Mathematical and Computer Modelling 51 (2010) 349–360

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Differential subordination and superordination for certain subclasses of p-valent functions

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ARTICLE INFO

Article history:
Received 22 March 2009
Received in revised form 27 December 2009
Accepted 28 December 2009

Keywords:
Analytic function
p-valent
Multiplier transformation
Differential subordination

ABSTRACT

In this paper we derive some subordination and superordination results for certain p-valent analytic functions in the open unit disc, which are acted upon by a class of extended multiplier transformations. Relevant connection of the results, which are presented in this paper with various known results are also considered.

1. Introduction

Let $H(U)$ denote the class of analytic functions in the open unit disc $U = \{z : z \in C : |z| < 1\}$ and let $H[a, p]$ denote the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (a \in C; p \in N = \{1, 2, \ldots\}).$$

Also, let $A(p)$ be the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N), \tag{1.1}$$

and set $A \equiv A(1)$. For functions $f(z) \in A(p)$, given by (1.1), and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in N), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z) \quad (z \in U; p \in N). \tag{1.3}$$

For $f, g \in H(U)$, we say that the function f is subordinate to g, if there exists a Schwarz function w, i.e.,$w \in H(U)$ with $w(0) = 0$ and $|w(z)| < 1, z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by

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0895-7177/$ – see front matter
doi:10.1016/j.mcm.2009.12.027

$f(z) \prec g(z)$. It is well known that, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Supposing that h and k are two analytic functions in U, let

$$\phi(r, s, t; z) : C^3 \times U \to C.$$

If h and $\varphi(h(z), zh'(z), z^2h''(z); z)$ are univalent functions in U and if h satisfies the second-order superordination

$$k(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z), \tag{1.4}$$

then h is called to be a solution of the differential superordination (1.4). A function $q \in H(U)$ is called a subordinant of (1.4), if $q(z) \prec h(z)$ for all the functions h satisfying (1.4). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (1.4), is said to be the best subordinant.

Recently, Miller and Mocanu [1] obtained sufficient conditions on the functions k, q and φ for which the following implication holds:

$$k(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z) \Rightarrow q(z) \prec h(z).$$

Using these results, Bulboaca [2] considered certain classes of first-order differential superordinations, as well as superordination-preserving integral operators [3]. Ali et al. [4], using the results from [2], obtained sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U.

Very recently, Shanmugam et al. [5–8] obtained the sandwich results for certain classes of analytic functions. Further subordination results can be found in [9–14].

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \notin \overline{Z_0} = \{0, -1, -2, \ldots\}$; $j = 1, 2, \ldots, s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [15, p.19])

$$ {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, \ldots, (\alpha_q)_k}{(\beta_1)_k, \ldots, (\beta_s)_k} . \frac{z^k}{k!} \quad (q \le s + 1; q, s \in N_0 = N \cup \{0\}; z \in U), \tag{1.5}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in C \setminus \{0\}), \\ \theta(\theta + 1) \ldots (\theta + \nu - 1) & (\nu \in N; \theta \in C). \end{cases} \tag{1.6}$$

Let

$$\begin{aligned} h_{p,q,s}(\alpha_1, \beta_1; z) &= z^p \, {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \\ &= z^p + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k (1)_k} z^{p+k}, \end{aligned} \tag{1.7}$$

and using the Hadamard product, we define the following operator $I^{m,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f : U \to U$ by

$$I^{0,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) = f(z) * h_{p,q,s}(\alpha_1, \beta_1; z);$$

$$I^{1,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) = (1 - \lambda)(f(z) * h_{p,q,s}(\alpha_1, \beta_1; z)) + \frac{\lambda}{(p + \ell)z^{\ell-1}} (z^\ell f(z) * h_{p,q,s}(\alpha_1, \beta_1; z))';$$

and

$$I^{m,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) = I^{1,\ell}_{p,q,s,\lambda}(I^{m-1,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z)). \tag{1.8}$$

If $f \in A(p)$, then from (1.1) and (1.8), we can easily see that

$$I^{m,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p + \ell + \lambda(k - p)}{p + \ell}\right]^m \frac{(\alpha_1)_{k-p} \ldots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \ldots (\beta_s)_{k-p}(1)_{k-p}} a_k z^k, \tag{1.9}$$

where $m \in N_0 = N \cup \{0\}$, $\ell \ge 0$, $\lambda \ge 0$ and $p \in N$.

We note that when $p = 1$ and $\ell = 0$, the operator $I^{m,0}_{1,q,s,\lambda}(\alpha_1, \beta_1)f(z) = D^m_\lambda(\alpha_1, \beta_1)f(z)$ was studied by Selvaraj and Karthikeyan [16].

We also note that:

(i) $I^{0,\ell}_{p,q,s,\lambda}f(z) = H_{p,q,s}(\alpha_1, \beta_1)f(z)$ (see Dziok and Srivastava [17,18]);
(ii) For $q = s+1, \alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, we get the operator $I_p(m, \lambda, \ell)$ (see Catas [19]);
(iii) For $q = s+1, \alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, $\ell = 0$ and $\lambda = 1$, we get the operator D^m_p (see Kamali and Orhan [20] and Aouf and Mostafa [21]);
(iv) For $q = s+1, \alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, and $\lambda = 1$, we get the operator $I_p(m, \ell)$ (see Kumar et al. [22]);
(v) For $q = s+1, \alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, $p = \lambda = 1$ and $\ell = 0$, we obtain the Salagean operator D^m (see Salagean [23]);
(vi) For $q = s+1, \alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, $p = \lambda = 1$, we get the operator I^m_ℓ (see Cho and Srivastava [24] and Cho and Kim [25]).
(vii) For $q = s+1, \alpha_i = 1$ $(i = 1, \dots, s+1)$, $\beta_j = 1$ $(j = 1, \dots, s)$, $p = 1$ and $\ell = 0$, we obtain the operator D^m_λ (see Al-Oboudi [26]).

By specializing the parameters $m, \lambda, \ell, p, q, s, \alpha_i$ $(i = 1, \dots, q)$ and β_j $(j = 1, \dots, s)$, we obtain various new operators, e.g.,

(i) $I^m_{p,2,1,\lambda}(n+p, 1; 1)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m \frac{(p+n)_{k-p}}{(1)_{k-p}} a_k z^k$ $(n > -p; p, n \in N)$;
(ii) $I^{m,\ell}_{p,2,1,\lambda}(a, 1; c)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k$ $(a \in R; c \in R \setminus \overline{Z_0})$;
(iii) $I^{m,\ell}_{p,2,1,\lambda}(p+1, 1; n+p)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m \frac{(p+1)_{k-p}}{(n+p)_{k-p}} a_k z^k$ $(n \in Z; p \in N; n > -p)$;
(iv) $I^{m,\ell}_{p,2,1,\lambda}(p+1, 1; p+1-\delta)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m \frac{(p+1)_{k-p}}{(p+1-\delta)_{k-p}} a_k z^k$ $(p \in N; 0 \le \delta < 1)$;
(v) $I^{m,\ell}_{p,2,1,\lambda}(p+\delta, c; a)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m \frac{(p+\delta)_{k-p}(c)_{k-p}}{(a)_{k-p}(1)_{k-p}} a_k z^k$ $(a, c \in R \setminus \overline{Z_0}; \delta > -p; p \in N)$;
(vi) $I^{m,\ell}_{p,2,1,\lambda}(p+\delta, 1; p+\delta+1)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m \frac{(p+\delta)_{k-p}}{(p+\delta+1)_{k-p}} a_k z^k$ $(\delta > -p; p \in N)$.

It can be easily verified from the definition (1.9) that:

$$z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z))' = \alpha_1 I^{m,\ell}_{p,q,s,\lambda}(\alpha_1+1, \beta_1)f(z) - (\alpha_1 - p)I^{m,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) \tag{1.10}$$

and

$$\lambda z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z))' = (p+\ell)I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) - [p(1-\lambda)+\ell]I^{m,\ell}_{p,q,s,\lambda}(\alpha_1, \beta_1)f(z) \quad (\lambda > 0). \tag{1.11}$$

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and results.

Definition ([1]). Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta : \zeta \in \partial \text{ and } \lim_{z \to \zeta} f(z) = \infty\right\} \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1 ([27]). *Let the function $q(z)$ be univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(i) *$Q(z)$ is starlike univalent in U,*
(ii) *$Re(\frac{zh'(z)}{Q(z)}) > 0$ for $z \in U$.*

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

Lemma 2 ([7]). *Let q be a convex univalent function in U and let $\psi \in C$, $\gamma \in C^* = C \setminus \{0\}$ with*

$$\mathrm{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{\psi}{\gamma}\right)\right\}.$$

If $p(z)$ is analytic in U with $p(0) = q(0)$ and

$$\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z), \tag{2.3}$$

then

$$p(z) \prec q(z) \quad (z \in U)$$

and q is the best dominant.

Lemma 3 ([28]). *Let $q(z)$ be convex univalent in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$;*
(ii) *$zq'(z)\varphi(q(z))$ is starlike univalent in U.*
If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.4}$$

then

$$q(z) \prec p(z) \quad (z \in U)$$

and $q(z)$ is the best subordinant.

Lemma 4 ([1]). *Let q be convex univalent in U and $\gamma \in C$. Further assume that $\mathrm{Re}(\gamma) > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + \gamma zp'(z)$ is univalent in U, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z), \tag{2.5}$$

implies

$$q(z) \prec p(z) \quad (z \in U)$$

and q is the best subordinant.

The last lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular case.

Lemma 5 ([29]). *The function $q(z) = (1-z)^{-2ab}$ is univalent in the unit disc U if and only if $|2ab - 1| \le 1$ or $|2ab + 1| \le 1$.*

3. Subordination results

Theorem 1. *Let q be univalent in U, with $q(0) = 1$, and suppose that*

$$\mathrm{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\frac{p(p+\ell)}{\lambda}\mathrm{Re}\left(\frac{1}{\alpha}\right)\right\}, \quad z \in U, \tag{3.1}$$

where $\ell \ge 0$, $\lambda > 0$, $\alpha \in C^*$ and $p \in N$. If $f \in A(p)$ satisfies the subordination

$$\frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) + \frac{p-\alpha}{p}\left(\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) \prec q(z) + \frac{\lambda\alpha zq'(z)}{p(p+\ell)}, \tag{3.2}$$

then

$$\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p} \prec q(z), \tag{3.3}$$

and q is the best dominant of (3.2).

Proof. If we consider the analytic function

$$h(z)=\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\quad(z\in U),\tag{3.4}$$

by differentiating (3.4) logarithmically with respect to z, we deduce that

$$\frac{zh'(z)}{h(z)}=\frac{z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z))'}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}-p.\tag{3.5}$$

From (3.5), by using the identity (1.11), a simple computation shows that

$$\frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right)=h(z)+\frac{\alpha\lambda}{p(p+\ell)}zh'(z),$$

hence the subordination (3.2) is equivalent to

$$h(z)+\frac{\lambda\alpha}{p(p+\ell)}zh'(z)\prec q(z)+\frac{\lambda\alpha}{p(p+\ell)}zq'(z).\tag{3.6}$$

An application of Lemma 2, with $\psi=1$ and $\gamma=\frac{\lambda\alpha}{p(p+\ell)}$, leads to (3.3). □

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 1, where $-1\le B<A\le 1$, the condition (3.1) becomes

$$\mathrm{Re}\,\frac{1-Bz}{1+Bz}>\max\left\{0;-\frac{p(p+\ell)}{\lambda}\mathrm{Re}\left(\frac{1}{\alpha}\right)\right\},\quad z\in U.\tag{3.7}$$

It is easy to check that the function $\varphi(\zeta)=\frac{1-\zeta}{1+\zeta}$, $|\zeta|<|B|$, is convex in U, and since $\varphi(\overline{\zeta})=\overline{\varphi(\zeta)}$ for all $|\zeta|<|B|$, it follows that the image $\phi(U)$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\mathrm{Re}\,\frac{1-Bz}{1+Bz};z\in U\right\}=\frac{1-|B|}{1+|B|}>0.\tag{3.8}$$

Then, the inequality (3.7) is equivalent to

$$\frac{p(p+\ell)}{\lambda}\mathrm{Re}\left(\frac{1}{\alpha}\right)\ge\frac{|B|-1}{|B|+1},$$

hence we obtain the following result:

Corollary 1. *Let $m\in N_0$, $\ell\ge 0$, $\lambda>0$, $\alpha\in C^*$, $-1\le B<A\le 1$ and $p\in N$ with*

$$\max\left\{0;-\frac{p(p+\ell)}{\lambda}\mathrm{Re}\left(\frac{1}{\alpha}\right)\right\}\le\frac{1-|B|}{1+|B|}.$$

If $f\in A(p)$, and

$$\frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right)+\left(\frac{p-\alpha}{p}\right)\left(\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right)\prec\frac{1+Az}{1+Bz}+\frac{\lambda\alpha}{p(p+\ell)}\frac{(A-B)z}{(1+Bz)^2},\tag{3.9}$$

then

$$\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\prec\frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.9).

Taking $p=A=1$ and $B=-1$ in Corollary 1, we obtain the following corollary.

Corollary 2. *Let $m\in N_0$, $\ell\ge 0$, $\lambda>0$ and $\alpha\in C^*$ with*

$$\frac{(1+\ell)}{\lambda}\mathrm{Re}\left(\frac{1}{\alpha}\right)\ge 0.$$

If $f \in A$, and

$$\alpha\left(\frac{I^{m+1,\ell}_{1,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z}\right)+(1-\alpha)\left(\frac{I^{m,\ell}_{1,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z}\right)\prec\frac{1+z}{1-z}+\frac{2\lambda\alpha z}{(\ell+1)(1-z)^2}, \tag{3.10}$$

then

$$\frac{I^{m,\ell}_{1,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z}\prec\frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ is the best dominant of (3.10).

Theorem 2. *Let $q(z)$ be univalent in U, with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$. Let $\gamma,\mu\in C^*$ and $\nu,\eta\in C$ with $\nu+\eta\neq 0$. Let $f\in A(p)$ and suppose that f and q satisfy the next conditions:*

$$\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)+\eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p}\neq 0\quad(z\in U), \tag{3.11}$$

and

$$\mathrm{Re}\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0\quad(z\in U). \tag{3.12}$$

If

$$1+\gamma\mu\left[\frac{\nu z(I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z))'+\eta z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z))'}{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)+\eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}-p\right]\prec 1+\gamma\frac{zq'(z)}{q(z)}, \tag{3.13}$$

then

$$\left[\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)+\eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q(z)$$

and q is the best dominant of (3.13). (The power is the principal one).

Proof. Let denotes

$$h(z)=\left[\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)+\eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu}\quad(z\in U). \tag{3.14}$$

According to (3.11) the function $h(z)$ is analytic in U, and differentiating (3.14) logarithmically with respect to z, we obtain

$$\frac{zh'(z)}{h(z)}=\mu\left[\frac{\nu z(I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z))'+\eta z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z))'}{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)+\eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}-p\right].$$

In order to prove our result we will use Lemma 1. In this lemma consider

$$\theta(w)=1\quad\text{and}\quad\varphi(w)=\frac{\gamma}{w},$$

then θ is analytic in C and $\varphi(w)\neq 0$ is analytic in C^*. Also, if we let

$$Q(z)=zq'(z)\varphi(q(z))=\gamma\frac{zq'(z)}{q(z)},$$

and

$$g(z)=\theta(q(z))+Q(z)=1+\gamma\frac{zq'(z)}{q(z)}.$$

From (3.12), we see that $Q(z)$ is starlike function in U. From (3.12), we also have

$$\mathrm{Re}\frac{zg'(z)}{Q(z)}=\mathrm{Re}\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0\quad(z\in U)$$

and then, by using Lemma 1 we deduce that the subordination (3.13) implies $h(z)\prec q(z)$, and the function q is the best dominant of (3.13). □

Taking $\nu=0$, $\eta=1$, $\gamma=1$ and $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 2, it is easy to check that the assumption (3.12) holds whenever $-1\leq A<B\leq 1$, hence we obtain the next result.

Corollary 3. *Let $-1 \leq A < B \leq 1$ and $\mu \in C^*$. Let $f \in A(p)$ and Suppose that*

$$\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p} \neq 0 \quad (z \in U)\ (m \in N_0;\ \ell \geq 0;\ \lambda > 0;\ p \in N).$$

If

$$1+\mu\left[\frac{z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z))'}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}-p\right] \prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{3.15}$$

then

$$\left[\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right]^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.15). (The power is the principal one).

Putting $\nu = 0$, $\eta = p = 1$, $m = 0$, $q = s+1$, $\alpha_i = 1$ $(i = 1,\ldots,s+1)$, $\beta_j = 1$ $(j = 1,\ldots,s)$, $\gamma = \frac{1}{ab}(a, b \in C^*)$, $\mu = a$, and $q(z) = (1-z)^{-2ab}$ in Theorem 2, then combining this to gather with Lemma 5 we obtain the next result due to Obradovic et al. [9, Theorem 1].

Corollary 4 *([9]). Let $a, b \in C^*$ such that $|2ab-1| \leq 1$ or $|2ab+1| \leq 1$. Let $f \in A$ and suppose that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If*

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right) \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^{a} \prec (1-z)^{-2ab} \tag{3.16}$$

and $(1-z)^{-2ab}$ is the best dominant of (3.17). (The power is the principal one).

Remark 1. For $a = 1$, Corollary 4 reduces to the recent result of Srivastava and Lashin [13].

Putting $\nu = 0$, $\eta = p = \gamma = 1$, $q = s+1$, $\alpha_i = 1$ $(i = 1,\ldots,s+1)$, $\beta_j = 1$ $(j = 1,\ldots,s)$ and $q(z) = (1+Bz)^{\frac{\mu(A-B)}{B}}$ in Theorem 2, and using Lemma 2 we obtain the next result.

Corollary 5. *Let $-1 \leq A < B \leq 1$ with $B \neq 0$, and suppose that $\left|\frac{\mu(A-B)}{B}-1\right| \leq 1$ or $\left|\frac{\mu(A-B)}{B}+1\right| \leq 1$. Let $f \in A$ such that $\frac{f(z)}{z} \neq 0$ for all $z \in U$, and let $\mu \in C^*$. If*

$$1+\mu\left(\frac{zf'(z)}{f(z)}-1\right) \prec \frac{1+[B+\mu(A-B)]z}{1+Bz}.$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}}, \tag{3.17}$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (3.17). (The power is the principal one).

Putting $\nu = 0$, $\eta = p = 1$, $q = s+1$, $\alpha_i = 1$ $(i = 1,\ldots,s+1)$, $\beta_j = 1$ $(j = 1,\ldots,s)$, $\gamma = \frac{e^{i\tau}}{ab\cos\tau}(a, b \in C^*;\ |\tau| < \frac{\pi}{2})$ and $q(z) = (1-z)^{-2ab\cos\tau e^{-i\tau}}$ in Theorem 2, we obtain the following result due to Aouf et al. [30, Theorem 1].

Corollary 6 *([30]). Let $a, b \in C^*$ and $|\tau| < \frac{\pi}{2}$ and suppose that $\left|2ab\cos\tau e^{-i\tau}-1\right| \leq 1$ or $\left|2ab\cos\tau e^{-i\tau}+1\right| \leq 1$. Let $f \in A$ and suppose that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If*

$$1+\frac{e^{i\tau}}{b\cos\tau}\left(\frac{zf'(z)}{f(z)}-1\right) \prec \frac{1+z}{1-z}$$

then

$$\left(\frac{f(z)}{z}\right)^{a} \prec (1-z)^{-2ab\cos\tau e^{-i\tau}} \tag{3.18}$$

and $(1-z)^{-2ab\cos\tau e^{-i\tau}}$ is the best dominant of (3.18). (The power is the principal one).

Theorem 3. *Let q be univalent in U, with* $q(0) = 1$*, let* $\mu, \gamma \in C^*$ *and let* $\delta, \Omega, \nu, \eta \in C$*. with* $\nu + \eta \neq 0$*. Let* $f(z) \in A(p)$ *and suppose that f and q satisfy the next two conditions:*

$$\frac{\nu I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z) + \eta I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)}{(\nu+\eta)z^p} \neq 0 \quad (z \in U),\ (m \in N_0;\ \ell \geq 0;\ \lambda > 0;\ p \in N), \tag{3.19}$$

and

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\delta}{\gamma}\right)\right\} \quad (z \in U). \tag{3.20}$$

If

$$\psi(z) = \left[\frac{\nu I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z) + \eta I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu} \cdot \left[\delta + \gamma\mu\left(\frac{\nu z(I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z))' + \eta z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z))'}{\nu I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z) + \eta I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)} - p\right)\right] + \Omega \tag{3.21}$$

and

$$\psi(z) \prec \delta q(z) + \gamma zq'(z) + \Omega, \tag{3.22}$$

then

$$\left[\frac{\nu I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z) + \eta I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu} \prec q(z),$$

and q is the best dominant of (3.22). (All the powers are the principal ones).

Proof. Let define the function h by

$$h(z) = \left[\frac{\nu I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z) + \eta I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu}. \tag{3.23}$$

According to (3.16), the function h is analytic in U, and differentiating (3.20) logarithmically with respect to z, we obtain

$$\frac{zh'(z)}{h(z)} = \mu\left[\frac{\nu z(I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z))' + \eta z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z))'}{\nu I_{p,,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z) + \eta I_{p,,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)} - p\right],$$

and hence

$$zh'(z) = \mu h(z)\left[\frac{\nu z(I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z))' + \eta z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z))'}{\nu I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1, \beta_1)f(z) + \eta I_{p,q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)} - p\right].$$

Let consider the next functions

$$\theta(w) = \delta w + \Omega, \qquad \varphi(w) = \gamma, \quad w \in C,$$
$$Q(z) = zq'(z)\varphi(q(z)) = \gamma zq'(z), \quad z \in U,$$

and

$$g(z) = \theta(q(z)) + Q(z) = \delta q(z) + \gamma zq'(z) + \Omega, \quad z \in U.$$

From the assumption (3.20) we see that Q is starlike in U and, that

$$\mathrm{Re}\frac{zg'(z)}{Q(z)} = \mathrm{Re}\left(\frac{\delta}{\gamma} + 1 + \frac{zq''(z)}{q'(z)}\right) > 0 \quad (z \in U),$$

thus, by applying Lemma 1, the proof is completed. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3, where $-1 \leq B < A \leq 1$ and according to (3.5), the condition (3.20) becomes

$$\max\left\{0; -\mathrm{Re}\left(\frac{\delta}{\gamma}\right)\right\} \leq \frac{1-|B|}{1+|B|}.$$

Hence, for the special case $\nu = \gamma = 1$ and $\eta = 0$, we obtain the following result.

Corollary 7. *Let* $-1 \leq A < B \leq 1$ *and let* $\delta \in C$ *with*

$$\max\{0; -\text{Re}(\delta)\} \leq \frac{1-|B|}{1+|B|}.$$

Let $f \in A(p)$ *and suppose that*

$$\frac{I_{p,q,s,\lambda}^{m+1,,\ell}(\alpha_1,\beta_1)f(z)}{z^p} \neq 0 \quad (z \in U)\ (m \in N_0;\ \ell \geq 0;\ \lambda > 0;\ p \in N),$$

and let $\mu \in C^*$. *If*

$$\left[\frac{I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1,\beta_1)f(z)}{z^p}\right]^{\mu}\left[\delta+\mu\left(\frac{z(I_{p,q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z))'}{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)}-p\right)\right]+\Omega \prec \delta\frac{1+Az}{1+Bz}+\Omega+\frac{(A-B)z}{(1+Bz)^2}, \tag{3.24}$$

then

$$\left[\frac{I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1,\beta_1)f(z)}{z^p}\right]^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of* (3.24). *(All the powers are the principal ones).*

Taking $p = \eta = \gamma = 1$, $\nu = m = 0$, $q = s + 1$, $\alpha_i = 1$ $(i = 1, \ldots, s + 1)$, $\beta_j = 1$ $(j = 1, \ldots, s)$ *and* $q(z) = \frac{1+z}{1-z}$ *in Theorem 3, we obtain the following result.*

Corollary 8. *Let* $f \in A$ *such that* $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$, *and let* $\mu \in C^*$. *If*

$$\left(\frac{f(z)}{z}\right)^{\mu}\left[\delta+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\right]+\Omega \prec \delta\frac{1+z}{1-z}+\Omega+\frac{2z}{(1-z)^2}, \tag{3.25}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1+z}$ *is the best dominant of* (3.25). *(All the powers are the principal ones).*

4. Superordination and sandwich results

Theorem 4. *Let* q *be convex in* U *with* $q(0) = 1$, *let* $m \in N_0$, $\ell \geq 0$, $\lambda > 0$, $\alpha \in C^*$ *and* $p \in N$ *with* $(\frac{\lambda}{p(p+\ell)})\text{Re}(\alpha) > 0$. *Let* $f \in A(p)$ *and suppose that* $\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)}{z^p} \in H[q(0);1] \cap Q$. *If the function*

$$\frac{\alpha}{p}\left(\frac{I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1,\beta_1)f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)}{z^p}\right)$$

is univalent in the unit disc U, *and*

$$q(z)+\frac{\lambda\alpha}{p(p+\ell)}zq'(z) \prec \frac{\alpha}{p}\left(\frac{I_{p,q,s,\lambda}^{m+1,\ell}(\alpha_1,\beta_1)f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)}{z^p}\right), \tag{4.1}$$

then

$$q(z) \prec \frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)}{z^p},$$

and q *is the best subordinant of* (4.1).

Proof. We define the function g by

$$g(z) = \frac{I_{p,q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)}{z^p} \quad (z \in U). \tag{4.2}$$

From the assumption of Theorem 4, the function g is analytic in U. Differentiating (4.2) logarithmically with respect to z, we obtain

$$\frac{zg'(z)}{g(z)} = \frac{z(I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z))'}{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)} - p. \tag{4.3}$$

After some computations, and using the identity (1.11) from (4.3), we get

$$g(z) + \frac{\lambda\alpha}{p(p+\ell)}zg'(z) = \frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) + \frac{p-\alpha}{p}\left(\frac{I^{m,\ell}_{p,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right),$$

and now, by using Lemma 4 we get the desired result. □

Taking $q(z) = \frac{1+Az}{1+Bz}(-1 \le B < A \le 1)$ in Theorem 4, we obtain the following corollary.

Corollary 9. *Let q be convex in U with $q(0) = 1$, let $m \in N_0$, $\ell \ge 0$, $\lambda > 0$, $\alpha \in C^*$ and $p \in N$ with $(\frac{\lambda}{p(p+\ell)})\mathrm{Re}(\alpha) > 0$. Let $f \in A(p)$ and suppose that $\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p} \in H[q(0),1] \cap Q$. If the function*

$$\frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) + \frac{p-\alpha}{p}\left(\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right)$$

is univalent in U, and

$$\frac{1+Az}{1+Bz} + \frac{\lambda\alpha(A-B)z}{p(p+\ell)(1+Bz)^2} \prec \frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) + \frac{p-\alpha}{p}\left(\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right), \tag{4.4}$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p},$$

and $\frac{1+Az}{1+Bz}(-1 \le B < A \le 1)$ is the best subordinant of (4.4).

Using arguments similar to those of the proof of Theorem 3, and then by applying Lemma 3, we obtain the following result.

Theorem 5. *Let q be convex in U with $q(0) = 1$, let $\mu, \gamma \in C^*$, and let $\delta, \Omega, \nu, \eta \in C$ with $\nu + \eta \ne 0$ and $\mathrm{Re}(\frac{\delta}{\gamma}) > 0$. Let $f \in A(p)$ and suppose that f satisfies the next conditions:*

$$\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z) + \eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p} \ne 0 \quad (z \in U; m \in N_0; \ell \ge 0; \lambda > 0; p \in N),$$

and

$$\left[\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z) + \eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu} \in H[q(0),1] \cap Q.$$

If the function ψ given by (3.21) is univalent in U, and

$$\delta q(z) + \gamma zq'(z) + \Omega \prec \psi(z), \tag{4.5}$$

then

$$q(z) \prec \left[\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z) + \eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu},$$

and q is the best subordinant of (4.5). (All the powers are the principal ones).

Combining Theorem 2 with Theorem 4 and Theorem 3 with Theorem 5, we obtain, respectively, the following two sandwich results:

Theorem 6. *Let q_1 and q_2 be two convex functions in U with $q_1(0) = q_2(0) = 1$, let $m \in N_0$, $\ell \ge 0$, $\lambda > 0$, $\alpha \in C^*$ and $p \in N$ with $\frac{\lambda}{p(p+\ell)}\mathrm{Re}(\alpha) > 0$. Let $f \in A(p)$ and suppose that $\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p} \in H[q(0),1] \cap Q$. If the function*

$$\frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) + \frac{p-\alpha}{p}\left(\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right)$$

is univalent in the unit disc U, and

$$q_1(z) + \frac{\lambda\alpha z q_1'(z)}{p(p+\ell)} \prec \frac{\alpha}{p}\left(\frac{I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) + \frac{p-\alpha}{p}\left(\frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p}\right) \prec q_2(z) + \frac{\lambda\alpha z q_2'(z)}{p(p+\ell)} \tag{4.6}$$

then

$$q_1(z) \prec \frac{I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{z^p} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (4.6).

Theorem 7. *Let q_1 and q_2 be two convex functions in U with $q_1(0) = q_2(0) = 1$, let $\mu, \gamma \in C^*$, and let $\delta, \Omega, \nu, \eta \in C$ with $\nu + \eta \neq 0$ and $\mathrm{Re}(\frac{\delta}{\gamma}) > 0$. Let $f \in A(p)$ suppose that f satisfies the next conditions:*

$$\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z) + \eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p} \neq 0 \quad (z \in U;\, m \in N_0;\, \ell \geq 0;\, \lambda > 0;\, p \in N),$$

and

$$\left[\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z) + \eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu} \in H[q(0), 1] \cap Q.$$

If the function ψ given by (3.18) is univalent in U, and

$$\delta q_1(z) + \gamma z q_1'(z) + \Omega \prec \psi(z) \prec \delta z q_2(z) + \gamma z q_2' + \Omega, \tag{4.7}$$

then

$$q_1(z) \prec \left[\frac{\nu I^{m+1,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z) + \eta I^{m,\ell}_{p,q,s,\lambda}(\alpha_1,\beta_1)f(z)}{(\nu+\eta)z^p}\right]^{\mu} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (4.7). (All the powers are the principal ones).

Acknowledgements

The authors are thankful to the referees for their comments and suggestions.

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