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Convergence in mean of some random Fourier series

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Abstract

For a symmetric stable process $X(t,\omega)$ with index $\alpha \in (1,2]$, $f \in L^p[0,2\pi]$, $p \geqslant \alpha$, $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\,dt$ and $A_n(\omega) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\,dX(t,\omega)$, we establish that the random Fourier–Stieltjes (RFS) series $\sum_{n=-\infty}^{\prime\infty} \frac{a_n A_n(\omega) e^{int}}{(in)^\beta}$ converges in the *mean* to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega)$, where f_β is the fractional integral of order β of the function f for $\frac{1}{p} < \beta < 1 + \frac{1}{p}$. Further it is proved that the RFS series $\sum_{n=-\infty}^{\prime\infty} \frac{a_n A_n(\omega) e^{int}}{(in)^\beta}$ is Abel summable to $\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega)$. Also we define fractional derivative of the sum $\sum_{n=-\infty}^{\infty} a_n A_n(\omega) e^{int}$ of order β for a_n, $A_n(\omega)$ as above and $\frac{1}{p} < 1-\beta < 1+\frac{1}{p}$. We have shown that the formal fractional derivative of the series $\sum_{n=-\infty}^{\infty} a_n A_n(\omega) e^{int}$ of order β exists in the sense of *mean*.

Keywords: Symmetric stable process; Random Fourier–Stieltjes series; Stochastic integral; Fractional integral

1. Introduction

Let $X(t,\omega)$, $t \in \mathbb{R}$ be a continuous stochastic process with independent increments and f be a continuous function in $[a,b]$. Then the stochastic integral $\int_a^b f(t)\,dX(t,\omega)$ is defined convergence in the probability and is a random variable (cf. Lukacs [3, p. 148, Theorem (6.2.3)]). If $X(t,\omega)$ is a symmetric stable process of index α, $\alpha \in (1,2]$ then the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dX(t,\omega)$ is defined convergence in the probability for $f \in L^p([0,2\pi])$, $p \geqslant \alpha$ (Nayak, Pattanayak and Mishra [4]).

If $X(t,\omega)$ is a symmetric stable process with independent increments of index $\alpha \in (1,2]$, then it is shown that the stochastic integral $\int_a^b f(t)\,dX(t,\omega)$ is defined in the sense of convergence in the mean (cf. Kwapień and Woyczyński [2]).

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Again if $X(t,\omega)$ is a symmetric stable process of index $\alpha \in (1,2]$ then it is shown that the RFS series

$$\sum_{n=-\infty}^{\infty} a_n A_n(\omega) e^{int} \tag{1}$$

converges in the mean to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)\,dX(u,\omega)$, where

$$A_n(\omega) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\,dX(t,\omega) \quad \text{for } n \in \mathbb{Z} \tag{2}$$

and $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt$ for $f \in L^p$, $p \geqslant \alpha$ and $n \in \mathbb{Z}$ (Pattanayak and Sahoo [6]).

In Pattanayak and Sahoo [5], it is shown that the RFS series $\sum_{n=-\infty}^{\prime\infty} \frac{a_n A_n(\omega)e^{int}}{(in)^\beta}$ (where $\sum'$ means, that the summation does not include the term $n = 0$) converges in the probability to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega)$ with $\beta \in (\frac{1}{p}, 1+\frac{1}{p})$, where f_β is the fractional integral of f. But we know that the fractional integral f_β of order β belongs to L^p, $\forall p \geqslant 1$, if $f \in L^p([0,2\pi])$ with $p \geqslant 1$, $\beta \in (\frac{1}{p}, 1+\frac{1}{p})$ (cf. Zygmund [7, vol. II, p. 138]). We have shown in this paper that the RFS series $\sum_{n=-\infty}^{\prime\infty} \frac{a_n A_n(\omega)e^{int}}{(in)^\beta}$ converges in the *mean* to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega)$, with $\beta \in (\frac{1}{p}, 1+\frac{1}{p})$.

If $f \in L^p([0,2\pi])$, $p > 1$ and

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\,dt \tag{3}$$

for $n \in \mathbb{Z}$ are the Fourier coefficients of f, then it is easy to see that for each r with $0 \leqslant r < 1$, the series

$$\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int} \tag{4}$$

converges uniformly and represents a continuous function on $[0,2\pi]$. Let us write

$$f_r(t) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}. \tag{5}$$

Then the stochastic integral

$$\int_0^{2\pi} f_r(t)\,dX(t,\omega) \tag{6}$$

is defined convergence in the mean. Since each $f_r(t)$ is continuous, it belongs to $L^p([0,2\pi])$ for all $p > 1$ and $a_n r^{|n|}$, $n \in \mathbb{Z}$ are the Fourier coefficients of $f_r(t)$. So the random series

$$\sum_{n=-\infty}^{\infty} a_n A_n r^{|n|} e^{int} \tag{7}$$

will converge to the stochastic integral

$$\frac{1}{2\pi}\int_0^{2\pi} f_r(t-u)\,dX(u,\omega) \tag{8}$$

in the sense of convergence in mean. Here $f_r(\cdot)$ is the harmonic extension of f to the disc $\{z\colon\ |z| < 1\}$ given by the Poission integral

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)f(t)\,dt}{1-2r\cos(\theta-t)+r^2}. \tag{9}$$

It is shown that $\int_0^{2\pi} f_r(t-u)\,dX(u,\omega)$ converges to $\int_0^{2\pi} f(t-u)\,dX(u,\omega)$ as $r\to 1^-$ in the sense of mean (Pattanayak and Sahoo [6]).

This would mean $\sum_{n=-\infty}^{\infty} a_n A_n e^{int}$ is Abel summable to $\int_0^{2\pi} f(t-u)\,dX(u,\omega)$. A Fourier series $\sum a_n e^{int}$ is said to be Abel summable to "s" if for r with $0\leqslant r<1$, $\lim_{r\to 1}\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}=s$.

We have shown that the RFS series $\sum_{n=-\infty}^{\prime\infty}\frac{a_n A_n(\omega)e^{int}}{(in)^\beta}$ is Abel summable to $\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega)$ in the sense of *mean*.

2. Definitions

Definition 2.1. Let f be defined in a closed interval I, and let

$$\omega(\delta; f)=\sup\bigl\{\bigl|f(x_2)-f(x_1)\bigr|:\ x_1,x_2\in I,\ |x_2-x_1|\leqslant\delta\bigr\}.$$

The function $\omega(\delta; f)$ is called the *modulus of continuity* of f.

Definition 2.2. The class λ_β, for $0\leqslant\beta<1$, is the class of functions f on the closed interval I whose modulus of continuity $\omega(\delta; f)$ satisfies the condition $\omega(\delta; f)=o(\delta^\beta)$.

Definition 2.3. Let $\sum_{n=-\infty}^{\infty} a_n A_n(\omega)e^{int}$ be a RFS series, where $A_n(\omega)$ are the random variables as defined in (2) with $X(t,\omega)$ a symmetric stable process of index α, $1<\alpha\leqslant 2$ and a_n are the Fourier coefficients of some $f\in L^p([0,2\pi])$, $p\geqslant\alpha$ with $\int_0^{2\pi} f(t)\,dt=0$. Then the *fractional integral* of this RFS series of order β such that $\frac{1}{p}<\beta<1+\frac{1}{p}$ is defined to be $\sum_{n=-\infty}^{\prime\infty}\frac{a_n A_n(\omega)e^{int}}{(in)^\beta}$ which converges in the sense of mean to the stochastic integral $\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega)$, where f_β is the fractional integral of f of order β.

Let us write

$$F_\beta(t,\omega)=\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega).$$

Now this definition of fractional integration leads to the following definition of *fractional differentiation* of the RFS series (1).

Definition 2.4. The RFS series $\sum_{n=-\infty}^{\infty} a_n A_n(\omega)e^{int}$ as in Definition(2.3) is said to have *fractional derivative* of order β in the sense of mean at $t=t_0$, if for $\beta>0$ with $\frac{1}{p}<1-\beta<1+\frac{1}{p}$, the stochastic integral $F_{1-\beta}(t,\omega)$ is *differentiable* in the sense of mean.

Denote this derivative by $F^\beta(t,\omega)$.

Definition 2.5. If the fractional derivative of the RFS series (1) exists at each $t\in[0,2\pi]$ then it is said to have fractional derivative of order β in mean in $[0,2\pi]$.

3. Results

Theorem 3.1. *Let $X(t,\omega)$ be a symmetric stable process of index α, $1<\alpha\leqslant 2$, and let $A_n(\omega)$ be defined as in* (2). *Suppose a_n are the Fourier coefficients of some $f\in L^p([0,2\pi])$, $p\geqslant\alpha$ with $\int_0^{2\pi} f(t)\,dt=0$. Then the RFS series*

$$\sum_{n=-\infty}^{\infty}{}' \frac{a_n A_n(\omega)e^{int}}{(in)^\beta} \tag{10}$$

converges in the mean to the stochastic integral

$$\frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\,dX(u,\omega) \tag{11}$$

where f_β is the fractional integral of f of order β.

Proof of this theorem requires the following lemma.

Lemma 3.2. *If $X(t,\omega)$ is a symmetric stable process with independent increment of index α, $1<\alpha\leqslant 2$ and $f\in L^p([a,b])$, $p\geqslant\alpha$ then the following inequality holds*:

$$E\left(\left|\int_a^b f(t)\,dX(t,\omega)\right|\right)\leqslant\frac{4}{\pi(\alpha-1)}\int_a^b\left|f(t)\right|^\alpha dt+\frac{2}{\pi}\int_{|u|>1}\frac{1-\exp(-|u|^\alpha\int_a^b|f(t)|^\alpha\,dt)}{u^2}\,du.$$

Proof. Let $X(t,.)$ be a symmetric stable process with independent increments and let the characteristic function of the increment $X(t_1)-X(t_2)$ is equal to $\exp(-|t_1-t_2||u|^\alpha)$. We know that the stochastic integral $\int_a^b f(t)\,dX(t)$ exists in the sense of convergence in the mean for $f\in L^\alpha([a,b])$ (cf. Kwapień and Woyczyński [2]), and the characteristic function of this stochastic integral is

$$\Psi(u)=\exp\left(-|u|^\alpha\int_a^b\left|f(t)\right|^\alpha dt\right).$$

Expressing the expectation of the absolute value of a random variable in terms of its characteristic function (cf. Chow and Teicher [1, p. 285]), we get:

$$\begin{aligned}E\left|\int_a^b f(t)\,dX(t)\right|&=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1-\operatorname{Re}\Psi(u)}{u^2}\,du\\&=\frac{2}{\pi}\int_{|u|\leqslant 1}\frac{1-\operatorname{Re}\Psi(u)}{u^2}\,du+\frac{2}{\pi}\int_{|u|>1}\frac{1-\operatorname{Re}\Psi(u)}{u^2}\,du.\end{aligned}$$

But

$$\begin{aligned}\int_{|u|\leqslant 1}\frac{1-\operatorname{Re}\Psi(u)}{u^2}\,du&=\int_{-1}^{1}\frac{1-\exp(-|u|^\alpha\int_a^b|f(t)|^\alpha\,dt)}{u^2}\,du\\&\leqslant\int_{-1}^{1}\frac{|u|^\alpha\int_a^b|f(t)|^\alpha\,dt}{u^2}\,du\quad\left(\because 1-e^{-x}\leqslant x,\ \text{for } x>0\right)\\&=2\int_0^1|u|^{\alpha-2}\,du\int_a^b\left|f(t)\right|^\alpha dt\\&=\frac{2}{\alpha-1}\int_a^b\left|f(t)\right|^\alpha dt.\end{aligned}$$

Therefore

$$E\left|\int_a^b f(t)\,dX(t)\right|\leqslant\frac{4}{\pi(\alpha-1)}\int_a^b\left|f(t)\right|^\alpha dt+\frac{2}{\pi}\int_{|u|>1}\frac{1-\exp(-|u|^\alpha\int_a^b|f(t)|^\alpha\,dt)}{u^2}\,du.$$

Hence the result. □

Proof of Theorem 3.1. Consider the fractional integral f_β of f of order $\beta > 0$. This $f_\beta \in \lambda_{\beta-\frac{1}{p}}$ (Definition 2.2) as $f \in L^p([0, 2\pi])$, $(p \geqslant 1)$ and $\beta > 0$ is such that $\frac{1}{p} < \beta < 1 + \frac{1}{p}$ (cf. Zygmund [7, vol. II, p. 138]). It is clear that $f_\beta \in \lambda_0$, the class of continuous functions and hence $f_\beta \in L^p([0, 2\pi])$ for all $p \geqslant 1$. Hence (cf. Kwapień and Woyczyński [2]), we have that $\int_0^{2\pi} f_\beta(t)\, dX(t, \omega)$ is defined in the sense of mean.

Now let

$$S_n(t) = \sum_{k=-n}^{n}{}' \frac{a_k A_k(\omega) e^{ikt}}{(ik)^\beta}$$

be the nth partial sum of the RFS series (Theorem 3.1) and that of f_β be

$$s_n(t) = \sum_{k=-n}^{n}{}' \frac{a_k e^{ikt}}{(ik)^\beta}.$$

Therefore

$$\begin{aligned} S_n(t) &= \sum_{k=-n}^{n}{}' \frac{a_k A_k(\omega) e^{ikt}}{(ik)^\beta} \\ &= \sum_{k=-n}^{n}{}' \frac{a_k}{(ik)^\beta} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-iku}\, dX(u, \omega) \right) e^{ikt} \\ &= \sum_{k=-n}^{n}{}' \frac{a_k}{(ik)^\beta} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{ik(t-u)}\, dX(u, \omega) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-n}^{n}{}' \frac{a_k}{(ik)^\beta} e^{ik(t-u)}\, dX(u, \omega) \\ &= \frac{1}{2\pi} \int_0^{2\pi} s_n(t-u)\, dX(u, \omega). \end{aligned}$$

Now

$$\begin{aligned} &E\left(\left| \frac{1}{2\pi} \int_0^{2\pi} f_\beta(t-u)\, dX(u, \omega) - S_n(t) \right| \right) \\ &\quad = E\left(\left| \frac{1}{2\pi} \int_0^{2\pi} f_\beta(t-u)\, dX(u, \omega) - \frac{1}{2\pi} \int_0^{2\pi} s_n(t-u)\, dX(u, \omega) \right| \right) \\ &\quad = E\left(\left| \frac{1}{2\pi} \int_0^{2\pi} \left[f_\beta(t-u) - s_n(t-u) \right] dX(u, \omega) \right| \right) \\ &\quad \leqslant \frac{2}{\pi^2(\alpha - 1)} \int_0^{2\pi} \left| f_\beta(t-u) - s_n(t-u) \right|^\alpha du \\ &\qquad + \frac{1}{\pi^2} \int_{|v|>1} \frac{1 - \exp(-|v|^\alpha \int_0^{2\pi} |f_\beta(t-u) - s_n(t-u)|^\alpha du)}{v^2}\, dv \quad \text{(by Lemma 3.2)}. \end{aligned}$$

It is known that (cf. Zygmund [7, p. 266]) for $f_\beta \in L^p([0, 2\pi])$, $p > 1$,

$$\lim_{n\to\infty} \int_0^{2\pi} \left| f_\beta(t-u) - s_n(t-u) \right|^p du = 0.$$

Now if $p \geqslant \alpha$ then we have:

$$\lim_{n\to\infty} E\left(\left| \frac{1}{2\pi} \int_0^{2\pi} f_\beta(t-u)\, dX(u,\omega) - S_n(t) \right| \right) = 0.$$

Hence the result. □

In the next theorem it is established that the RFS series

$$\sum_{n=-\infty}^{\infty}{}' \frac{a_n A_n(\omega) e^{int}}{(in)^\beta}$$

is Abel summable to

$$\frac{1}{2\pi} \int_0^{2\pi} f_\beta(t-u)\, dX(u,\omega)$$

in the sense of mean.

Theorem 3.3. *Let $X(t,\omega)$ be a symmetric stable process of index α, with $1 < \alpha \leqslant 2$, and $f(t) \in L^p([0, 2\pi])$, $p \geqslant \alpha$. If $A_n(\omega) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\, dX(t,\omega)$ and $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\, dt$, then the RFS series*

$$\sum_{n=-\infty}^{\infty}{}' \frac{a_n A_n(\omega) e^{int}}{(in)^\beta}$$

is Abel summable to

$$\frac{1}{2\pi} \int_0^{2\pi} f_\beta(t-u)\, dX(u,\omega)$$

in the sense of mean.

Proof. As we know, for each r with $0 \leqslant r < 1$, the series

$$\sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}$$

converges uniformly and represents a continuous function and hence belongs to L^p, for all $p > 1$.

Therefore the series

$$\sum_{n=-\infty}^{\infty}{}' \frac{a_n r^{|n|} e^{int}}{(in)^\beta}$$

also converges uniformly and represents a continuous function and hence belongs to L^p, for all $p > 1$. Denote

$$f_{r_\beta}(t) = \sum_{n=-\infty}^{\infty}{}' \frac{a_n r^{|n|} e^{int}}{(in)^\beta}, \quad 0 \leqslant r \leqslant 1.$$

Since $f_\beta \in L^p,\ p \geqslant \alpha$, a_n are the Fourier coefficients of $f \in L^p$, and $0 \leqslant r < 1$, each $f_{r_\beta} \in L^p,\ p \geqslant \alpha$. So by Theorem 3.1 the RFS series

$$\sum_{n=-\infty}^{\infty}{}' \frac{a_n A_n r^{|n|} e^{int}}{(in)^\beta}$$

will converge to the stochastic integral

$$\frac{1}{2\pi} \int_0^{2\pi} f_{r_\beta}(t-u)\, dX(u,\omega)$$

in the sense of mean. Since the RFS series

$$\sum_{n=-\infty}^{\infty}{}' \frac{a_n A_n e^{int}}{(in)^\beta}$$

converges to the stochastic integral

$$\frac{1}{2\pi} \int_0^{2\pi} f_\beta(t-u)\, dX(u,\omega)$$

in the sense of mean, we have:

$$\begin{aligned}
&E\left(\left|\frac{1}{2\pi}\int_0^{2\pi} f_{r_\beta}(t-u)\, dX(u,\omega) - \frac{1}{2\pi}\int_0^{2\pi} f_\beta(t-u)\, dX(u,\omega)\right|\right) \\
&\quad = E\left(\left|\frac{1}{2\pi}\int_0^{2\pi} \bigl[f_{r_\beta}(t-u) - f_\beta(t-u)\bigr]\, dX(u,\omega)\right|\right) \\
&\quad \leqslant \frac{2}{\pi^2(\alpha-1)} \int_0^{2\pi} \bigl|f_{r_\beta}(t-u) - f_\beta(t-u)\bigr|^\alpha\, du \\
&\qquad + \frac{1}{\pi^2} \int_{|v|>1} \frac{1-\exp(-|v|^\alpha \int_0^{2\pi} |f_{r_\beta}(t-u) - f_\beta(t-u)|^\alpha du)}{v^2}\, dv \quad \text{(by Lemma 3.2).}
\end{aligned}$$

We know that the integral $\int_0^{2\pi} |f_{r_\beta}(t-u) - f_\beta(t-u)|^\alpha\, du$ tends to 0 as $r \to 1$ if $f_\beta \in L^p,\ p > 1$ (cf. Zygmund [7, p. 150]). As $\frac{1}{v^2}$ in the integrand of the second integral is dominated by "1," the second integral also tends to "0."

Hence the result. □

A sufficient condition for the existence of fractional derivative of order β in the sense of mean of the RFS series (1) is obtained in the following theorem.

Theorem 3.4. *The RFS series* (1) *having conditions as stated in Definition* 2.3 *has fractional derivative of order β in the sense of mean, for $\beta > 0$ with $\frac{1}{p} < 1 - \beta < 1 + \frac{1}{p}$ if*

$$\sum_{n=-\infty}^{\infty} \bigl|n^\beta a_n\bigr|^2 < \infty.$$

Proof of this theorem requires the following lemma.

Lemma 3.5. *Let $X(t,\omega)$ be a symmetric stable process, $A_n(\omega)$, a_n as defined above. Then the sum function of the RFS series* (1) *is differentiable in the sense of mean if a_n satisfy the condition*

$$\sum_{n=-\infty}^{\infty} |na_n|^2 < \infty.$$

Proof. By the condition on the coefficients, we have that there exists a function $g \in L^2$, such that

$$na_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} g(t)\,dt.$$

Let

$$S(y,\omega) = \sum_{n=-\infty}^{\infty} a_n A_n(\omega) e^{iny}.$$

Then

$$\begin{aligned}\frac{S(y+h,\omega)-S(y,\omega)}{h} &= \sum_{n=-\infty}^{\infty} a_n A_n(\omega)\left(\frac{e^{in(y+h)}-e^{iny}}{h}\right)\\ &= \sum_{n=-\infty}^{\infty} a_n A_n(\omega) e^{iny}\left(\frac{e^{inh}-1}{h}\right)\\ &= \sum_{n=-\infty}^{\infty} ina_n A_n(\omega) e^{iny}\left(\frac{e^{inh}-1}{inh}\right)\\ &= i\sum_{n=-\infty}^{\infty} d_n A_n(\omega) e^{iny}\end{aligned}$$

which is a RFS series with weights d_n, where $d_n = b_n(\frac{e^{inh}-1}{inh})$ and $b_n = na_n$.

Again

$$d_n = b_n\left(\frac{e^{inh}-1}{inh}\right) = b_n\frac{1}{h}\int_{-h}^{0} e^{-int}\,dt = na_n\frac{1}{h}\int_{-h}^{0} e^{-int}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{h}\int_{-h}^{0} g(y-t)\,dt\, e^{-iny}\,dy.$$

Thus d_n is the Fourier coefficients of an integral which is absolutely continuous and hence belongs to L^p, $p > 0$. So we have

$$i\sum_{n=-\infty}^{\infty} d_n A_n(\omega) e^{iny}$$

converges in the mean to

$$\frac{i}{2\pi}\int_0^{2\pi}\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi\, dX(t,\omega)$$

by the result of Pattanayak and Sahoo [6].

Thus

$$E\left(\left|\frac{S(y+h,\omega)-S(y,\omega)}{h} - \frac{1}{2\pi}\int_0^{2\pi} g(y-t)\,dX(t,\omega)\right|\right)$$

$$= E\left(\left|\frac{S(y+h,\omega)-S(y,\omega)}{h} - \frac{i}{2\pi}\int_0^{2\pi}\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi\,dX(t,\omega)\right.\right.$$
$$\left.\left. + \frac{i}{2\pi}\int_0^{2\pi}\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi\,dX(t,\omega) - \frac{1}{2\pi}\int_0^{2\pi} g(y-t)\,dX(t,\omega)\right|\right)$$
$$= E\left(\left|\frac{i}{2\pi}\int_0^{2\pi}\left(\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi - g(y-t)\right)dX(t,\omega)\right|\right)$$

($\because$ $\frac{S(y+h,\omega)-S(y,\omega)}{h}$ converges in the mean to $\frac{i}{2\pi}\int_0^{2\pi}\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi\,dX(t,\omega)$).

But we know that the characteristic function of

$$\int_0^{2\pi}\left(\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi - g(y-t)\right)dX(t,\omega)$$

is:

$$e^{-c|u|^\alpha \int_0^{2\pi} |\frac{1}{h}\int_{-h}^0 g(y-t-\xi)\,d\xi - g(y-t)|^\alpha\,dt}.$$

Therefore

$$E\left(\left|\frac{i}{2\pi}\int_0^{2\pi}\left(\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi - g(y-t)\right)dX(t,\omega)\right|\right)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{-c|u|^\alpha \int_0^{2\pi} |\frac{1}{h}\int_{-h}^0 g(y-t-\xi)\,d\xi - g(y-t)|^\alpha\,dt}}{u^2}\,du,$$

and by Lemma 3.2, we have:

$$E\left(\left|\frac{i}{2\pi}\int_0^{2\pi}\left(\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi - g(y-t)\right)dX(t,\omega)\right|\right)$$
$$\leqslant \frac{2}{\pi^2(\alpha-1)}\int_0^{2\pi}\left|\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi - g(y-t)\right|^\alpha dt$$
$$+ \frac{1}{\pi^2}\int_{|v|>1}\frac{1-e^{-|v|^\alpha \int_0^{2\pi} |\frac{1}{h}\int_{-h}^0 g(y-t-\xi)\,d\xi - g(y-t)|^\alpha\,dt}}{v^2}\,dv.$$

It is known that for $g \in L^p([0,2\pi])$, $p > 1$,

$$\lim_{h\to 0}\int_0^{2\pi}\left|\frac{1}{h}\int_{-h}^{0} g(y-t-\xi)\,d\xi - g(y-t)\right|^\alpha dt = 0.$$

Now if $p \geqslant \alpha$, then we have:

$$\lim_{h\to 0} E\left(\left|\frac{S(y+h,\omega)-S(y,\omega)}{h} - \frac{1}{2\pi}\int_0^{2\pi} g(y-t)\,dX(t,\omega)\right|\right) = 0.$$

Hence the result. □

Proof of Theorem 3.4. If β is such that $\frac{1}{p} < 1-\beta < 1+\frac{1}{p}$, the fractional integration of (1) of order $1-\beta$ is defined, which is:

$$F_{1-\beta}(t,\omega) = \sum_{n=-\infty}^{\infty}{}' \frac{a_n A_n(\omega) e^{int}}{(in)^{1-\beta}}.$$

By Lemma 3.5, $F_{1-\beta}$ is differentiable in the sense of mean if

$$\sum_{n=-\infty}^{\infty} \left| \frac{n a_n}{(in)^{1-\beta}} \right|^2 < \infty,$$

that is:

$$\sum_{n=-\infty}^{\infty} \left| n^{\beta} a_n \right|^2 < \infty.$$

Hence the result. □

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