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Distributed Control Schemes for Large-Scale Interconnected Discrete-Time Linear Systems

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Abstract—In this paper, we derive distributed control schemes for large-scale interconnected discrete-time linear systems. Our work is a discrete-time analog of the results of Aldeen (1991) for large-scale interconnected power systems.

Keywords—Distributed control, Model reduction, Dominant dynamics, Discrete-time linear systems.

1. INTRODUCTION

During the past four decades, there has been an impressive variety of new techniques developed for obtaining low-order models for large-scale linear systems [1–5]. In [5], Aldeen has presented a class of distributed control schemes suitable for the order reduction of large-scale interconnected continuous-time power systems. In this paper, we derive similar results of distributed control schemes suitable for the order reduction of large-scale interconnected discrete-time linear systems.

This paper is organized as follows. In Section 2, we describe the large-scale global discrete-time linear systems that arises in various applications. We are interested in reducing the order of this plant. In Section 3, we describe the identification of dominant and nondominant modes in the plant under consideration using the modal approach. In Section 4, we describe the model reduction schemes. In Section 5, we describe the distributed control schemes utilizing the modal approach to model reduction detailed in Section 4.

2. GLOBAL CONTROL SYSTEM

Consider a large-scale interconnected linear discrete-time system described by the following equation [6],

$$x(k+1) = Ax(k) + Bu(k) + Hd(k), \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state*, $u \in \mathbb{R}^m$ is the *input* or *control*, and $d \in \mathbb{R}^p$ is the *disturbance* vector. Also, note that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $H \in \mathbb{R}^{n\times p}$ are constant matrices.

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Suppose that system (1) is composed of N subsystems with the i^{th} subsystem having x_i, u_i, and d_i as the state, control, and disturbance vectors, respectively. Let $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $d_i \in \mathbb{R}^{p_i}$ so that

$$\sum_{i=1}^{N} n_i = n, \qquad \sum_{i=1}^{N} m_i = m, \qquad \text{and} \qquad \sum_{i=1}^{N} p_i = p.$$

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \qquad \text{and} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix}.$$

Also, let

$$A = [A_{ij}], \qquad B = \operatorname{diag}[B_1, B_2, \ldots, B_N], \qquad \text{and} \qquad H = \operatorname{diag}[H_1, H_2, \ldots, H_N],$$

where A_{ij} is an $n_i \times n_j$ submatrix of A, B_i is an $n_i \times m_i$ submatrix of B, and H_i is an $n_i \times p_i$ submatrix of H, for $i, j = 1, 2, \ldots, N$.

Hence, the global control system described by (1) may be decomposed into N subsystems described by the following equations,

$$x_i(k+1) = A_{ii} x_i(k) + B_i u_i(k) + H_i d_i(k) + \sum_{\substack{j=1 \\ j \neq i}}^{N} A_{ij} x_j(k). \tag{2}$$

Note that the summation term on the right-hand side of (2) describes the interactions between the subsystem i and the rest of the global control system.

In this paper, we assume that system (1) is completely controllable. We assume also that using the existing state feedback control methods, we have found a satisfactory feedback control law of the form,

$$u(k) = F x(k), \tag{3}$$

where $F \in \mathbb{R}^{m \times n}$ is the global state feedback control matrix with the property that the eigenvalues of the matrix $A+BF$ of the closed-loop control system lie in preassigned locations in the complex plane. For simplifying the subsequent calculations, we assume without loss of generality that all the eigenvalues of the closed-loop system matrix $A + BF$ are distinct. Since we are usually interested in stabilizing feedback control laws in applications, we also assume that $A + BF$ is a *convergent* matrix, i.e., it has all eigenvalues inside the open unit disc of the complex plane.

Setting $E = A + BF$, the closed-loop control system may be expressed as

$$x(k+1) = E x(k) + H d(k). \tag{4}$$

In this paper, we derive distributed control schemes so that the combined performance of the local control actions is equivalent to or near that of the global control action described by the feedback control law (3).

For this purpose, let us first partition the feedback control matrix F as

$$F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1N} \\ F_{21} & F_{22} & \cdots & F_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ F_{N1} & F_{N2} & \cdots & F_{NN} \end{bmatrix},$$

where $F_{ij} \in \mathbb{R}^{m_i \times n_j}$, for $i, j = 1, 2, \ldots, N$.

Hence, the i^{th} subsystem of the closed-loop control system (4) may now be expressed as

$$x_i(k+1) = [A_{ii} + B_i F_{ii}]\, x_i(k) + H_i d_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{N} [A_{ij} + B_i F_{ij}]\, x_j(k). \tag{5}$$

Comparing equations (2) and (5), we conclude that the control input u_i for the i^{th} subsystem may be decomposed as

$$u_i = u_{il} + u_{ic},$$

where u_{il} is the local control component, generated from local state feedback, i.e., $u_{il}(k) = F_{ii}x_i(k)$, and u_{ic} is the corrective control component, generated from remote state feedback, i.e.,

$$u_{ic}(k) = \sum_{\substack{j=1 \\ j\neq i}}^{N} F_{ij}x_j(k).$$

Our design procedure for distributed control schemes are carried out in the following three phases.

(I) Identification of the dominant and nondominant modes of the global closed-loop control system.
(II) Finding a reduced-order model of the global control system.
(III) Modelling of the interactions between the control subsystems.

We discuss Phases I, II, and III in Sections 3, 4, and 5, respectively.

3. IDENTIFICATION OF DOMINANT AND NONDOMINANT MODES

For determinining the dominant and nondominant modes of linear time-invariant discrete-time systems, we use the combined eigenvalue participation measure outlined in this section. The advantage of this approach lies in the fact that the physical meaning of the state is preserved.

For the global closed-loop control system,

$$x(k+1) = Ex(k) + Hd(k), \tag{6}$$

consider the similarity transformation,

$$x(k) = Mz(k). \tag{7}$$

Since we have assumed that the closed-loop feedback matrix E has distinct eigenvalues, we can easily find a nonsingular modal matrix M so that the similarity transformation (7) simplifies the closed-loop system (6) as

$$z(k+1) = \Lambda z(k) + \Gamma d(k), \tag{8}$$

where

$$\Lambda = M^{-1}EM = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad \Gamma = M^{-1}H. \tag{9}$$

From (9), it is easy to see that the step steady-state response of the i^{th} mode, z^i, is given by

$$z^i = \frac{1}{1-\lambda_i}\sum_{j=1}^{p} \gamma_{ij}\beta_j, \qquad \text{for } i = 1, 2, \ldots, n, \tag{10}$$

where γ_{ij} is the $(i,j)^{\text{th}}$ entry of the matrix Γ and β_j are the weighting factors associated with the disturbance d. We note that the weighting factors β_j should be chosen so as to reflect the relative importance of each control input to the performance of the global control system.

To determine the steady-state response of the original system, we substitute (10) into the transformation equation (7), which yields

$$x^i = \sum_{j=1}^{n} \frac{m_{ij}}{1-\lambda_j} \sum_{k=1}^{r} \gamma_{ik}\beta_k, \tag{11}$$

where m_{ij} is the $(i,j)^{\text{th}}$ entry of the modal matrix M.

We note that the right-hand side of (11) provides the state controllability measure of the i^{th} state variable, which is a function of all the eigenvalues of the global closed-loop control system. To determine the dominance of the k^{th} eigenvalue in the i^{th} state, we use the measure,

$$\Omega_{ik} = \frac{m_{ik}}{1-\lambda_k} \sum_{j=1}^{p} \gamma_{kj}\beta_j. \tag{12}$$

To determine the dominance of the k^{th} eigenvalue in all the n states, we use the measure

$$\Theta_k = \sum_{i=1}^{n} \Omega_{ik}. \tag{13}$$

To determine the relative dominance of the k^{th} eigenvalue in the i^{th} state, we use the measure,

$$\phi_{ik} = \left| \frac{\Omega_{ik}}{\Theta_k} \right| \times 100, \tag{14}$$

and finally, the relative contribution of the first l $(l = 1,2,\dots,n)$ eigenvalues in the i^{th} state is given by the measure,

$$\psi_{il} = \sum_{k=1}^{l} \phi_{ik}. \tag{15}$$

Based on the above definitions, we present the following procedure to identify the most dominant modes of the system.

Step 1. Using (12), calculate Ω_{ik}, for $i,k = 1,2,\dots,n$. This yields an $n \times n$ matrix, whose $(i,k)^{\text{th}}$ entry gives the participation measure of the i^{th} eigenvalue in the k^{th} state.

Step 2. Using (13), calculate Θ_k, for $k = 1,2,\dots,n$, where Θ_k represents the dominance of k^{th} eigenvalue in all the states.

Step 3. Sort the values of Θ_k in the order of dominance, starting from the most dominant to the least dominant. Accordingly, sort the associated eigenvalues in the order of dominance. Denote these eigenvalues as $\lambda_1, \lambda_2, \dots, \lambda_n$ in order of dominance so that λ_1 is the most dominant and λ_n is the least dominant. Also, reorder the columns of the modal matrix M so that $z_1, z_2, \dots, z_n$ correspond to the renamed eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ in the order of dominance.

Step 4. Using (14), calculate ϕ_{ik}, for all values of i and k, where ϕ_{ik} gives the relative participation of the k^{th} eigenvalue in the i^{th} state variable.

Step 5. Using (15), calculate ψ_{il}, for all values of i and l.

Step 6. Finally, test the condition,

$$\psi_{il} = \sum_{k=1}^{l} \phi_{ik} \geq \epsilon_i, \qquad \text{for } i = 1,2,\dots,n, \tag{16}$$

for values of l starting from $l = 1$ to $l = n$ and where ϵ_i is chosen, arbitrarily, between 80% and 100% according to the degree of approximation required. When condition (15) holds, the first l eigenvalues are the dominant ones. The rest are nondominant. Correspondingly, the first l states in the new coordinates, namely, $z_1, z_2, \dots, z_l$ will be the dominant states, while the rest will be nondominant.

4. MODEL REDUCTION SCHEMES

For the linear plant,

$$z(k+1) = \lambda z(k) + \Gamma d(k), \tag{17}$$

discussed in Section 3, suppose that we have made an identification of the dominant (or slow) and nondominant (or fast) modes of the system. Without loss of generality, suppose that the first q_1 eigenvalues of the diagonal matrix λ are dominant and the rest are nondominant. Define $q_2 = n - q_1$ so that plant (17) has q_1 dominant modes and q_2 nondominant modes.

Define

$$\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix},$$

where λ_1 is a $q_1 \times q_1$ diagonal matrix consisting of the dominant eigenvalues of the system, λ_2 is a $q_2 \times q_2$ diagonal matrix consisting of the nondominant eigenvalues of the system, and Γ_1 and Γ_2 are $q_1 \times p$ and $q_2 \times p$ matrices, respectively.

Accordingly, we may write plant (17) as

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} d(k). \tag{18}$$

Thus, the dominant dynamics is governed by

$$z_1(k+1) = \lambda_1 z_1(k) + \Gamma_1 d(k) \tag{19}$$

and the nondominant dynamics is governed by

$$z_2(k+1) = \lambda_2 z_2(k) + \Gamma_2 d(k). \tag{20}$$

Hence, the transformation equation $x(k) = Mz(k)$ (see Section 3) may be partitioned as

$$\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_N(k) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ \vdots & \vdots \\ M_{N1} & M_{N2} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \tag{21}$$

where M_{i1} and M_{i2} are $(n_i \times q_1)$- and $(n_i \times q_2)$-dimensional submatrices of M, respectively.

Next, we describe a set of three-model reduction schemes.

4.1. Neglecting the Nondominant Dynamics

A simple way of reducing the model of the plant in question is to neglect the fast modes of the plant altogether, i.e., $z_2 = 0$, since the contribution of the fast modes to the plant dynamics is significant only at the beginning of the system response whereas the slower modes are important throughout the entire system response [1]. This leads to the following approximate model of the state of the i^{th} subsystem,

$$x_i(k) = M_{i1} z_1(k). \tag{22}$$

We note that because the nondominant part of the system is neglected altogether, the reduced-order model thus derived will have a steady-state response different from that of the original system.

4.2. Approximating the Nondominant Dynamics

Another way of reducing the model of the plant is to suppose that $z_2(k)$ takes a constant value in the steady-state.

This gives

$$z_2(k) \approx \lambda_2 z_2(k) + \Gamma_2 d(k).$$

Since $I - \lambda_2$ is invertible, it is immediate that

$$z_2(k) = (I - \lambda_2)^{-1} \Gamma_2 d(k). \tag{23}$$

From (23) and (21), we get the following approximate model of the state of the i^{th} subsystem,

$$x_i(k) = M_{i1} z_1(k) + M_{i2} (I - \lambda_2)^{-1} \Gamma_2 d(k). \tag{24}$$

4.3. Approximating the Dominant Dynamics by the Nondominant Dynamics

We note that the nondominant dynamics may also be approximated by estimating the state z_2 by an optimal linear combination of the dominant modes, i.e., by setting

$$z_2 \approx E z_1.$$

A number of optimization techniques are available for calculating the proportionality matrix E [3]. This leads to the following approximate model of the state of the i^{th} subsystem,

$$x_i(k) = (M_{i1} + M_{i2} E) z_1(k). \tag{25}$$

5. DISTRIBUTED CONTROL SCHEMES

In this section, we present a set of three different distributed control schemes. First, we restate equation (5) of the i^{th} subsystem of the global closed-loop control system discussed in Section 2,

$$x_i(k+1) = [A_{ii} + B_i F_{ii}] x_i(k) + H_i d_i(k) + \sum_{\substack{j=1 \\ j \neq i}}^{N} [A_{ij} + B_i F_{ij}] x_j(k). \tag{26}$$

As pointed out in Section 2, the control input u_i for the i^{th} subsystem may be decomposed as

$$u_i = u_{il} + u_{ic},$$

where u_{il} is the local control component, generated from local state feedback, i.e.,

$$u_{il}(k) = F_{ii} x_i(k),$$

and u_{ic} is the corrective control component, generated from remote state feedback, i.e.,

$$u_{ic}(k) = \sum_{\substack{j=1 \\ j \neq i}}^{N} F_{ij} x_j(k). \tag{27}$$

From equation (27), it is clear that in order for the global closed-loop control system to work well, all the subsystems of the global system in question must share information about their current states. Basically, we eliminate the need for such a requirement by making use of the set of three different model-reduction schemes detailed in Section 4. Accordingly, we derive a set of three different distributed control schemes detailed as below.

5.1. Scheme 1

From (22), we have

$$x_j(k) = M_{j1} z_1(k). \tag{28}$$

Substituting (28) into (27), we get

$$u_{ic}(k) = Y_i z_1(k),$$

where

$$Y_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} F_{ij} M_{j1}.$$

5.2. Scheme 2

From (24), we have

$$x_j(k) = M_{j1} z_1(k) + M_{j2} (I - \lambda_2)^{-1} \Gamma_2 d(k). \tag{29}$$

Substituting (29) into (27), we get

$$u_{ic}(k) = Y_i z_1(k) + Q_i d(k),$$

where

$$Y_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} F_{ij} M_{j1}$$

and

$$Q_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} F_{ij} M_{j2} (I - \lambda_2)^{-1} \Gamma_2.$$

5.3. Scheme 3

From (25), we have

$$x_j(k) = (M_{j1} + M_{j2} E) z_1(k). \tag{30}$$

Substituting (30) into (27), we get

$$u_{ic}(k) = Y_i z_1(k),$$

where

$$Y_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} F_{ij} (M_{j1} + M_{j2} E).$$

REFERENCES

1. E.J. Davison, A method for simplifying linear dynamic systems, *IEEE Trans. Auto. Control* **AC-11**, 93–101, (1966).
2. J. Hickin and N.K. Sinha, Model reduction for linear multivariable systems, *IEEE Trans. Auto. Control* **AC-25**, 1121–1127, (1980).
3. L. Litz and H. Roth, State decomposition for singular perturbation order reduction—A modal approach, *International J. Control* **34**, 937–954, (1981).
4. G.J. Lastman, N.K. Sinha and P. Rozsa, On the selection of states to be retained in a reduced-order model, *IEE Proceedings* **131**, 15–24, (1984).
5. M. Aldeen, Interaction modelling approach to distributed control with application to power systems, *International J. Control* **53**, 1035–1054, (1991).
6. C.E. Fosha and O.I. Elgerd, The megawatt-frequency control problem: A new approach via optimal control theory, *IEEE Trans. Power Apparatus and Systems* **89**, 563–571, (1970).