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Exact wirelength of hypercubes on a grid

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ABSTRACT

Grid embeddings are used not only to study the simulation capabilities of a parallel architecture but also to design its VLSI layout. In addition to dilation and congestion, wirelength is an important measure of an embedding. There are very few papers in the literature which provide the exact wirelength of grid embedding. As far as the most versatile architecture hypercube is concerned, only approximate estimates of the wirelength of grid embedding are available. In this paper, we give an exact formula of minimum wirelength of hypercube layout into grids and thereby we solve completely the wirelength problem of hypercubes into grids.

We introduce a new technique to estimate the wirelength of a grid embedding. This new technique is based on a Congestion Lemma and a Partition Lemma which we study in this paper.

1. Introduction and terminology

A parallel algorithm or a massively parallel computer can each be modeled by a graph, in which the vertices of the graph represent the processes or processing elements, and the edges represent the communications among processes or processors. Thus, the problem of efficiently executing a parallel algorithm A on a parallel computer M can be often reduced to the problem of mapping the graph G, representing A, on the graph H, representing M, so that the mapping satisfies some predefined constraints. This is called graph embedding [25], which is defined more precisely as follows:

Let G and H be finite graphs with n vertices. $V(G)$ and $V(H)$ denote the vertex sets of G and H respectively. $E(G)$ and $E(H)$ denote the edge sets of G and H respectively. An embedding [4] f of G into H is defined as follows:

(i). f is a bijective map from $V(G) \to V(H)$;
(ii). f is a one-to-one map from $E(G)$ to $\{P_f(f(u), f(v)) : P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)\}$.

See Fig. 1. A set of edges of H is said to be an *edge cut* of H if the removal of these edges results in a disconnection of H.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H. The vertex congestion $VC(G, H)$ is the minimum over all embeddings $f : V(G) \to V(H)$ of the maximum number of wires that pass any point of H [4].

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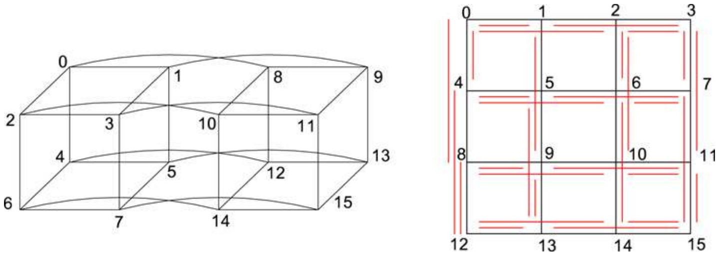

Fig. 1. Embedding of a hypercube onto a grid.

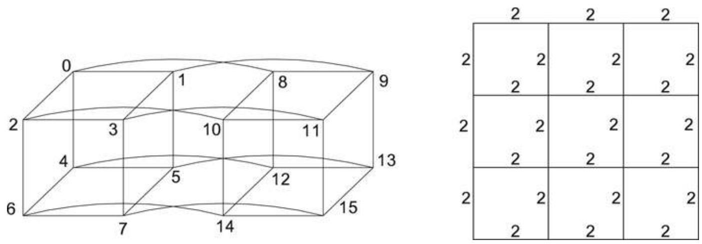

Fig. 2. For the embedding mentioned in Fig. 1, the edge congestions are marked on the respective edges of the grid.

The *congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u, v) of G such that e is in the path $P_f(f(u), f(v))$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(G, H(e)) = \left|\left\{(u, v) \in E(G) : e \in P_f(f(u), f(v))\right\}\right|$$

where $P_f(f(u), f(v))$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

In the same way, let $VC_f(G, H(w))$ denote the number of edges (u, v) of G such that w is a start vertex or an internal vertex of the path $P_f(f(u), f(v))$ between $f(u)$ and $f(v)$ in H and w is not an end vertex of the path $P_f(f(u), f(v))$. In other words,

$$VC_f(G, H(w)) = \left|\left\{(u, v) \in E(G) : w \in P_f(f(u), f(v)) \setminus \{f(v)\}\right\}\right|.$$

The edge congestion problem. The *edge congestion* [28] of an embedding f of G into H is given by

$$EC_f(G, H) = \max EC_f(G, H(e))$$

where the maximum is taken over all edges e of H. Then, the *minimum edge congestion* of G into H is defined as

$$EC(G, H) = \min EC_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. See Fig. 2. The *edge congestion problem* of a graph G into H is to find an embedding of G into H that induces the minimum edge congestion $EC(G, H)$.

The concept of cutwidth is a special case of edge congestion when H is a path or a cycle [5,9,23,27]. There are several results on the congestion problem of various architectures such as trees into cycles [9], trees into hypercubes [24], hypercubes into grids [4,5], complete binary trees into grids [25], ladders and caterpillars into hypercubes [6,8]. □

The vertex congestion problem. The *vertex congestion* of a vertex w [4,5,25] of an embedding f of G into H is given by

$$VC_f(G, H) = \max VC_f(G, H(w))$$

where the maximum is taken over all vertices w of H. See Fig. 3. Then, the *minimum vertex congestion* of G into H is defined as

$$VC(G, H) = \min VC_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. The *vertex congestion problem* [4,5,9,25] of a graph G into H is to find an embedding of G into H that induces the minimum vertex congestion $VC(G, H)$. The vertex congestion problem and the edge congestion problem are the same if H is a path or a cycle. □

The wirelength problem. The *wirelength* of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} d_H(f(u), f(v))$$

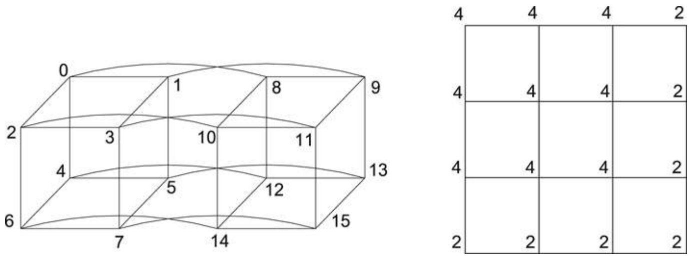

Fig. 3. For the embedding mentioned in Fig. 1, the vertex congestions are marked at the respective vertices of the Grid.

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(f(u), f(v))$ in H. Then, the *minimum wirelength* of G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. The *wirelength problem* [4,5,9,25,19] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$. The wirelength problem is studied for binary trees into paths [22], hypercubes into paths [4,5], generalized wheels into arbitrary trees [19], and complete graphs into hypercubes [21]. □

There are several ways one can calculate the wirelength of an embedding. Here we list two other equivalent ways to compute the wirelength of an embedding, which are rather straightforward. The following result states that the edge congestion and the vertex congestion of an embedding f of G into H contribute the same wirelength [5,9,19].

Lemma 1. *For an embedding f of G into H, the wirelength of f is*

$$WL_f(G, H) = \sum_{e \in E(H)} EC_f(G, H(e)) = \sum_{w \in V(H)} VC_f(G, H(w)). \quad \square$$

2. Overview of the paper

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [28]. Grid embedding plays an important role in computer architecture. VLSI Layout Problem [1], Crossing Number Problem [12], Graph Drawing [11] and Edge Embedding Problem [10,15] are all a part of grid embedding. Embedding problems have been considered for star networks into hypercubes [2], complete binary trees into hypercubes [3], hypercubes into grids [4,5], generalized ladders into hypercubes [8], complete graphs into hypercubes [21], honeycomb into hypercubes [13], grids into grids [26], and binary trees into grids [25].

Chavez and Trapp [9] have studied the embedding of hypercubes into cycles. They have conjectured that the cyclic cutwidth of a hypercube is minimized with the Grey code numbering. This conjecture is called CT conjecture [14,16]. A team, which has studied elaborately the embedding of hypercubes into grids, is Bezrukov et al. [4,5]. They [5] have completely solved the vertex congestion problem of hypercubes into grids using a lexicographic labeling. According to our literature survey, the wirelength problem is not solved for hypercubes into grids. Only tight estimates are available [4,5].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [4,9]. The embeddings discussed in this paper produce an exact wirelength. Our technique is different from the existing ones and we apply the maximum subgraph problem to estimate the edge congestion of each edge cut of a grid. □

3. Maximum subgraph problem

A maximum subgraph problem [15] of a graph $G(V, E)$ is to find a maximum induced subgraph A of G on k vertices for a given integer k. We extensively make use of the results related to the maximum subgraph problem of hypercubes. There is a striking relationship between the maximum subgraph problem and the wirelength problem. The maximum subgraph problem for hypercubes is well researched [7,18,20]. These results help us to estimate edge congestion of grid embedding of hypercubes. Using Congestion Lemma and Partition Lemma, we construct a simple and elegant proof of correctness of the algorithm which produces minimum edge congestion and minimum wirelength.

Definition 1 (*[28]*). For $r \geq 1$ let Q^r denote the graph of r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary representations. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit. See Fig. 4(a). □

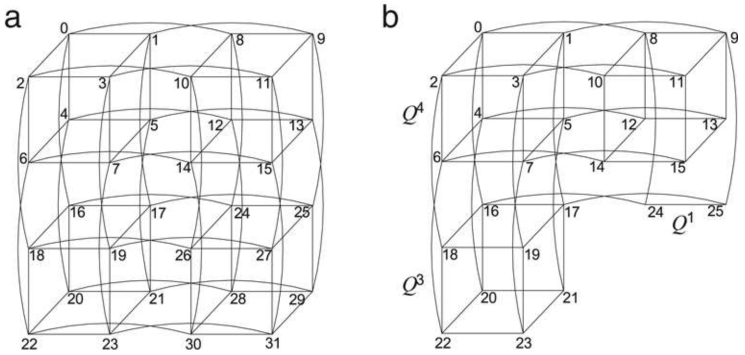

Fig. 4. (a) Hypercube Q^5. (b) Composite subcube I^5_{26}.

Definition 2 (*[18]*). Given an integer m and $k = \lceil \log m \rceil$, a set I^k_m of m vertices of Q^r is said to be a composite set if the number of edges of the subgraph induced by I^k_m is not less than the number of edges of a subgraph induced by any other set S_m of m vertices of Q^r. A composite hypercube of Q^r is defined to be a subgraph of Q^r, which is induced by some composite set of Q^r. □

In other words, a set I^k_m of m vertices of Q^r is a composite set [7,17,18] if $\left|E(G\left[I^k_m\right])\right| \geq |E(G[S_m])|$ for any set S_m of m vertices of Q^r. Here $|E(G[S_m])|$ denotes the number of edges of the subgraph induced by S_m. The set $\{0, 1, 2, 3\}$ of vertices of Q^5 of Fig. 4(a) is a composite set whereas the set $\{0, 6, 8, 31\}$ of vertices is not a composite set.

In this section we discuss a method [7,18] to construct I^k_m of m vertices of Q^r. A maximum k-dimensional composite subgraph I^k_m of Q^r on m vertices with $2^{k-1} < m \leq 2^k$ comprises two components Q^{k-1} and $I^t_{m-2^{k-1}}$ where $t = \lceil \log_2(m - 2^{k-1}) \rceil$ with vertices in Q^{k-1} numbered from 0 to $2^{k-1} - 1$ and vertices in $I^t_{m-2^{k-1}}$ numbered from 2^{k-1} to $m - 1$. Thus I^k_m comprises a set of complete cubes of dimensions $k-1$ and below, and no two constituent cubes are of the same size. For example, the composite hypercube I^5_{26} depicted in Fig. 4(b) comprises Q^4 and I^4_{10}, which, in turn, contains Q^3 and $I^2_2 = Q^1$. In Fig. 4(b), $m = 26 = 2^4 + 2^3 + 2^1$. Moreover $\left|E(I^5_{26})\right| = 57$. We infer two important results from the above discussion and present them in a mathematical format. □

Definition 3 (*[20]*). An incomplete hypercube on i vertices of Q^r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i. □

Theorem 1 (*[7,17,18]*). *Let Q^r be an r-dimensional hypercube. For $i = 0, 1, \ldots, 2^r - 1$, L_i is a composite set.* □

Definition 4. Let Q^s denote an s-dimensional subcube of Q^r. Two subcubes Q^{s_1} and Q^{s_2} of Q^r are said to be adjacent if they satisfy the following condition: If $s_1 \leq s_2$, then for every vertex u of Q^{s_1}, there is a unique vertex v of Q^{s_2} such that u and v are adjacent and if $s_2 \leq s_1$, then for every vertex u of Q^{s_2}, there is a unique vertex v of Q^{s_1} such that u and v are adjacent. □

Theorem 2. *Let Q^r be an r-dimensional hypercube. Let $m = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$ such that $k_1 > k_2 > \cdots > k_l$ and $k_1 + 1 = \lceil \log m \rceil$. Then*

$$\left|\left(E\left[I^{k_1+1}_m\right]\right)\right| = k_1 2^{k_1-1} + (k_2+2)2^{k_2-1} + (k_3+4)2^{k_3-1} + \cdots + (k_l + 2(l-1))2^{k_l-1}.$$

Proof. Let us revisit the structure of $I^{k_1+1}_m$ where $m = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$. According to the discussion in the preceding paragraph [7,18], $I^{k_1+1}_m$ contains subcubes $Q^{k_1}, Q^{k_2}, \ldots, Q^{k_l}$ such that Q^{k_i} is adjacent to $Q^{k_1}, Q^{k_2}, \ldots, Q^{k_{i-1}}$ for $i = 2, 3, \ldots, l$. This means that there are 2^{k_i} edges between Q^{k_i} and Q^{k_j} for all $j = 1, 2, \ldots, i-1$. Thus there are $(i-1)2^{k_i}$ edges from Q^{k_i} to Q^{k_j}, for all $j = 1, 2, \ldots, i-1$. Also, Q^{k_i} has $k_i 2^{k_i-1}$ edges within itself. Thus Q^{k_i} contributes $k_i 2^{k_i-1} + (i-1)2^{k_i}$ edges to $I^{k_1+1}_m$. In other words, each Q^{k_i} contributes $(k_i + 2(i-1))2^{k_i-1}$ edges to $I^{k_1+1}_m$. Hence the theorem. □

4. A few basic results

Here onwards, for the sake of simplicity, $EC_f(G, H(e))$ will be represented by $EC_f(e)$.

Notation. For any set S of edges of H, $EC_f(S) = \sum_{e \in S} EC_f(e)$. □

The following lemma is a generalization of the Fundamental Lemma [19] and it will be used throughout this paper. We apply this result to estimate the edge congestion and wirelength.

Lemma 2 (*Congestion Lemma*). *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies*

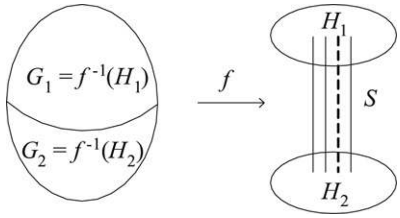

Fig. 5. S is an edge cut of H and $f : G \to H$ is an embedding.

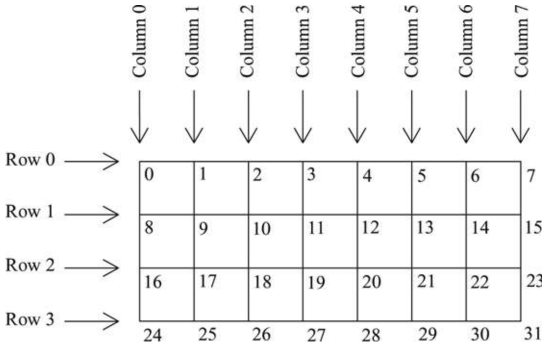

Fig. 6. Labels of rows and columns of 4×8 grid.

the following conditions:

(i) *For every edge* $(a, b) \in G_i, i = 1, 2, P_f(f(a), f(b))$ has no edges in S.
(ii) *For every edge* (a, b) in G with $a \in G_1$ and $b \in G_2, P_f(f(a), f(b))$ has exactly one edge in S.
(iii) G_1 *is a maximum subgraph on* k vertices where $k = |V(G_1)|$.
Then $EC_f(S)$ *is minimum*, that is, $EC_f(S) \le EC_g(S)$ for any other embedding g of G into H.

Proof. We have $S = \{(u, v) \in E(H) : u \in H_1, v \in H_2\}$. Let $R = \{(a, b) \in E(G) : a \in G_1, b \in G_2\}$. By condition (i), an edge of $G_i, i = 1, 2$, contributes nothing to $EC_f(S)$. By condition (ii), every edge (a, b) of R increments $EC_f(S)$ by 1. Therefore, $EC_f(S) = |R|$. It is straightforward to compute that $|R| = r\,|V(G_1)| - 2\,|E(G_1)|$, since G is r-regular. Hence $EC_f(S) = r\,|V(G_1)| - 2\,|E(G_1)|$. By condition (iii), $|E(G_1)|$ is maximum. Since $|E(G_1)|$ is maximum and $EC_f(S) = r\,|V(G_1)| - 2\,|E(G_1)|$, $EC_f(S)$ is minimum. See Fig. 5. □

The next Lemma follows immediately from Lemma 1.

Lemma 3 (*Partition Lemma*). *Let* $f : G \to H$ *be an embedding. Let* $\{S_1, S_2, \ldots, S_p\}$ *be a partition of* $E(H)$*. Then*

$$WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i).$$

The following lemma is an application of Lemma 3.

Lemma 4. *Let* $\{S_1, S_2, \ldots, S_p\}$ *be a partition of* $E(H)$*. Let* $f : G \to H$ *and* $g : G \to H$ *be two embeddings such that* $EC_f(S_i) \le EC_g(S_i)$ *for all i. Then* $WL_f(G, H) \le WL_g(G, H)$. □

Notation. An $n \times m$ grid with n rows and m columns is represented by $M[n \times m]$ where the rows are labeled $0, 1, \ldots, n-1$ and the columns are labeled $0, 1, \ldots, m-1$. See Fig. 6. □

Here is our strategy. We partition the grid into vertical and horizontal edge cuts $S_1, S_2, \ldots, S_p$, and apply the Partition Lemma to compute the wirelength $WL(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}])$. For each S_i, we have the option to maximize any one of $|E(G_1)|$ and $|E(G_2)|$, for any regular graph G. While maximizing one of $|E(G_1)|$ and $|E(G_2)|$, we always choose one of G_1 and G_2 which has vertices less than or equal to $(1/2)\,|V(G)|$. We apply Congestion Lemma to minimize $EC_f(S_i)$.

5. The wirelength problem of Q^r into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$

Bezrukov et al. [4,5] have solved the vertex congestion problem of Q^r into grids. They define a lexicographic embedding and show that it induces minimum vertex congestion of Q^r into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$. In this section, we show that this lexicographic embedding also solves the wirelength problem of Q^r into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$.

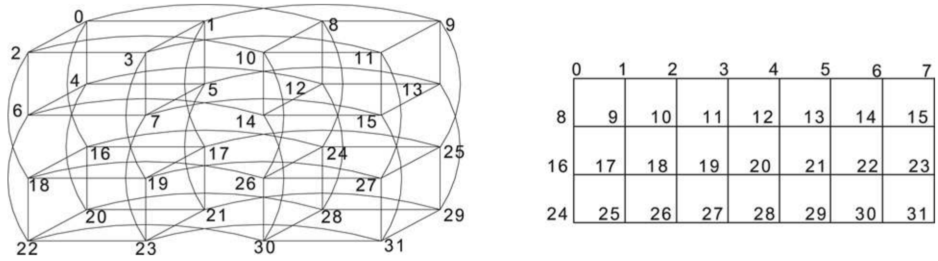

Fig. 7. Lexicographic embedding of Q^5 into $M[2^2 \times 2^3]$ grid.

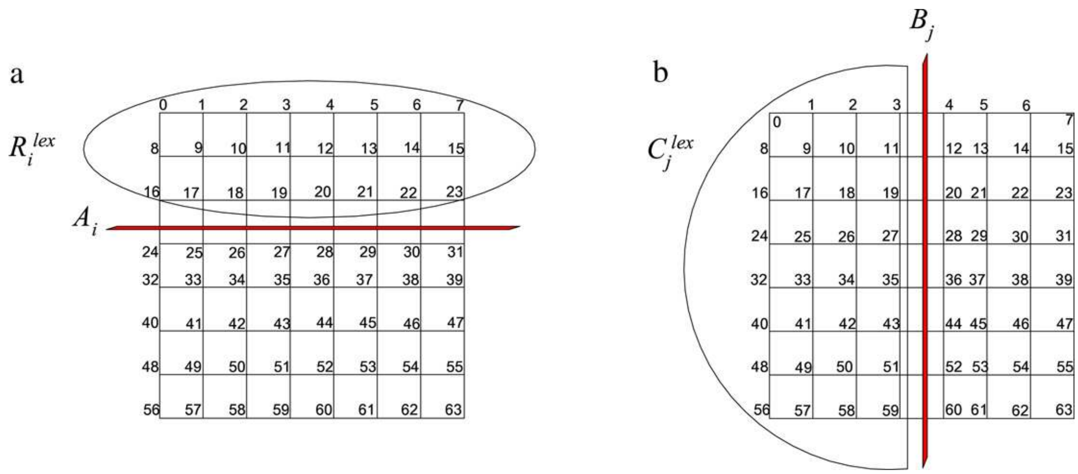

Fig. 8. (a) Grid $M[2^3 \times 2^3]$ with lexicographic labeling and R_i^{lex} denotes the set of vertices of the first i rows. (b) Grid $M[2^3 \times 2^3]$ with lexicographic labeling and C_j^{lex} denotes the set of vertices of the first j columns.

Lexicographic embedding (*[4]*). The lexicographic embedding of Q^r with the labeling 0 to $2^r - 1$ into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ is an assignment of labels to the nodes of $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ as follows: The 0th row is labeled 0 to $2^{\lceil r/2 \rceil} - 1$ from left to right. The $(i-1)$th row is labeled $(i-1)2^{\lceil r/2 \rceil}, (i-1)2^{\lceil r/2 \rceil} + 1, \ldots, i2^{\lceil r/2 \rceil} - 1$ from left to right where $i = 0, 1, \ldots, 2^{\lfloor r/2 \rfloor} - 1$. See Figs. 7 and 9. This lexicographic embedding is denoted by *lex*. □

Now we shall prove that the lexicographic embedding solves the edge congestion problem and the wirelength problem of hypercubes into grids. There are two important observations about the lexicographic embedding of hypercubes into grids, which help us to build an elegant proof of correctness of the embedding algorithm. Let us first discuss these two results before we analyze the proof of correctness.

Notation. $R_i^{lex} = \{0, 1, \ldots, i2^{\lceil r/2 \rceil} - 1\}$ for $i = 0, 1, \ldots, 2^{\lfloor r/2 \rfloor} - 1$. □

Readers may observe that R_i^{lex} is the set of vertices of the first i rows of $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ with the lexicographic embedding of Q^r into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$. See Fig. 8(a). Also, it may be observed that $R_i^{lex} = L_{i2^{\lceil r/2 \rceil}}$. □

We shall prove that the inverse image of a component formed by the deletion of a horizontal or vertical edge cut of $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ corresponding to *lex* from Q^r to $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ is a composite set. The following result is a particular case of Theorem 1 and hence it requires no proof.

Lemma 5. *R_i^{lex} is a composite set in Q^r for $i = 0, 1, \ldots, 2^{\lfloor r/2 \rfloor} - 1$.* □

Notation.

$$C_j^{lex} = \{$$
$$\begin{array}{lllll} 0, & 1 \times 2^{\lceil r/2 \rceil}, & 2 \times 2^{\lceil r/2 \rceil} & \cdots & (2^{\lfloor r/2 \rfloor} - 1) \times 2^{\lceil r/2 \rceil}, \\ 1, & 1 \times 2^{\lceil r/2 \rceil} + 1, & 2 \times 2^{\lceil r/2 \rceil} + 1 & \cdots & (2^{\lfloor r/2 \rfloor} - 1) \times 2^{\lceil r/2 \rceil} + 1, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ j-1, & 1 \times 2^{\lceil r/2 \rceil} + j - 1, & 2 \times 2^{\lceil r/2 \rceil} + j - 1 & \cdots & (2^{\lfloor r/2 \rfloor} - 1) \times 2^{\lceil r/2 \rceil} + j - 1 \end{array}$$
$$\}$$

for $j = 0, 1, \ldots, 2^{\lceil r/2 \rceil} - 1$. □

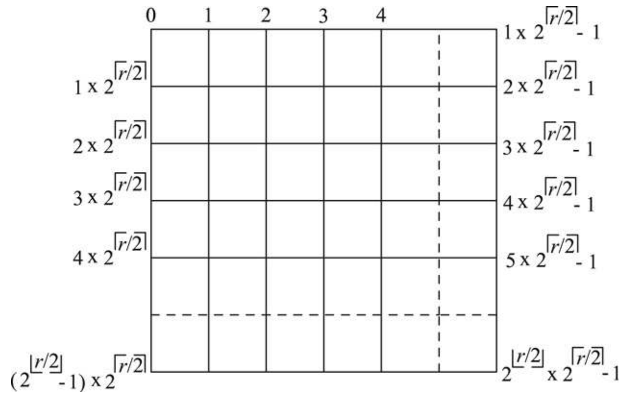

Fig. 9. Grid $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ with lexicographic labeling.

It may be observed that C_j^{lex} is the set of vertices of the first j columns of $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ with the lexicographic embedding *lex* of Q^r into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$. See Fig. 8(b).

Lemma 6. *The subgraphs of Q^r induced by C_j^{lex} and $L_{j2^{\lfloor r/2 \rfloor}}$ are isomorphic.*

Proof. Here in this proof, we make use of the binary representations of Q^r.

Define $\phi : C_j^{lex} \to L_{j2^{\lfloor r/2 \rfloor}}$ such that

$$\phi(k \times 2^{\lceil r/2 \rceil} + l) = \begin{cases} l \times 2^{\lceil r/2 \rceil} + k & \text{if } r \text{ is even} \\ l \times 2^{\lfloor r/2 \rfloor} + k & \text{if } r \text{ is odd.} \end{cases}$$

It is enough to prove that ϕ is an isomorphism. The binary representations of integers $k \times 2^{\lceil r/2 \rceil} + l$ and $l \times 2^{\lceil r/2 \rceil} + k$ (or $l \times 2^{\lfloor r/2 \rfloor} + k$) have some nice relationship. If the binary representation of $k \times 2^{\lceil r/2 \rceil} + l$ is $(\alpha_1, \alpha_2, \ldots, \alpha_{\lceil r/2 \rceil}, \beta_1, \beta_2, \ldots, \beta_{\lfloor r/2 \rfloor})$ then the binary representation of $l \times 2^{\lceil r/2 \rceil} + k$ (or $l \times 2^{\lfloor r/2 \rfloor} + k$) is $(\beta_1, \beta_2, \ldots, \beta_{\lfloor r/2 \rfloor}, \alpha_1, \alpha_2, \ldots, \alpha_{\lceil r/2 \rceil})$. Thus the binary representation of two numbers x and y differ in exactly 1 bit if and only if the binary representation of $\phi(x)$ and $\phi(y)$ differ in the same bit. Therefore (x, y) is an edge in Q^r if and only if $(\phi(x), \phi(y))$ is an edge in Q^r. In other words, (x, y) is an edge in C_j^{lex} if and only if $(\phi(x), \phi(y))$ is an edge in $L_{j2^{\lfloor r/2 \rfloor}}$. Hence C_j^{lex} and $L_{j2^{\lfloor r/2 \rfloor}}$ are isomorphic. □

Lemma 7. *C_j^{lex} is a composite set in Q^r for $j = 0, 1, \ldots, 2^{\lceil r/2 \rceil} - 1$.*

Proof. By Theorem 1, $L_i = \{0, 1, \ldots, i - 1\}$ is a composite set of Q^r for every i. This implies that C_j^{lex} is a composite set in Q^r by Lemma 6. □

Theorem 3. *The lexicographic embedding lex of Q^r into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ induces a minimum wirelength $WL(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}])$.*

Proof. Let A_i be an edge cut of the grid $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ such that A_i disconnects $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ into two components X_i and $X_{i'}$ where $V(X_i)$ is R_i^{lex}. See Fig. 10(a). Let B_j be an edge cut of the grid $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ such that B_j disconnects $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ into two components Y_j and $Y_{j'}$ where $V(Y_j)$ is C_j^{lex}. See Fig. 10(b). Let G_i and $G_{i'}$ be the inverse images of X_i and $X_{i'}$ under *lex* respectively. The edge cuts A_i and B_j of the partition, satisfy conditions (i) and (ii) of the Congestion Lemma. In order to show that $EC_{lex}(A_i)$ is minimum, by condition (iii), it is enough to show that $|E(G_i)|$ is maximum.

We know that G_i is a subcube induced by the vertices of R_i^{lex}. By Lemma 5, it is true that G_i is a composite hypercube. Thus by the Congestion Lemma, $EC_{lex}(A_i)$ is minimum for $i = 0, 1, \ldots, 2^{\lfloor r/2 \rfloor} - 1$.

Similarly, let G_j and $G_{j'}$ be inverse images of Y_j and $Y_{j'}$ under *lex* respectively. By Lemma 7, it is true that G_j is a composite hypercube induced by the vertices of C_j^{lex}. Thus by Congestion Lemma, $EC_{lex}(B_j)$ is minimum for $j = 0, 1, \ldots, 2^{\lceil r/2 \rceil} - 1$.

Thus by Partition Lemma, $WL_{lex}(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}])$ is minimum. □

6. Derivation of wirelength $WL(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}])$

So far we have demonstrated that the lexicographic embedding *lex* of Q^r into the grid $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ provides minimum wirelength. Let P_k denote a path on k vertices. Here we provide a mathematical formula for $WL(Q^r, P_{2^r})$ which is more straightforward.

Theorem 4. $WL(Q^r, P_{2^r}) = 2^{2r-1} - 2^{r-1}$.

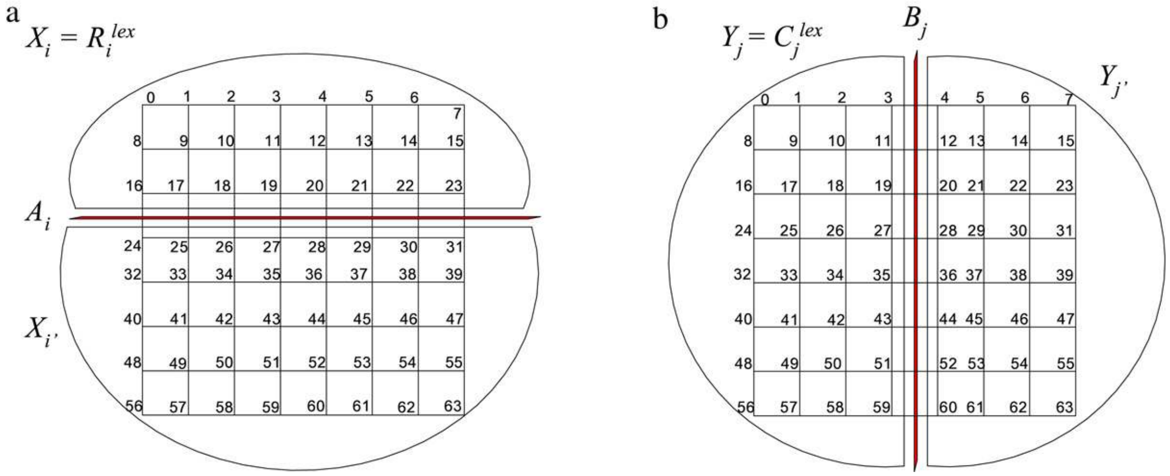

Fig. 10. (a) Each A_i is an edge cut of $M\left[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}\right]$ which disconnects $M\left[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}\right]$ into two components X_i and $X_{i'}$ where X_i is R_i^{lex}. A_i is called a horizontal edge cut. (b) Each B_j is an edge cut of $M\left[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}\right]$ which disconnects $M\left[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}\right]$ into two components Y_j and $Y_{j'}$ where Y_j is C_j^{lex}. B_j is called a vertical edge cut.

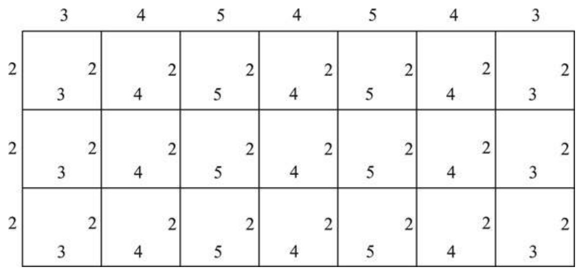

Fig. 11. The edge congestion of the edges of $M\left[2^2 \times 2^3\right]$ induced by the lexicographic embedding of Q^5 into $M\left[2^2 \times 2^3\right]$.

Proof. Bezrukov et al. [4,5] have proved that the lexicographic embedding *lex* of Q^r into P_{2^r} induces a minimum wirelength. In other words, $WL(Q^r, P_{2^r}) = WL_{lex}(Q^r, P_{2^r})$. Thus it is enough to show that $WL_{lex}(Q^r, P_{2^r}) = 2^{2r-1} - 2^{r-1}$. We prove the result by induction on r. The base case is trivial. Assume that the result is true for $WL_{lex}(Q^{k-1}, P_{2^{k-1}})$. Then

$$\begin{aligned} WL_{lex}(Q^k, P_{2^k}) &= 2WL_{lex}(Q^{k-1}, P_{2^{k-1}}) + 2^{k-1} \times 2^{k-1} \\ &= 2(2^{2(k-1)-1} - 2^{(k-1)-1}) + 2^{k-1} \times 2^{k-1} \\ &= 2^{2k-1} - 2^{k-1}. \quad \square \end{aligned}$$

Now we derive a similar expression for $WL(Q^r, M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}])$.

Theorem 5. *The wirelength of Q^r into $M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}]$ is given by*

$$WL(Q^r, M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}]) = 2^{\lfloor r/2\rfloor}(2^{2\lceil r/2\rceil-1} - 2^{\lceil r/2\rceil-1}) + 2^{\lceil r/2\rceil}(2^{2\lfloor r/2\rfloor-1} - 2^{\lfloor r/2\rfloor-1}).$$

Proof. The lexicographic embedding has a nice symmetric property. The edges of Q^r are stretched in the grid $M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}]$ either vertically or horizontally. Moreover each edge of B_j (vertical edge cut) has the same edge congestion. In the same way, each edge of A_i (horizontal edge cut) has the same edge congestion. Thus the wirelength of each row is the same. The same holds for columns of $M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}]$. See Fig. 11. The wirelength of each row of $M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}]$ is $2^{2\lceil r/2\rceil-1} - 2^{\lceil r/2\rceil-1}$ by Theorem 4 and there are $2^{\lfloor r/2\rfloor}$ rows in $M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}]$. In the same way the wirelength along the columns is $2^{\lceil r/2\rceil}(2^{2\lfloor r/2\rfloor-1} - 2^{\lfloor r/2\rfloor-1})$. Hence the theorem. $\square$

As a by product of this work, we observe a solution to the edge congestion problem of Q^r into $M[2^{\lfloor r/2\rfloor} \times 2^{\lceil r/2\rceil}]$. It is known that [5]

$$EC(Q^r, P_{2^r}) = \begin{cases} \dfrac{2^{r+1}-2}{3} & \text{if } r \text{ is even} \\[2ex] \dfrac{2^{r+1}-1}{3} & \text{if } r \text{ is odd.} \end{cases}$$

From the proof of Theorem 5, we observe that

$$EC_{lex}(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]) = \begin{cases} \dfrac{2^{\lceil r/2 \rceil+1} - 2}{3} & \text{if } \lceil r/2 \rceil \text{ is even} \\ \dfrac{2^{\lceil r/2 \rceil+1} - 1}{3} & \text{if } \lceil r/2 \rceil \text{ is odd.} \end{cases}$$

Since $EC(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]) = EC_{lex}(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}])$, we have

$$EC(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]) = \begin{cases} \dfrac{2^{\lceil r/2 \rceil+1} - 2}{3} & \text{if } \lceil r/2 \rceil \text{ is even} \\ \dfrac{2^{\lceil r/2 \rceil+1} - 1}{3} & \text{if } \lceil r/2 \rceil \text{ is odd.} \end{cases}$$

7. Further research

The hypercube has many desirable and attractive properties. However it has its own intrinsic drawbacks; for instance its diameter is large. As a result, several enhancements of the hypercube have been proposed to improve some properties such as diameter. These include crossed cubes, folded hypercubes and enhanced hypercubes [28]. Folded hypercube is one of the interesting variants of the hypercube.

Definition 5. The r-dimensional folded hypercube, denoted by FQ^r is an undirected graph obtained from Q^r by adding all complementary edges. For two vertices $x = x_1x_2, \ldots, x_r$ and $y = y_1y_2, \ldots, y_r$ of FQ^r, (x, y) is a complementary edge if and only if their bits are complements of each other. □

The authors leave the following results as conjectures for further research.
Conjecture 1: $WL(FQ^r, P_{2^r}) = 3 \times 2^{2r-2} - 2^{r-1}$.
Conjecture 2: The wirelength of FQ^r into $M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]$ is given by

$$WL(FQ^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}]) = 2^{\lfloor r/2 \rfloor}(3 \times 2^{2\lceil r/2 \rceil-1} - 2^{\lceil r/2 \rceil-1}) + 2^{\lceil r/2 \rceil}(3 \times 2^{2\lfloor r/2 \rfloor-1} - 2^{\lfloor r/2 \rfloor-1}). \quad \square$$

8. Conclusion

We identify an exact layout of hypercubes and folded hypercubes on a grid. This layout solves the wirelength problem and edge congestion problem of hypercubes into grids. We also provide formulae for $WL(Q^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}])$ and $WL(FQ^r, M[2^{\lfloor r/2 \rfloor} \times 2^{\lceil r/2 \rceil}])$. Apart from these results, there are other interesting results in this paper. We analyze the maximum subgraph problem of hypercubes. Another significant result is the Congestion Lemma and Partition Lemma which yield a new technique to estimate the lower bound of wirelength. The interested readers will observe that it is not easy to extend these results further to other hypercube-like topologies such as crossed cubes, enhanced cubes, augmented cubes, and Fibonacci cubes. It is also an interesting research topic to verify whether this technique can be employed to solve the wirelength problem for architectures such as butterfly, torus, star, and pancake. □

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References

[1] S.N. Bhatt, F.T. Leighton, A framework for solving VLSI graph layout problems, Journal of Computer and System Sciences 28 (1984) 300–343.
[2] S. Bettayeb, B. Cong, M. Girou, I.H. Sudborough, Embedding of star networks into hypercubes, IEEE Transactions on Computers 45 (1996) 186–194.
[3] S.L. Bezrukov, Embedding complete trees into the hypercube, Discrete Applied Mathematics 110 (2001) 101–119.
[4] S.L. Bezrukov, J.D. Chavez, L.H. Harper, M. Röttger, U.P. Schroeder, Embedding of hypercubes into grids, in: MFCS, 1998, pp. 693–701.
[5] S.L. Bezrukov, J.D. Chavez, L.H. Harper, M. Röttger, U.P. Schroeder, The congestion of n-cube layout on a rectangular grid, Discrete Mathematics 213 (2000) 13–19.
[6] S. Bezrukov, B. Monien, W. Unger, G. Wechsung, Embedding ladders and caterpillars into the hypercube, Discrete Applied Mathematics 83 (1998) 21–29.
[7] A.J. Boals, A.K. Gupta, N.A. Sherwani, Incomplete hypercubes: Algorithms and embeddings, The Journal of Supercomputing 8 (1994) 263–294.
[8] R. Caha, V. Koubek, Optimal embeddings of generalized ladders into hypercubes, Discrete Mathematics 233 (2001) 65–83.
[9] J.D. Chavez, R. Trapp, The cyclic cutwidth of trees, Discrete Applied Mathematics 87 (1998) 25–32.
[10] David R. Wood, Degree constrained book embeddings, Journal of Algorithms 45 (2002) 144–154.
[11] G. Di Battista, P. Eades, R. Tamassia, I.G. Tollis, Algorithms for drawing graphs: An annotated bibliography, Computational Geometry: Theory and Applications 4 (1994) 235–282.
[12] H.N. Djidjev, I. Vrto, Crossing numbers and cutwidths, Journal of Graph Algorithms and Applications 7 (2003) 245–251.
[13] B. Doina, W.B. Wolfgang, B. Natasa, L. Shahram, An optimal embedding of honeycomb networks into hypercubes, Parallel Processing Letters 14 (2004) 367–375.
[14] J. Erbele, J. Chavez, R. Trapp, The Cyclic Cutwidth of Q, Manuscript, California State University, San Bernardino USA, 2003.

[15] M.R. Garey, D.S. Johnson, Computers and Intractability, A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.
[16] C. Guu, The Circular Wirelength Problem for Hyper-cubes Ph.D. dissertation, University of California, Riverside, 1997.
[17] L.H. Harper, Global Methods for Combinatorial Isoperimetric Problems, Cambridge University Press, 2004.
[18] Hsing-Lung Chen, Nian-Feng Tzeng, A Boolean expression-based approach for maximum incomplete subcube identification in faulty hypercubes, IEEE Transactions on Parallel and Distributed Systems 8 (1997).
[19] Indra Rajasingh, Jasintha Quadras, Paul Manuel, Albert William, Embedding of cycles into arbitrary trees, Networks 44 (2004) 173–178.
[20] H. Katseff, Incomplete Hypercubes, IEEE Transactions on Computers 37 (1988) 604–608.
[21] M. Klugerman, A. Russell, R. Sundaram, On embedding complete graphs into hypercubes, Discrete Mathematics 186 (1998) 289–293.
[22] Y.L. Lai, K. Williams, A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs, Journal of Graph Theory 31 (1999) 75–94.
[23] Lin Yixun, Bandwidth and cyclic bandwidth, Systems Science and Systems Engineering 5 (1996) 296–302.
[24] A. Matsubayashi, S. Ueno, Small congestion embedding of graphs into hypercubes, Networks 33 (1999) 71–77.
[25] J. Opatrny, D. Sotteau, Embeddings of complete binary trees into grids and extended grids with total vertex-congestion 1, Discrete Applied Mathematics 98 (2000) 237–254.
[26] M. Rottger, U.P. Schroeder, Efficient embeddings of grids into grids, Discrete Applied Mathematics 108 (1-2) (2001) 143–173.
[27] I. Vrto, Cutwidth of the r-dimensional mesh of d-ary trees, Theoretical Informatics and Applications 34 (2000) 515–519.
[28] J. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, 2001.