^s(x_{2m(k)-1}, x_{2n(k)}), \qquad p^s(x_{2m(k)-1}, x_{2n(k)+1})$$

tending to some $2\varepsilon > 0$ when $k \to \infty$. Using the definition of the associated metric

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

and (3.1) (which implies that also $\lim_{n\to\infty} p(x_{2n}, x_{2n}) = 0$), we get that the sequences (3.2) tend to ε when $k \to \infty$. □

3.2. Main results

The first result of the paper is the following:

Theorem 3.6. *Let $(\mathcal{X}, p, \preceq)$ be a complete partially ordered partial metric space. Let $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ be two mappings such that*

$$p(\mathcal{T}x, \mathcal{S}y) \le \mathbf{M}(x, y) \tag{3.3}$$

for all comparable $x, y \in \mathcal{X}$, where

$$\mathbf{M}(x, y) = \max\left\{\varphi(p(x, y)), \varphi(p(x, \mathcal{T}x)), \varphi(p(y, \mathcal{S}y)), \varphi\left(\frac{p(y, \mathcal{T}x) + p(x, \mathcal{S}y)}{2}\right)\right\} \tag{3.4}$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous function with $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$. We suppose the following:

(i) *$\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing,*
(ii) *$\mathcal{S}$ and $\mathcal{T}$ are continuous.*

Then the set $F(\mathcal{T}, \mathcal{S})$ of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is nonempty, and $p(z, z) = p(\mathcal{T}z, \mathcal{T}z) = p(\mathcal{S}z, \mathcal{S}z) = p(z, \mathcal{S}z) = p(z, \mathcal{T}z) = 0$ for $z \in F(\mathcal{T}, \mathcal{S})$. Moreover, the set $F(\mathcal{T}, \mathcal{S})$ is well ordered if and only if $\mathcal{T}$ and $\mathcal{S}$ have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in $\mathcal{X}$. If $x_0 = \mathcal{S}x_0$ or $x_0 = \mathcal{T}x_0$ the proof can be easily finished using contractive condition (3.3), so we assume that $x_0 \ne \mathcal{S}x_0$ and $x_0 \ne \mathcal{T}x_0$. We can define a sequence $\{x_n\}$ in $\mathcal{X}$ as follows:

$$x_{2n+1} = \mathcal{S}x_{2n} \quad \text{and} \quad x_{2n+2} = \mathcal{T}x_{2n+1} \quad \text{for } n \in \{0, 1, \ldots\}. \tag{3.5}$$

Without loss of generality we can suppose that the successive terms of $\{x_n\}$ are different. Otherwise we have again finished. Note that, since $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing, we have

$$x_1 = \mathcal{S}x_0 \preceq \mathcal{T}\mathcal{S}x_0 = \mathcal{T}x_1 = x_2 \preceq \mathcal{S}\mathcal{T}\mathcal{S}x_0 = \mathcal{S}\mathcal{T}x_1 = \mathcal{S}x_2 = x_3,$$
$$x_3 = \mathcal{S}x_2 \preceq \mathcal{T}\mathcal{S}x_2 = \mathcal{T}x_3 = x_4 \preceq \mathcal{S}\mathcal{T}\mathcal{S}x_2 = \mathcal{S}\mathcal{T}x_3 = \mathcal{S}x_4 = x_5,$$

and continuing this process we get

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{3.6}$$

Now we claim that for all $n \in \mathbb{N}$, we have

$$p(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1}). \tag{3.7}$$

From (3.6) we have that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Then from (3.3) with $x = x_{2n+1}$ and $y = x_{2n}$, we get

$$p(x_{2n+1}, x_{2n+2}) = p(\mathcal{T}x_{2n+1}, \mathcal{S}x_{2n}) \leq \mathbf{M}(x_{2n+1}, x_{2n}). \tag{3.8}$$

By (3.4) and (3.5), we have

$$\begin{aligned}\mathbf{M}(x_{2n+1}, x_{2n}) &= \max\Bigg\{\varphi(p(x_{2n}, x_{2n+1})), \varphi(p(x_{2n}, \mathcal{T}x_{2n})), \varphi(p(x_{2n+1}, \mathcal{S}x_{2n+1})),\\ &\qquad \varphi\left(\frac{p(x_{2n+1}, \mathcal{T}x_{2n}) + p(x_{2n}, \mathcal{S}x_{2n+1})}{2}\right)\Bigg\}\\ &= \max\left\{\varphi(p(x_{2n}, x_{2n+1})), \varphi(p(x_{2n+1}, x_{2n+2})), \varphi\left(\frac{p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{2}\right)\right\}.\end{aligned}$$

- If $\mathbf{M}(x_{2n+1}, x_{2n}) = \varphi(p(x_{2n+1}, x_{2n+2}))$, by (3.8) and using the fact that $\varphi(t) < t$ for all $t > 0$, we have

  $$p(x_{2n+1}, x_{2n+2}) \leq \varphi(p(x_{2n+1}, x_{2n+2})) < p(x_{2n+1}, x_{2n+2}),$$

  a contradiction.
- If $\mathbf{M}(x_{2n+1}, x_{2n}) = \varphi(p(x_{2n}, x_{2n+1}))$, we get

  $$p(x_{2n+1}, x_{2n+2}) \leq \varphi(p(x_{2n}, x_{2n+1})) < p(x_{2n}, x_{2n+1}).$$
- If $\mathbf{M}(x_{2n+1}, x_{2n}) = \varphi\left(\frac{p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{2}\right)$, we get

  $$\begin{aligned}p(x_{2n+1}, x_{2n+2}) &\leq \varphi\left(\frac{p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{2}\right)\\ &< \frac{p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{2}.\end{aligned}$$

By (p_4), we have

$$p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1}) \leq p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}).$$

Therefore we have

$$p(x_{2n+1}, x_{2n+2}) < \frac{p(x_{2n}, x_{2n+1})}{2} + \frac{p(x_{2n+1}, x_{2n+2})}{2}$$

which implies that

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n}, x_{2n+1}).$$

Thus, in all possible cases, we have $p(x_{2n+1}, x_{2n+2}) < p(x_{2n}, x_{2n+1})$ for all $n \in \mathbb{N}$. Similarly, we can prove that $p(x_{2n}, x_{2n+1}) < p(x_{2n-1}, x_{2n})$ for all $n \in \mathbb{N}$. Therefore, we conclude that (3.7) holds.

Now, from (3.7) it follows that the sequence $\{p(x_n, x_{n+1})\}$ is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = \delta. \tag{3.9}$$

We are able to prove that $\delta = 0$. By (p_4), we have

$$p(x_n, x_{n+2}) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}).$$

Therefore

$$\frac{1}{2}p(x_n, x_{n+2}) + \frac{1}{2}p(x_{n+1}, x_{n+1}) \leq \frac{1}{2}p(x_n, x_{n+1}) + \frac{1}{2}p(x_{n+1}, x_{n+2}).$$

By (3.7), we have

$$\frac{1}{2}p(x_n, x_{n+2}) + \frac{1}{2}p(x_{n+1}, x_{n+1}) \leq p(x_n, x_{n+1}). \tag{3.10}$$

From (3.10), taking the upper limit as $n \to \infty$, we get

$$\limsup_{n\to\infty}\left[\frac{1}{2}p(x_{2n},x_{2n+2})+\frac{1}{2}p(x_{2n+1},x_{2n+1})\right]\le\lim_{n\to\infty}p(x_{2n},x_{2n+1}).$$

If we set

$$\limsup_{n\to\infty}\left[\frac{1}{2}p(x_{2n},x_{2n+2})+\frac{1}{2}p(x_{2n+1},x_{2n+1})\right]=b, \tag{3.11}$$

then clearly $0\le b\le\delta$. As φ is continuous and taking the upper limit on both sides of (3.8), we get

$$\limsup_{n\to\infty}p(x_{2n+1},x_{2n+2}) \le \max\Bigg\{\varphi(\limsup_{n\to\infty}p(x_{2n+1},x_{2n+2})),\varphi(\limsup_{n\to\infty}(p(x_{2n+1},x_{2n})),\varphi\left(\limsup_{n\to\infty}\left(\frac{1}{2}p(x_{2n},x_{2n+2})+\frac{1}{2}p(x_{2n+1},x_{2n+1})\right)\right)\Bigg\}.$$

Hence by (3.9) and (3.11), we deduce

$$\delta\le\max\{\varphi(\delta),\varphi(b)\}.$$

If we suppose that $\delta>0$, then we have

$$\delta\le\max\{\varphi(\delta),\varphi(b)\}<\max\{\delta,b\}=\delta,$$

a contradiction. Thus $\delta=0$ and consequently

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0. \tag{3.12}$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence in the metric space $(\mathcal{X},p^s)$ (and so also in the space $(\mathcal{X},p)$ by Lemma 2.5). For this it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose that this is not the case. Then, using Lemma 3.5 we get that there exist $\varepsilon>0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k)>m(k)>k$ and sequences (3.2) tend to ε when $k\to\infty$. Applying condition (3.3) to (comparable) elements $x=x_{2m(k)-1}$ and $y=x_{2n(k)}$ we get that

$$\begin{aligned}p(x_{2n(k)},x_{2m(k)}) &\le p(x_{2n(k)},x_{2n(k)+1})+p(x_{2n(k)+1},x_{2m(k)})-p(x_{2n(k)+1},x_{2n(k)+1})\\ &\le p(x_{2n(k)},x_{2n(k)+1})+p(\mathcal{S}x_{2n(k)},\mathcal{T}x_{2m(k)-1})\\ &\le p(x_{2n(k)},x_{2n(k)+1})+\mathbf{M}(x_{2m(k)-1},x_{2n(k)})\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\mathbf{M}(x_{2m(k)-1},x_{2n(k)}) &= \max\Bigg\{\varphi(p(x_{2m(k)-1},x_{2n(k)})),\varphi(p(x_{2m(k)-1},x_{2m(k)})),\\ &\qquad\varphi(p(x_{2n(k)},x_{2n(k)+1})),\varphi\left(\frac{p(x_{2n(k)},x_{2m(k)})+p(x_{2m(k)-1},x_{2n(k)+1})}{2}\right)\Bigg\}\\ &\to \max\{\varphi(\varepsilon),0,0,\varphi(\varepsilon)\}\\ &= \varphi(\varepsilon)\quad\text{as } k\to\infty.\end{aligned}$$

Letting $k\to\infty$ in (3.13), we obtain

$$\varepsilon\le\varphi(\varepsilon)<\varepsilon,$$

a contradiction. Thus $\{x_{2n}\}$ is a Cauchy sequence and so $\{x_n\}$ is a Cauchy sequence both in $(\mathcal{X},p)$ and in $(\mathcal{X},p^s)$.

Since $(\mathcal{X},p)$ is complete then from Lemma 2.5, the sequence $\{x_n\}$ converges in the metric space $(\mathcal{X},p^s)$, say $\lim_{n\to\infty}p^s(x_n,z)=0$. Again from Lemma 2.5, we have

$$p(z,z)=\lim_{n\to\infty}p(x_n,z)=\lim_{n,m\to\infty}p(x_n,x_m). \tag{3.14}$$

Moreover, since $\{x_n\}$ is a Cauchy sequence in the metric space $(\mathcal{X},p^s)$, we have $\lim_{n,m\to\infty}p^s(x_n,x_m)=0$ and so, by the definition of p^s, we have $\lim_{n,m\to\infty}p(x_n,x_m)=0$. Then (3.14) implies that $p(z,z)=0$ and

$$\lim_{n\to\infty}p(x_n,z)=p(z,z)=0. \tag{3.15}$$

By (p_4), we have

$$\begin{aligned} p(z, \mathcal{T}z) &\le p(z, x_{2n+2}) + p(x_{2n+2}, \mathcal{T}z) - p(x_{2n+2}, x_{2n+2}) \\ &\le p(z, x_{2n+2}) + p(x_{2n+2}, \mathcal{T}z). \end{aligned} \tag{3.16}$$

Suppose that $\mathcal{T}$ is continuous. Letting $n \to \infty$ in (3.16) and applying (3.15) we get

$$\begin{aligned} p(z, \mathcal{T}z) &\le \lim_{n\to\infty} p(z, x_{2n+2}) + \lim_{n\to\infty} p(\mathcal{T}x_{2n+1}, \mathcal{T}z) \\ &= p(\mathcal{T}z, \mathcal{T}z). \end{aligned}$$

But from (p_2), we have $p(\mathcal{T}z, \mathcal{T}z) \le p(z, \mathcal{T}z)$. Hence

$$p(z, \mathcal{T}z) = p(\mathcal{T}z, \mathcal{T}z). \tag{3.17}$$

Similarly, if $\mathcal{S}$ is continuous, we have

$$p(z, \mathcal{S}z) = p(\mathcal{S}z, \mathcal{S}z). \tag{3.18}$$

By (p_4) and using (3.18), we have

$$\begin{aligned} p(z, \mathcal{T}z) &\le p(z, \mathcal{S}z) + p(\mathcal{S}z, \mathcal{T}z) - p(\mathcal{S}z, \mathcal{S}z) \\ &= p(z, \mathcal{S}z) + p(\mathcal{S}z, \mathcal{T}z) - p(z, \mathcal{S}z) \\ &= p(\mathcal{S}z, \mathcal{T}z), \end{aligned}$$

that is,

$$p(z, \mathcal{T}z) \le p(\mathcal{S}z, \mathcal{T}z). \tag{3.19}$$

Similarly, by (p_4) and using (3.17), we can obtain

$$p(z, \mathcal{S}z) \le p(\mathcal{S}z, \mathcal{T}z). \tag{3.20}$$

Suppose that $p(\mathcal{T}z, \mathcal{S}z) > 0$. Then, since $z \preceq z$, by inequality (3.3), we have

$$\begin{aligned} p(\mathcal{T}z, \mathcal{S}z) &\le \mathbf{M}(z, z) \\ &\le \max\left\{\varphi(p(z, z)), \varphi(p(z, \mathcal{T}z)), \varphi(p(z, \mathcal{S}z)), \varphi\left(\frac{p(z, \mathcal{T}z) + p(z, \mathcal{S}z)}{2}\right)\right\} \\ &< p(\mathcal{S}z, \mathcal{T}z), \quad \text{by (3.19), (3.20)} \end{aligned}$$

which is a contradiction (assumption $\varphi(t) < t$ was used). Thus, we get that $p(\mathcal{S}z, \mathcal{T}z) = 0$. By (p_1), we conclude that $\mathcal{S}z = \mathcal{T}z$, that is, z is a coincidence point of $\mathcal{T}$ and $\mathcal{S}$.

Moreover, $p(\mathcal{S}z, \mathcal{T}z) = 0$, together with (3.19) and (3.20), implies that

$$p(z, \mathcal{S}z) = 0 = p(z, \mathcal{T}z).$$

By (p_1), we conclude that $\mathcal{S}z = z$ and $\mathcal{T}z = z$, that is, z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Also, by (p_2), we can obtain

$$p(\mathcal{S}z, \mathcal{S}z) = 0 = p(\mathcal{S}z, \mathcal{T}z).$$

Thus, we have

$$p(\mathcal{T}z, \mathcal{T}z) = p(\mathcal{S}z, \mathcal{S}z) = p(z, \mathcal{T}z) = p(z, \mathcal{S}z) = p(z, z) = 0.$$

Now suppose that the set of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is well ordered. We claim that common fixed point of $\mathcal{T}$ and $\mathcal{S}$ is unique. Assume on contrary that $\mathcal{S}u = \mathcal{T}u = u$ and $\mathcal{T}v = \mathcal{S}v = v$ but $u \ne v$. By supposition, we can replace x by u and y by v in (3.3) to obtain

$$p(u, v) = p(\mathcal{T}u, \mathcal{S}v) \le \mathbf{M}(u, v)$$

where

$$\begin{aligned} \mathbf{M}(u, v) &= \max\left\{\varphi(p(u, v)), \varphi(p(u, \mathcal{T}u)), \varphi(p(v, \mathcal{S}v)), \varphi\left(\frac{p(v, \mathcal{T}u) + p(u, \mathcal{S}v)}{2}\right)\right\} \\ &= \max\left\{\varphi(p(u, v)), \varphi(p(u, u)), \varphi(p(v, v)), \varphi\left(\frac{p(v, u) + p(u, v)}{2}\right)\right\} \\ &< p(u, v). \end{aligned}$$

Therefore

$$p(u, v) < p(u, v),$$

a contradiction. Hence $p(u, v) = 0$ that is $u = v$. Conversely, if $\mathcal{T}$ and $\mathcal{S}$ have only one common fixed point then the set of common fixed point of $\mathcal{T}$ and $\mathcal{S}$ being singleton is well ordered. Thus, the proof is complete. □

Now, referring to the paper of Jachymski [14], we give some remarks on the contractive condition (3.3).

Remark 3.7. The following condition

$$p(\mathcal{T}x, \mathcal{S}y) \le \varphi(\mathbf{M_1}(x, y)), \tag{3.21}$$

where

$$\mathbf{M_1}(x, y) = \max\left\{p(x, y), p(x, \mathcal{T}x), p(y, \mathcal{S}y), \frac{p(y, \mathcal{T}x) + p(x, \mathcal{S}y)}{2}\right\},$$

implies condition (3.3). We observe that these two conditions are equivalent if we suppose that φ is a nondecreasing function. However, if φ is not monotone, condition (3.3) is more general. We demonstrate it by the following example.

Example 3.8. Let $\mathcal{X} = [0, 1]$ be endowed with the standard order and with the partial metric $p(x, y) = \max\{x, y\}$. Consider the mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}x = \mathcal{S}x = x - \frac{3}{4}x^2, \quad x \in \mathcal{X},$$

and let $\varphi : [0, +\infty) \to [0, +\infty)$ be given as

$$\varphi(t) = \begin{cases} t - \frac{3}{4}t^2, & t \in [0, 1] \\ \frac{1}{4}, & t > 1. \end{cases}$$

Function φ satisfies all the requirements. We will prove that mappings $\mathcal{T}$ and $\mathcal{S}$ satisfy condition (3.3) but they do not satisfy condition (3.21).

Let, e.g., $x \ge y$. Then

$$p(\mathcal{T}x, \mathcal{S}y) = \max\left\{x - \frac{3}{4}x^2, y - \frac{3}{4}y^2\right\} = \begin{cases} x - \frac{3}{4}x^2, & x + y \le \frac{4}{3} \vee x = y \\ y - \frac{3}{4}y^2, & x + y \ge \frac{4}{3} \vee x = y. \end{cases}$$

On the other hand,

$$\mathbf{M}(x, y) = \max\left\{\varphi(x), \varphi(x), \varphi(y), \varphi\left(\frac{1}{2}\left(x + \max\left\{y, x - \frac{3}{4}x^2\right\}\right)\right)\right\}.$$

If $x = y$, or $x + y \le \frac{4}{3}$, then $p(\mathcal{T}x, \mathcal{S}y) = \varphi(x) \le \mathbf{M}(x, y)$. If $x + y \ge \frac{4}{3}$, then $p(\mathcal{T}x, \mathcal{S}y) = \varphi(y) \le \mathbf{M}(x, y)$. Hence, in all possible cases condition (3.3) is satisfied.

Take $x = 1$ and $y = \frac{2}{3}$. Then $p(\mathcal{T}x, \mathcal{S}y) = p(\frac{1}{4}, \frac{1}{3}) = \frac{1}{3}$ and

$$\varphi(\mathbf{M_1}(x, y)) = \varphi\left(\max\left\{1, 1, \frac{2}{3}, \frac{1}{2}\left(1 + \frac{2}{3}\right)\right\}\right) = \varphi(1) = \frac{1}{4} < \frac{1}{3} = p(\mathcal{T}x, \mathcal{S}y).$$

Hence, condition (3.21) does not hold for all $x, y \in \mathcal{X}$.

Remark 3.9. Clearly, from our Theorem 3.6 we can derive a corollary involving condition (3.21). Moreover, under the hypothesis that φ is a nondecreasing function, we can state many other corollaries using the equivalences established in [14]. To avoid repetition, these results are omitted.

From Theorem 3.6 and Remark 3.3, we deduce the following corollary.

Corollary 3.10. *The conclusion of Theorem 3.6 holds if we suppose that*

$\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ *are two weakly increasing mappings*,

instead of

$\mathcal{S}$ *is* $\mathcal{T}$*-weakly isotone increasing*.

In the following theorem we prove the existence of a common fixed point of two mappings without using the continuity of $\mathcal{S}$ or $\mathcal{T}$.

Theorem 3.11. *Let $(\mathcal{X}, p, \preceq)$ be a complete partially ordered partial metric space. Assume that there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two mappings such that*

$$p(\mathcal{T}x, \mathcal{S}y) \le \mathbf{M}(x, y), \tag{3.22}$$

for all comparable $x, y \in \mathcal{X}$, where

$$\mathbf{M}(x, y) = \max\left\{\varphi(p(x, y)), \varphi(p(x, \mathcal{T}x)), \varphi(p(y, \mathcal{S}y)), \varphi\left(\frac{p(y, \mathcal{T}x) + p(x, \mathcal{S}y)}{2}\right)\right\}. \tag{3.23}$$

We suppose the following hypotheses:

(i) *$\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing,*
(ii) *$\mathcal{X}$ is regular.*

Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point, that is, there exists $z \in \mathcal{X}$ such that $\mathcal{T}z = \mathcal{S}z = z$, and $p(z, z) = p(\mathcal{T}z, \mathcal{T}z) = p(\mathcal{S}z, \mathcal{S}z) = p(z, \mathcal{S}z) = p(z, \mathcal{T}z) = 0$. Moreover, the set $F(\mathcal{T}, \mathcal{S})$ of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is well ordered if and only if $\mathcal{T}$ and $\mathcal{S}$ have one and only one common fixed point.

Proof. Using the same arguments as in the proof of Theorem 3.6, we deduce that $\{x_n\}$ is a Cauchy sequence, tending in $(\mathcal{X}, p^s)$ to some z. Since $\{x_n\}$ is a non-decreasing sequence, if $\mathcal{X}$ is regular, it follows that $x_n \preceq z$ for all n. Therefore, for all n, we can use the inequality (3.23) for x_{2n+1} and z. Since

$$\begin{aligned}\mathbf{M}(x_{2n+1}, z) &= \max\left\{\varphi(p(x_{2n+1}, z)), \varphi(p(x_{2n+1}, \mathcal{T}x_{2n+1})), \varphi(p(z, \mathcal{S}z)), \varphi\left(\frac{p(z, \mathcal{T}x_{2n+1}) + p(x_{2n+1}, \mathcal{S}z)}{2}\right)\right\}\\ &= \max\left\{\varphi(p(x_{2n+1}, z)), \varphi(p(x_{2n+1}, x_{2n+2})), \varphi(p(z, \mathcal{S}z)), \varphi\left(\frac{p(z, x_{2n+2}) + p(x_{2n+1}, \mathcal{S}z)}{2}\right)\right\},\end{aligned}$$

we have that $\lim_{n\to\infty} \mathbf{M}(z, x_{2n+1}) = \max\{\varphi(p(z, \mathcal{S}z)), \varphi(p(z, \mathcal{S}z)/2)\}$.

Using (p_4) and (3.22), we have

$$\begin{aligned}p(z, \mathcal{S}z) &\le p(z, x_{2n+2}) + p(\mathcal{T}x_{2n+1}, \mathcal{S}z) - p(x_{2n+2}, x_{2n+2})\\ &\le p(z, x_{2n+2}) + \mathbf{M}(x_{2n+1}, z).\end{aligned}$$

Passing to the limit when $n \to \infty$ we get that

$$p(z, \mathcal{S}z) \le \max\{\varphi(p(z, \mathcal{S}z)), \varphi(p(z, \mathcal{S}z)/2)\}.$$

Hence $p(z, \mathcal{S}z) = 0$ and so $\mathcal{S}z = z$. Analogously, for $x = z$ and $y = x_{2n}$, one can prove that $\mathcal{T}z = z$. It follows that $z = \mathcal{S}z = \mathcal{T}z$, that is, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. Also by (p_2), we can obtain $p(\mathcal{T}z, \mathcal{T}z) = 0$ and $p(\mathcal{S}z, \mathcal{S}z) = 0$. Following the line of proof of Theorem 3.6, we can show that the set $F(\mathcal{T}, \mathcal{S})$ of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is well ordered if and only if $\mathcal{T}$ and $\mathcal{S}$ have one and only one common fixed point. □

Corollary 3.12. *The conclusion of Theorem 3.11 holds if we suppose that*

$\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two weakly increasing mappings,

instead of

$\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing.

Putting $\mathcal{S} = \mathcal{T}$ in Corollary 3.12, we obtain

Corollary 3.13. *Let $(\mathcal{X}, p, \preceq)$ be a complete partially ordered partial metric space. Assume that there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a mapping such that*

$$p(\mathcal{T}x, \mathcal{T}y) \le \max\left\{\varphi(p(x, y)), \varphi(p(x, \mathcal{T}x)), \varphi(p(y, \mathcal{T}y)), \varphi\left(\frac{p(y, \mathcal{T}x) + p(x, \mathcal{T}y)}{2}\right)\right\}$$

for all comparable $x, y \in \mathcal{X}$. We suppose the following hypotheses:

(a) *$\mathcal{T}x \preceq \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$*
(b) *$\mathcal{X}$ is regular.*

Then $\mathcal{T}$ has a fixed point and $p(\mathcal{T}z, \mathcal{T}z) = 0 = p(z, z)$. Moreover, the set $F(\mathcal{T})$ of fixed points of $\mathcal{T}$ is well ordered if and only if it is a singleton.

We demonstrate the use of Theorems 3.6 and 3.11 with the help of the following example. It will show also that these theorems are more general than some other known fixed point results.

Example 3.14. Let $\mathcal{X} = [0, +\infty)$ be endowed with the usual partial metric $p : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ defined by $p(x, y) = \max\{x, y\}$. We give the partial order on $\mathcal{X}$ by

$$x \preceq y \Leftrightarrow x = y \quad \text{or} \quad (x, y \in [0, 1] \text{ with } x \le y).$$

The partial metric space $(\mathcal{X}, p)$ is complete because $(\mathcal{X}, p^s)$ is complete. Indeed, for any $x, y \in \mathcal{X}$,

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) = 2\max\{x, y\} - (x + y) = |x - y|.$$

Thus, $(\mathcal{X}, p^s) = ([0, +\infty), |\cdot|)$ is the usual metric space, which is complete.

Define $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{T}x = \begin{cases} \dfrac{x^2}{2(1+x)}, & \text{if } x \in [0, 1] \\ \dfrac{x}{4}, & \text{if } x > 1, \end{cases} \qquad \mathcal{S}x = \begin{cases} \dfrac{x^2}{4(1+x)}, & \text{if } x \in [0, 1] \\ \dfrac{x}{8}, & \text{if } x > 1. \end{cases}$$

Let us take $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$\varphi(t) = \begin{cases} \dfrac{t^2}{2(1+t)}, & t \in [0, 1] \\ \dfrac{t}{4}, & t > 1. \end{cases}$$

Then φ has the properties mentioned in Theorem 3.6.

Take arbitrary elements, say $y \le x$ from $\mathcal{X}$. Then there are two possibilities. If $x \in [0, 1]$ (and so $y \in [0, 1]$), then

$$p(\mathcal{T}x, \mathcal{S}y) = \max\left\{\frac{x^2}{2(1+x)}, \frac{y^2}{4(1+y)}\right\} = \frac{x^2}{2(1+x)}.$$

On the other hand,

$$\begin{aligned} \mathbf{M}(x, y) &= \max\Bigg\{\varphi(p(x, y)), \varphi\left(p(x, \frac{x^2}{2(1+x)})\right), \\ &\quad \varphi\left(p\left(y, \frac{y^2}{4(1+y)}\right)\right), \varphi\left(\frac{1}{2}\left[p\left(x, \frac{y^2}{4(1+y)}\right) + p\left(y, \frac{x^2}{2(1+x)}\right)\right]\right)\Bigg\} \\ &= \max\left\{\varphi(x), \varphi(x), \varphi(y), \varphi\left(\frac{1}{2}\left[x + \max\left\{y, \frac{x^2}{2(1+x)}\right\}\right]\right)\right\} \\ &= \varphi(x). \end{aligned}$$

(It was used that the function φ is increasing and, since $x \ge y$ and $x \ge \frac{x^2}{2(1+x)}$, that $\frac{x+\max\{y, \frac{x^2}{2(1+x)}\}}{2} \le x$.) Hence in this case

$$p(\mathcal{T}x, \mathcal{S}y) \le \mathbf{M}(x, y)$$

is satisfied.

If $x > 1$ (and so $y = x$), then $p(\mathcal{T}x, \mathcal{S}y) = \frac{x}{4}$ and $\mathbf{M}(x, y) = \varphi(x) = \frac{x}{4}$. Hence, in all possible cases condition (3.3) holds. Also, it is clear that the condition of regularity of $\mathcal{X}$ is satisfied. Therefore, all conditions of Theorems 3.6 and 3.11 are satisfied, and so $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point $z = 0$ such that $p(z, z) = p(\mathcal{T}z, \mathcal{T}z) = p(\mathcal{S}z, \mathcal{S}z) = 0$.

On the other hand, consider the same problem in the standard metric $d(x, y)$ and take $x = 1$ and $y = \frac{1}{2}$. Then $d(\mathcal{T}x, \mathcal{S}y) = \left|\frac{1}{4} - \frac{1}{24}\right| = \frac{5}{24}$ and $\mathbf{M}(x, y) = \max\left\{\varphi\left(\frac{1}{2}\right), \varphi\left(\frac{3}{4}\right), \varphi\left(\frac{11}{24}\right), \varphi\left(\frac{29}{48}\right)\right\} = \varphi\left(\frac{3}{4}\right)$ and so

$$\mathbf{M}(x, y) = \varphi\left(\frac{3}{4}\right) = \frac{9}{56} < \frac{5}{24}.$$

Hence, $d(\mathcal{T}x, \mathcal{S}y) \le \mathbf{M}(x, y)$ does not hold and the existence of a common fixed point of $\mathcal{T}$ and $\mathcal{S}$ cannot be obtained from the known results in standard metric spaces (see, e.g., [20, Theorems 4 and 5]).

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