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Excellent γ - stable graphs

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Abstract. A graph G is said to be γ – stable if γ (G_{xy}) = γ (G), for all x, y ∈ V (G), x not adjacent to y, where G_{xy} denotes the graph obtained by identifying the vertices x, y. In this paper we have obtained conditions under which a graph G is γ - stable and very excellent. We have proved that if G is γ - stable and very excellent, then it is domination subdivision stable. We have obtained method of generating γ - stable graphs when G is γ - stable and just excellent. Properties satisfied by a γ - stable graph having cut vertices are proved.

1. Introduction

We consider only simple connected undirected graphs G = (V, E). We say that H is a subgraph of G, if V (H) ⊆ V (G) and (uv)∈ E(H) implies (uv)∈ E(G). If a subgraph H satisfies the added property that for every pair u, v of vertices, (uv)∈ E (H) if and only if (uv)∈ E(G), then H is called an induced subgraph of G and is denoted by < H >. The open neighborhood of vertex v ∈V (G) is denoted by N (v) = { u ∈ V (G) / (uv) ∈ E (G) } while its closed neighborhood is the setN [v] = N (v) ∪ { v }. A cut vertex of a graph G is a vertex whose deletion increases the number of components. We write G – v or G – S for the subgraph obtained by deleting a vertex v or set of vertices S. We indicate that u is adjacent to v by writing u ⊥ v. For details on graph properties we refer to [1].

A set of vertices D in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set – abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by γ(G). γ - set denotes a dominating set for G with minimum cardinality. A set of vertices D in a graph G is called a clique dominating set if every two vertices in D are adjacent.

The private neighborhood of v ∈ D is defined by pn[v, D] = N [v] – N [D − { v }]. The open private neighborhood of v ∈ D is denoted by pn(v, D), is defined by pn(v, D) = N (v) – N (D − { v }). A vertex v is said to be a, level vertex if γ (G – v) = γ (G), up vertex if γ (G – v) > γ (G) and down vertex if γ (G – v) < γ (G). A vertex v is said to be selfish in the γ- set D, if v is needed only to dominate itself. A vertex in V – D is k – dominated if it is dominated by at least k – vertices in D that is | N (v) ∩ D | ≥ k. If every vertex in V – D is k – dominated then D is called k – dominating set. For details of on domination we refer to [2, 3].

2. Materials and methods

A vertex v is said to be good if there is a γ – set of G containing v. If there is no γ – set of G containing v, then v is said to be a bad vertex. A graph G is said to be excellent if given any vertex v then there is a γ - set of G containing v. In [4], Yamuna and Sridharan have introduced the concept of very

excellent graphs. An excellent graph G is said to be very excellent (VE), if there is a γ - set D of G such that to each vertex $u \in V - D$ there is a vertex $v \in D$ such that $D - \{ v \} \cup \{ u \}$ is a γ - set of G. A γ - set D of G satisfying this property is called a very excellent γ - set of G. In this case we say that u and v are vertex exchangeable.
In all the figures encircled vertices denote a γ - set for G.
Example

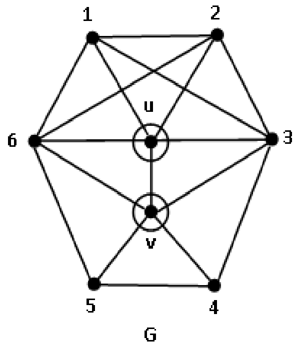


Figure 1.
In Fig. 1, $D = \{ u, v \}$ is a γ - set for G. $D - \{ u \} \cup \{ 1 \}$, $D - \{ u \} \cup \{ 2 \}$, $D - \{ u \} \cup \{ 3 \}$, $D - \{ u \} \cup \{ 6 \}$, $D - \{ v\} \cup \{ 4 \}$, $D - \{ v \} \cup \{ 5 \}$, are γ - sets for G. In this figure u is exchangeable with 1, 2, 3 and v is exchangeable with 6, 4 and 5 respectively.
In [4], they have proved the following result
R$_1$. A graph G is VE if and if only there exist a γ - set D of G such that to each $u \notin D$, there is any $v \in D$ such that $pn[v, D] \subset N [u]$.
For a given non – adjacent pair { x, y } in a graph G, we denote by G_{xy} the graph obtained by deleting x and y and adding a new vertex xy adjacent to precisely those vertices of $G - \{ x, y \}$ which were adjacent to at least one of x or y in G. We say that $G_{xy}$is obtained by contracting on {x, y} [5].
In [6], Yamuna and Karthika have introduced the γ - stable graphs. A γ - set $D \subseteq V$ is said to graph domination set if D is a clique dominating set for G, that is $\gamma (G_{xy}) = \gamma (G)$.
Example

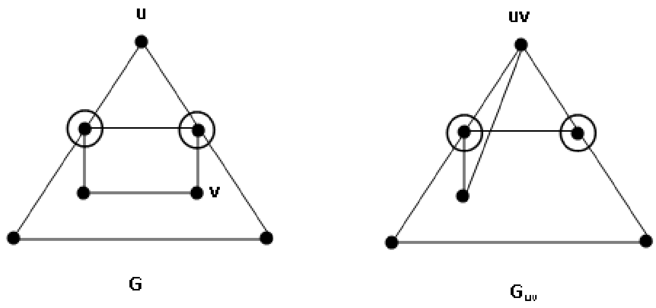
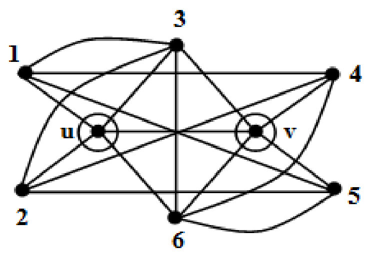


Figure 2.
In Fig. 2, $\gamma (G) = \gamma (G_{uv}) = 2$, this is true for all $x, y \in V (G)$, where x is not adjacent to y.
In [6], they have proved the following results
R$_2$. A graph G is γ - stable if and only if every γ - set D of G is clique dominating.
R$_3$. If G is γ - stable, then $| pn[u, D] | \geq 2$, for all $u \in V (G)$.

3. Results and discussion
In this section we have obtained results when a γ - stable graph is very excellent. We have given a procedure of generating γ - stable graphs, when a γ - stable is just excellent. Also properties satisfied by a γ - stable graph with cut vertices are discussed.

Theorem 1
Let G be a γ – stable graph. G is VE if and if only there exists a γ – set D of G such that to each $v \notin D$, there is any $u \in D$ such that $pn[u, D] \subset N[v]$ and v is adjacent to $D - \{u\}$.
Proof
Let G be a VE graph. Then by R1, $pn[u, D] \subset N[v]$. Since G is γ – stable, $< D - \{u\} \cup \{v\} >$ is a clique, implies v is adjacent to $D - \{u\}$.
Conversely, if the conditions of the Theorem are satisfied, then D is a VE γ – set for G (by R1), implies G is VE.
Theorem 2
If G is γ – stable, then there is no $v \in pn(u, D)$ such that v is adjacent to all $w \in pn(u, D)$, $w \neq v$.
Proof
Let G be a γ – stable graph. Let D be a γ – set for G. Suppose there exist one $v \in pn(u, D)$ such that v is adjacent to all $w \in pn(u, D)$, $w \neq v$, then $D' = D - \{u\} \cup \{v\}$ is a γ – set for G. This implies $\langle D' \rangle$ is not a clique for G (since $v \in pn(u, D)$, v is not adjacent any other vertex in D except u). Hence there is no $v \in pn(u, D)$ such that v adjacent to all $w \in pn(u, D)$, $w \neq v$, that is $pn(u, D) \not\subset N[v]$, for all $v \in pn(u, D)$.
Remark
If G is γ - stable and VE, then by Theorem 2 we observe that for any $v \in pn(u, D)$, $D' = D - \{u\} \cup \{v\}$ is not a γ - set, that is v cannot be exchanged with u.
Theorem 3
There is no graph G that is γ - stable VE such that $\gamma(G) \geq 3$.
Proof
If possible assume that there is a γ - stable VE graph such that $\gamma(G) \geq 3$. Let $D = \{v_1, v_2, \ldots, v_k\}$ be a γ - set for G, where $k \geq 3$. Since G is γ - stable, by R_3 we know that $|pn[v_i, D]| \geq 2$, for all $v_i \in D$, $i = 1, 2, \ldots, k$. Let $u \in pn[v_i, D]$ for some $v_i \in D$. Since G is VE, there is one $v_j \in D$ such that $D' = D - \{v_j\} \cup \{u\}$ is a γ - set for G. By Remark of Theorem 2, we know that $v_j \neq v_i$, that is there is one $v_j \in D$ such that D' is a γ - set for G and $v_j \neq v_i$. Since G is γ - stable and $u \in D'$, $D \cap D' = \{v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k\}$. By Theorem 1, we know that u adjacent to every vertex in $D \cap D'$. Since $\gamma(G) \geq 3$, $|D \cap D'| \geq 2$. This means that u is atleast 2 – dominated, which implies $u \notin pn[v_i, D]$, a contradiction. This is true for all $u \in pn[v_i, D]$. This implies $|pn[v_i, D]| = \phi$, which is not possible as G is a γ – stable graph. Hence there is no graph G, which is γ - stable VE such that $\gamma(G) \geq 3$.
Conclusion
From the above discussion we conclude that if G is VE and γ - stable, then $\gamma(G) \leq 2$.
If $\gamma(G) = 1$, K_n is the only possible graph.
Example



G

Figure 3.

In Fig. 3, $\gamma(G) = 2$, $D - \{v\} \cup \{1\}$, $D - \{v\} \cup \{2\}$, $D - \{v\} \cup \{6\}$, $D - \{u\} \cup \{4\}$, $D - \{u\} \cup \{5\}$ and $D - \{u\} \cup \{3\}$ are γ - sets for G and every γ - set is a clique, implies G is γ - stable and VE with $\gamma(G) = 2$.
Observation 1

A γ – stable graph does not have a down vertex.

Proof

If $u \in V(G)$ is a down vertex, then there is a γ – set D for G such that $|pn[u, D]| = \phi$, which is not possible by R3.

Theorem 4

If G is VE and γ - stable, then every vertex of G is a level vertex.

Proof

Let G be a VE and γ – stable graph. By Observation 1 we know that G does not have any down vertex. By Theorem 3, we know that if G is VE and γ – stable, $\gamma(G) \leq 2$. Let D = { u, v } be a VE γ – set for G such that | D | = 2. Let $V_1 = \{u_1, u_2, \ldots, u_p\}$ be the set of vertices in V – D such that $D - \{u\} \cup \{u_i\}$ are γ – sets for G, i = 1, 2, …, p. Let $V_2 = \{v_1, v_2, \ldots, v_q\}$ be the set of vertices in V – D such that $D - \{v\} \cup \{v_j\}$ are γ – sets for G, j = 1, 2, …, q, p, q ≥ 2 (by R3).

Let D' = { u_i, v }, D'' = { u, v_j }, i = 1, 2, …, p, j = 1, 2, …, q. D' are γ – sets for G not containing u, v_1, v_2, …, v_p. D'' are γ – sets for G not containing v, u_1, u_2, …,u_q. For every v in G, there is a γ – set not containing v, implies G has no up vertices [2].

Let D = { u } be a VE γ – set for G such that | D | = 1. Since $\gamma(G) = 1$ and G is VE, any vertex is a γ – set for G. For all $v \in V(G)$, there is a γ – set not containing v, implies G has no up vertices [2].

In [8], Yamuna and Karthika have introduced the concept of domination subdivision stable graphs. A graph G is said to be domination subdivision stable (DSS) if the γ - value of G does not change by subdividing any edge of G. We shall denote the graph obtained by subdividing any edge e = (uv) of a graph G, by $G_{sd}uv$. Let w be a vertex introduced by subdividing (uv). We shall denote this by $G_{sd}uv$ = w.

In [9], they have proved the following results

R_4. A graph G is DSS if and only if for every u, $v \in V(G)$, either

i. There is a γ - set containing u and v or there is a γ - set D such that
ii. pn[u, D] = { v }, or
iii. v is atleast 2 – dominated.

Example

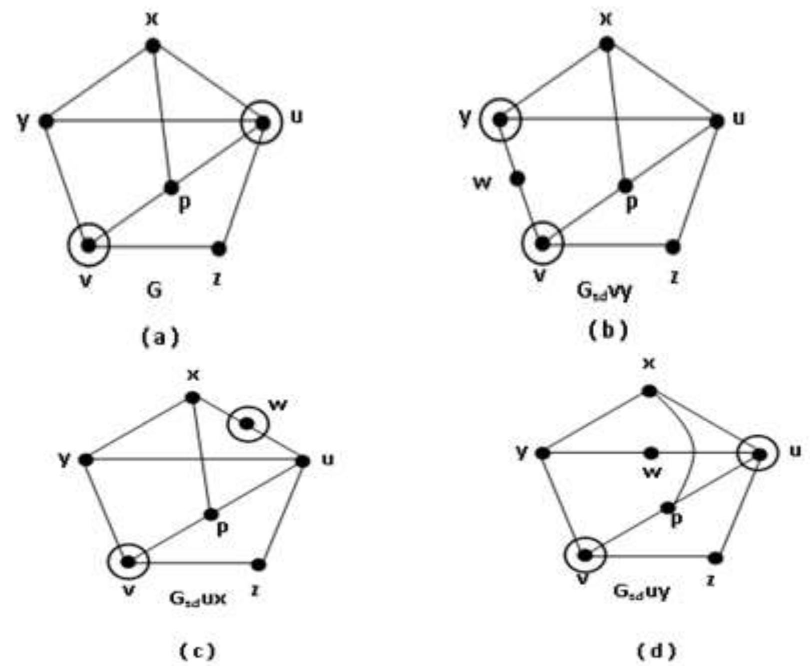


Figure 4.

In Fig. 4, D = γ (G) = { u, v } = 2. In Fig. 4 (b), there is a γ - set containing v, y and $G_{sd}vy$ = 2. In Fig. 4 (c), x $\in$pn [u, D] and $G_{sd}ux$ = 2. In Fig. 4 (d), y is 2 - dominated and $G_{sd}uy$ = 2. This is true for all e = (a b) $\in$ E (G), which implies that G is a DSS graph.

Theorem 5

Let G be any γ - stable VE graph. Then G is DSS.

Proof

If G is VE and γ – stable, we know that γ (G) $\leq$ 2. Let D = { u, v } be a VE γ – set for G. Let V_1 = { $u_1, u_2, \ldots, u_p$ } be a set of vertices in V – D such that D – { u } $\cup$ { u_i } are γ – sets for G, i = 1, 2, …, p. Let V_2 = { $v_1, v_2, \ldots, v_q$ } be the set of vertices in V – D such that D – { v } $\cup$ { v_j } are γ – sets for G, j = 1, 2, …, q.

D' = { u_i, v } and D'' = { u, v_j } are γ – sets for G, for all i = 1, 2, …, p, j = 1, 2, …, q.

Claim

For all $u_i \in$pn [v, D], $v_j \in$pn [u, D], 2 $\leq$ i $\leq$ p, 2 $\leq$ j $\leq$ q, { u_i, v_j } is a γ – set for G, if u_i adjacent to v_j.

Proof

u_i dominates private neighbors of u, v_j dominates private neighbors of v. Let x $\in$ V (G) such that x is 2 – dominated with respect to D. Since G is VE, either D – { u } $\cup$ { x } or D – { v } $\cup$ { x } is a γ – set for G. Assume that D''' = D – { u } $\cup$ { x } is a γ – set for G. Since v_j is a private neighbor of u, v_j can be dominated only by x. If x was exchangeable with v, then by similar arguments u_i can be dominated by x only. So, every k – dominated vertex is adjacent to either u_i or v_j. So { u_i, v_j } is a γ – set for G, whenever $u_i \perp v_j$.

Let D^{iv} = { u_i, v_j }, for all $u_i \in$pn [v, D], $v_j \in$pn [u, D], $u_i \perp v_j$.

D = { u, v }, D' = { u_i, v }, D'' = { u, v_j }, D^{iv} = { u_i, v_j } are γ – sets for G, for all i =1, 2, …, p and j = 1, 2, …, q.

To prove that, G is DSS we consider all possible edges of G. The possible edges of G are (u_i u), (u_i v), (u_iu_j), (v_i v), (v_i u), (v_iv_j), (u_iv_j), (u v).

We prove that G is DSS, by showing that for all (u, v) $\in$ V (G), u $\perp$ v, atleast one of the conditions of R4 is satisfied.

1. D' is a γ – set for $G_{sd}u_iu$, since u is 2 – dominated by (u_i, v), implies γ ($G_{sd}u_iu$) = γ (G).
2. D' is a γ – set for $G_{sd}u_iv$, since u_i, v $\in$ D, for all i = 1, 2, …, p, implies γ ($G_{sd}u_iv$) = γ (G).
3. D' is a γ – set for $G_{sd}u_iu_j$, i $\neq$ j, i, j = 1, 2, …, p, since every u_j is dominated by (u_i, v), that is each u_j is a 2 – dominated vertex, implies γ ($G_{sd}u_iu_j$) = γ (G).
4. a) D^{iv} is a γ – set for $G_{sd}u_iv_j$, for all $u_i \in$pn [v, D], $v_j \in$pn [u, D], $u_i \perp v_j$, i $\neq$ j, i = 1, 2, …, p, j = 1, 2, …, q.
   b) D'' is a γ – set for $G_{sd}u_iv_j$, if u_i is 2 – dominated.
   c) D' is a γ – set for $G_{sd}u_iv_j$, if v_j is 2 – dominated.
   d) D' and D'' are γ – sets for $G_{sd}u_iv_j$, if u_i and v_j are both 2 – dominated.

In all cases a – d, γ ($G_{sd}u_iv_j$) = γ (G).

5. D is a γ – set for $G_{sd}uv$, since u, v $\in$ D, implies γ ($G_{sd}uv$) = γ (G).
6. D'' is a γ – set for $G_{sd}v_jv$, since v is 2 – dominated by (v_j, u), implies γ ($G_{sd}v_jv$) = γ (G).
7. D'' is a γ – set for $G_{sd}v_ju$, since v_j, u $\in$ D, for all j = 1, 2, …, q, implies γ ($G_{sd}v_ju$) = γ (G).
8. D'' is a γ – set for $G_{sd}v_iv_j$, i $\neq$ j, i, j = 1, 2, …, q, since every v_i is dominated by (v_j, u), that is each v_i is a 2 – dominated vertex, implies γ ($G_{sd}v_iv_j$) = γ (G).

So we conclude that, for all (u, v) $\in$ V (G), u $\perp$ v, atleast one of the conditions of R4 is satisfied, implies G is DSS.

In [10], Yamuna and Sridharan had defined a graph G to be Just excellent (JE), if to each u $\in$ V, there is a unique γ - set of G containing u.

Example

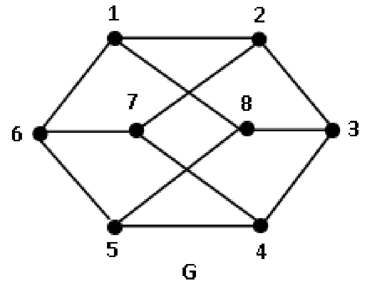


Figure 5.

In Fig. 5, G is a JE graph and { 1, 4 }, { 2, 5 }, { 3, 6 }, { 7, 8 } are the distinct γ - sets for G.
In [10], they have proved the following results

R$_5$. A graph G is JE if and only if,

i. γ (G) divides n.
ii. d (G) = n / γ (G), where d (G) denotes the domatic partition of G.
iii. G has exactly n / γ (G) distinct γ - sets.

R$_6$.. In a JE graph, $G \neq \bar{K}_n$ every vertex u is a level vertex, and also γ(G – u) = γ (G).

Theorem 6

If G is a JE and γ - stable graph, then G – u is γ - stable for every u ∈ V(G).

Proof

If G is JE, we know that every u ∈ V (G) is a level vertex by R$_6$. Let d (G) = | { V_1 }, { V_2 }, …, { V_k } |. By second condition in R$_5$ we know that k = n / γ (G). Let u ∈ V_i. Consider G – u. By third condition in R$_5$, we know that { V_1, V_2, …, V_{i-1}, V_{i+1}, …, V_k } are γ - sets for G – u also. We claim that these are the only possible γ - sets for G – u. If possible assume that D is a γ - set for G – u but not for G. Let v be any vertex in D. Since G is JE there is a unique γ - set for G containing v, say v ∈V_j. Since D ≠ V_j, D and V_j are two γ - sets for G containing v, a contradiction as every γ - set of G is unique. This means that { V_1, V_2, …, V_{i-1}, V_{i+1}, …, V_k } are the only possible γ - sets for G – u. This implies G – u is γ - stable, for all u ∈V (G).

Remark

By Theorem 6 we can generate γ - stable graphs from G, if G is a JE γ - stable graph with n vertices. We can generate a maximum of n - γ - stable graphs if G – u is distinct, for all u ∈ V (G).

Example

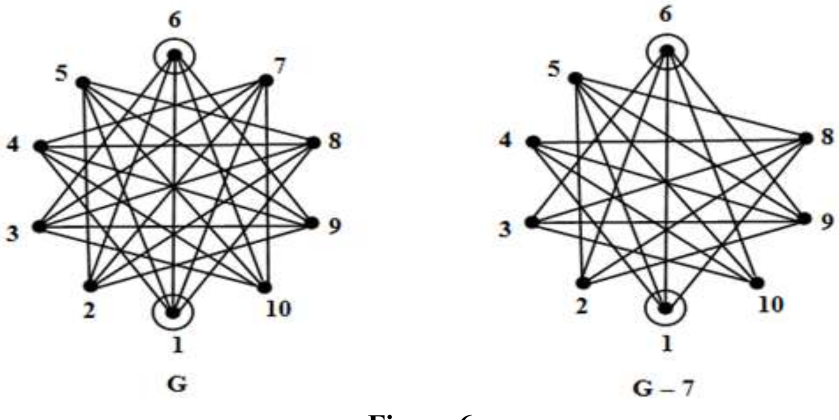


Figure 6.

In Fig. 6, γ (G) = γ (G – 7) = { 6, 7 } and G – 7 is also γ – stable. γ(G – u) is isomorphic toγ (G – 7), for all u ∈ V (G).

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