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He's variational iteration method for treating nonlinear singular boundary value problems

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ABSTRACT

This paper applies He's variational iteration method for solving nonlinear singular boundary value problems. The solution process is illustrated and various physically relevant results are obtained. Comparison of the obtained results with exact solutions shows that the method used is an effective and highly promising method for treating various classes of both linear and nonlinear singular boundary value problems.

1. Introduction

The aim of this paper is to introduce He's variational iteration method for the numerical solution of the following class of singular boundary value problems:

$$y'' + \frac{\alpha}{x} y' + f(x, y) = 0 \tag{1}$$

subject to the boundary conditions

$$y(0) = A \text{ (or } y'(0) = B), \qquad y(1) = C \text{ (or } \eta y(1) + \beta y'(1) = \mu). \tag{2}$$

If $\alpha = 1$, (1) becomes a cylindrical problem, and if $\alpha = 2$, then it becomes a spherical problem, where A, B, C, η, β and μ are real constants. It is well known that (1) has a unique solution if $f(x, y)$ is a continuous function, $\partial f/\partial y$ exists and is continuous and $\partial f/\partial y \geq 0$ [1]. Accurate and fast numerical solution of two-point singular boundary value problems for ordinary differential equations is necessary in many important scientific and engineering applications, e.g. reactant concentration in a chemical reactor, boundary layer theory, control and optimization theory, and flow networks in biology, areas of astrophysics such as the theory of stellar interiors, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents. In recent years, seeking numerical solutions of singular differential equations has been the focus of a number of authors. In [2] the original differential equation is modified at a singular point and then the boundary value problem is treated by using cubic splines. Kamel Al-Khaled [3] used the Sinc–Galerkin method and homotopy perturbation method for finding the approximate solution of a certain class of singular two-point boundary value problems. In [4] a method based on *B*-splines for solving a class of singular boundary value problems was presented. Sami Bataineh et al. [5] used the modified homotopy analysis method to obtain the approximate solutions of singular two-point boundary value problems. Ravi Kanth and Bhattacharya [6] used a quasilinearization technique to reduce a class of nonlinear singular boundary value problems arising in physiology to a sequence of linear problems; the resulting set of

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differential equations are modified at the singular point and the spline technique is utilized to obtain a numerical solution. Recently, the application of the differential transform method [7] was extended to singular boundary value problems, and the homotopy perturbation method [8] was extended to singular initial value problems.

In this paper, we applied He's variational iteration method for treating linear and nonlinear singular two-point boundary value problems. The variational iteration method was first proposed by He [9–11] and has been proved by many authors to be a powerful mathematical tool for treating various kinds of nonlinear problems [12–25]. The idea of the method is based on constructing a correction functional using a general Lagrange multiplier and the multiplier is chosen in such away that its correction solution is improved with respect to the initial approximation or to the trial function. Salkuyeh [26] studied the convergence of the variational iteration method for solving a linear system of ordinary differential equations. Recently, Nicolae Herişanu and Vasile Marinca [27] presented a modified variational iteration method for treating strongly nonlinear problems. For a more comprehensive survey on this method and its applications, the reader can read the review articles [28–30] and references therein.

2. He's variational iteration method

Now, to illustrate the basic concept of the method, we consider the following general nonlinear differential equation given in the form

$$Ly(x) + Ny(x) = g(x) \tag{3}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is a known analytical function; we can construct a correction functional according to the variational method as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(Ly_n(\xi) + N\tilde{y}_n(\xi) - g(\xi))\mathrm{d}\xi, \quad n \geq 0 \tag{4}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, the subscript n denotes the nth approximation, and $\tilde{y}_n$ is considered as a restricted variation, namely $\delta\tilde{y}_n = 0$. Successive approximations, $y_{n+1}(x)$, will be obtained by applying the Lagrange multiplier obtained and a properly chosen initial approximation $y_0(x)$.

3. Convergence analysis of nonlinear singular boundary value problems

Before we begin the simulations of the iterative formulas in (4) we shall analyze the convergence of a general nonlinear singular differential equation of the form (1) and (2). We shall then adopt the variational iteration method strategy in constructing the correction functional as below:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left[(y_n)_{ss} + \frac{\alpha}{s}(y_n)_s + \tilde{f}(s, y_n) - \tilde{g}(s)\right]\mathrm{d}s, \quad n \geq 0. \tag{5}$$

To find the optimal value of $\lambda(s)$, we proceed as follows; we take the variation with respect to $y_n(x)$

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(s)\left[(y_n)_{ss} + \frac{\alpha}{s}(y_n)_s + \tilde{f}(s, y_n) - \tilde{g}(s)\right]\mathrm{d}s \tag{6}$$

or

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \lambda(s)\left[(y_n)_{ss} + \frac{\alpha}{s}(y_n)_s\right]\mathrm{d}s \tag{7}$$

which gives

$$\delta y_{n+1}(x) = \left[1 - \lambda'(x) + \frac{\alpha}{x}\lambda(x)\right]\delta y_n(x) + \delta\lambda(x).(y_n)_s(x) + \int_0^x \delta y_n\left[\lambda''(s) - \alpha\frac{s\lambda'(s) - \lambda(s)}{s^2}\right]\mathrm{d}s = 0. \tag{8}$$

Hence, we obtain the stationary conditions:

$$1 - \lambda'(x) + \frac{\alpha}{x}\lambda(x) = 0, \quad \lambda(x) = 0, \qquad \lambda''(x) - \alpha\frac{x\lambda'(x) - \lambda(x)}{x^2} = 0. \tag{9}$$

Case (i): when $\alpha = 1$, the Lagrange multiplier is obtained as

$$\lambda(s) = s\log\left(\frac{s}{x}\right) \tag{10}$$

and then the correction functional (5) can be written as

$$y_{n+1}(x) = y_n(x) + \int_0^x s\log\left(\frac{s}{x}\right)\left[(y_n)_{ss} + \frac{\alpha}{s}(y_n)_s + \tilde{f}(s, y_n) - \tilde{g}(s)\right]\mathrm{d}s. \tag{11}$$

The variational iteration formula (12) produces recurrence sequences, i.e. $\{y_n(x)\}$. Obviously, the limit of these sequences will be the solution of (1) and (2) if the sequences are convergent. In order to prove that the sequences $\{y_n(x)\}$ are convergent, we construct the series

$$y_0(x) + [y_1(x) - y_0(x)] + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots. \tag{12}$$

Note that

$$s_{n+1}(x) = y_0(x) + [y_1(x) - y_0(x)] + \cdots + [y_n(x) - y_{n-1}(x)] = y_n(x). \tag{13}$$

The sequences $\{y_n(x)\}$ will be convergent if all the series are convergent. Now we show that the sequences $\{y_n(x)\}$ defined with $y_0(x) = a$ converge to $\{y_n(x)\}$. To do this, we state and prove the following theorem.

Theorem 1. *Suppose that $\{y_n(x)\} \in [0, 1]$, $n = 0, 1, 2, \ldots$. The sequences defined by (12) with $y_0(x) = a$ will converge to $\{y_n(x)\}$, the exact solution of the boundary value problems for (1) and (2) (if $\alpha = 1$).*

Proof. According to (12), note that

$$|y_1(x) - y_0(x)| = \left|\int_0^x s\log\left(\frac{s}{x}\right)\left\{(y_0)_{ss} + \frac{1}{s}(y_0)_s + \tilde{f}(s, y_n) - \tilde{g}(s)\mathrm{d}s\right\}\right| \tag{14}$$

$$\leq \int_0^x \left|s\log\left(\frac{s}{x}\right)\right|\left\{\left|(y_0)_{ss} + \frac{1}{s}(y_0)_s\right| + \left|\tilde{f}(s, y_n) - \tilde{g}(s)\right|\right\}\mathrm{d}s \tag{15}$$

$$\leq \int_0^x B_i\left\{\|C_i\|_\infty + \|D_i\|_\infty + \|E_i\|_\infty\right\}\mathrm{d}s \tag{16}$$

$$\leq M\int_0^x \mathrm{d}s \tag{17}$$

$$= Mx. \tag{18}$$

Since $s \leq x \leq 1$, we deduce that

$$\left|s\log\left(\frac{s}{x}\right)\right| \leq |s|\,|\log s - \log x| \tag{19}$$

$$= B_i \tag{20}$$

and

$$M = \mathrm{Max}\left\{B_i\left(\|C_i\|_\infty + \|D_i\|_\infty + \|E_i\|_\infty\right), \|C_i\|_\infty\right\}. \tag{21}$$

We have $\|C_i\|_\infty \leq \max_i |C_i|$, $\|D_i\|_\infty \leq \max_i |D_i|$ and $\|E_i\|_\infty \leq \max_i |E_i|$.

From (12) and (19), it follows that

$$|y_2(x) - y_1(x)| = \left|\int_0^x s\log\left(\frac{s}{x}\right)\left\{(y_1)_{ss} + \frac{1}{s}(y_1)_s + \tilde{f}(s, y_1) - \tilde{g}(s)\mathrm{d}s\right\}\right| \tag{22}$$

$$|y_2(x) - y_1(x)| \leq \int_0^x \left|s\log\left(\frac{s}{x}\right)\right|\left\{\left|(y_1)_{ss} + \frac{1}{s}(y_1)_s\right| + \left|\tilde{f}(s, y_1) - \tilde{g}(s)\right|\right\}\mathrm{d}s. \tag{23}$$

The right hand side of the inequality can be reduced by eliminating the irrelevant terms in (23), and by considering the absolute value, the following result is obtained:

$$|y_2(x) - y_1(x)| \leq \frac{M^2x^2}{2}. \tag{24}$$

From (18) and (24), we suppose that

$$|y_n(x) - y_{n-1}(x)| \leq \frac{M^nx^n}{n!}. \tag{25}$$

According to mathematical induction, we assume that (25) is valid; then we write

$$|y_{n+1}(x) - y_n(x)| = \left|\int_0^x s\log\left(\frac{s}{x}\right)\left[(y_n)_{ss} + \frac{\alpha}{s}(y_n)_s + \tilde{f}(s, y_n) - \tilde{g}(s)\right]\mathrm{d}s\right| \tag{26}$$

$$\leq M^nB_i\|C_i\|_\infty\int_0^x \frac{s^n}{n!}\mathrm{d}s \tag{27}$$

$$\leq M^{n+1}\frac{x^{n+1}}{(n+1)!}. \tag{28}$$

As we know, the series of $\sum_{n=0}^{\infty}\frac{M^nx^n}{n!}$ is convergent for the whole solution domain $x \in (-\infty, \infty)$; therefore the series of (13) is absolutely convergent, i.e. the sequence $\{y_n(x)\}$ is convergent for $x \in (0, 1)$. □

Case (ii): If $0 \geq \alpha > 1$, the Lagrange multiplier is obtained as

$$\lambda(s) = \frac{s}{1-\alpha} - \frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}. \tag{29}$$

Then the correction functional (5) can be written as

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{s}{1-\alpha} - \frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}\right)\left[(y_n)_{ss} + \frac{\alpha}{s}(y_n)_s + \tilde{f}(s, y_n) - \tilde{g}(s)\right] \mathrm{d}s. \tag{30}$$

The variational iteration formula (30) produces recurrence sequences, i.e. $\{y_n(x)\}$. Obviously, the limit of these sequences will be the solution of (1) and (2) if the sequences are convergent. In order to prove that the sequences $\{y_n(x)\}$ are convergent, we construct the series

$$y_0(x) + [y_1(x) - y_0(x)] + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots. \tag{31}$$

Note that

$$s_{n+1}(x) = y_0(x) + [y_1(x) - y_0(x)] + \cdots + [y_n(x) - y_{n-1}(x)] = y_n(x). \tag{32}$$

The sequences $\{y_n(x)\}$ will be convergent if all the series are convergent. Now we show that the sequences $\{y_n(x)\}$ defined with $y_0(x) = a$ converge to $\{y_n(x)\}$. To do this, we state and prove the following theorem.

Theorem 2. *Suppose that $\{y_n(x)\} \in [0, 1]$, $n = 0, 1, 2, \ldots$. The sequences defined by (30) with $y_0(x) = a$ will converge to $\{y_n(x)\}$, the exact solution of the boundary value problems for (1) and (2) (if $0 \geq \alpha > 1$).*

Proof. According to (30), note that

$$|y_1(x) - y_0(x)| = \left|\int_0^x \left(\frac{s}{1-\alpha} - \frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}\right)\left\{(y_0)_{ss} + \frac{\alpha}{s}(y_0)_s + \tilde{f}(s, y_n) - \tilde{g}(s)\mathrm{d}s\right\}\right| \tag{33}$$

$$\leq \int_0^x \left|\frac{s}{1-\alpha} - \frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}\right| \left\{\left|(y_0)_{ss} + \frac{\alpha}{s}(y_0)_s\right| + \left|\tilde{f}(s, y_n) - \tilde{g}(s)\right|\right\} \mathrm{d}s \tag{34}$$

$$\leq \int_0^x B_i \left\{\|C_i\|_\infty + \|D_i\|_\infty + \|E_i\|_\infty\right\} \mathrm{d}s \tag{35}$$

$$\leq M \int_0^x \mathrm{d}s \tag{36}$$

$$= Mx. \tag{37}$$

Since $s \leq x \leq 1$, we deduce that

$$\left|\frac{s}{1-\alpha} - \frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}\right| \leq \left|\frac{s}{1-\alpha}\right| + \left|\frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}\right| \tag{38}$$

$$= B_i \tag{39}$$

and

$$M = \mathrm{Max}\left\{B_i\left(\|C_i\|_\infty + \|D_i\|_\infty + \|E_i\|_\infty\right), \|C_i\|_\infty\right\} \tag{40}$$

with $\|C_i\|_\infty \leq \max_i |C_i|$, $\|D_i\|_\infty \leq \max_i |D_i|$ and $\|E_i\|_\infty \leq \max_i |E_i|$.

From (32) and (38), it follows that

$$|y_2(x) - y_1(x)| = \left|\int_0^x \left(\frac{s}{1-\alpha} - \frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}\right)\left\{(y_1)_{ss} + \frac{\alpha}{s}(y_1)_s + \tilde{f}(s, y_1) - \tilde{g}(s)\mathrm{d}s\right\}\right| \tag{41}$$

$$|y_2(x) - y_1(x)| \leq \int_0^x \left|\left(\frac{s}{1-\alpha} - \frac{s^{\alpha}}{x^{\alpha-1}(1-\alpha)}\right)\right| \left\{\left|(y_1)_{ss} + \frac{\alpha}{s}(y_1)_s\right| + \left|\tilde{f}(s, y_1) - \tilde{g}(s)\right|\right\} \mathrm{d}s. \tag{42}$$

The right hand side of the inequality can be reduced by eliminating the irrelevant terms in (42), and by considering the absolute value, the following result is obtained:

$$|y_2(x) - y_1(x)| \leq \frac{M^2 x^2}{2}. \tag{43}$$

From (37) and (43), we suppose that

$$|y_n(x) - y_{n-1}(x)| \leq \frac{M^n x^n}{n!}. \tag{44}$$

According to mathematical induction, we assume that (44) is valid; then we write

$$|y_{n+1}(x) - y_n(x)| = \left| \int_0^x \left[(y_n)_{ss} + \frac{\alpha}{s}(y_n)_s + \tilde{f}(s, y_n) - \tilde{g}(s) \right] ds \right| \tag{45}$$

$$\leq M^n B_i \|C_i\|_\infty \int_0^x \frac{s^n}{n!} ds \tag{46}$$

$$\leq M^{n+1} \frac{x^{n+1}}{(n+1)!}. \tag{47}$$

As we know, the series of $\sum_{n=0}^{\infty} \frac{M^n x^n}{n!}$ is convergent for the whole solution domain $x \in (-\infty, \infty)$; therefore the series of (32) is absolutely convergent, i.e., the sequence $\{y_n(x)\}$ is convergent for $x \in (0, 1)$. □

4. Numerical results

To demonstrate the applicability of He's variational iteration method, we have solved several singular boundary value problems. These problems have been chosen because they have been widely discussed in the literature.

Example 1. First we consider the linear singular two-point boundary value problem [23]

$$y'' + \frac{1}{x}y' + y - \frac{5}{4} - \frac{x^2}{16} = 0 \tag{48a}$$

subject to the boundary conditions

$$y'(0) = 0, \qquad y(1) = \frac{17}{16}. \tag{48b}$$

The exact solution of this problem is $y(x) = 1 + \frac{x^2}{16}$. (49)

According to (5), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[s \log\left(\frac{s}{x}\right) \left[(y_n)_{ss} + \frac{1}{s}(y_n)_s + y_n - \frac{5}{4} - \frac{x^2}{16} \right] \right] ds, \quad n \geq 0. \tag{50}$$

We start with the initial approximation $y_0 = a$. The next iterates $y_1, y_2, \ldots$ are given below:

$$y_1 = a - \frac{1}{4}ax^2 + \frac{5}{16}x^2 + \frac{1}{256}x^4 \tag{51}$$

$$y_2 = a - \frac{1}{4}ax^2 + \frac{5}{16}x^2 - \frac{1}{64}x^4 + \frac{1}{64}ax^4 - \frac{1}{9216}x^6. \tag{52}$$

Incorporating the boundary condition at $x = 1$ in (51) and (52), we get

$$y_1 = 0.99479166666667 + 0.06380208333333x^2 + 0.003906250x^4. \tag{53}$$

$$y_2 = 1.00014172335601 + 0.06246456916100x^2 + 0.000002214427437655098x^4 - 0.0001085069444444444x^6. \tag{54}$$

Tables 1(a) and 1(b) exhibits the errors obtained by the proposed method, the solutions obtained in [23] and the exact solution.

Example 2. Consider the nonlinear singular two-point boundary value problem

$$y'' + \frac{1}{x}y' - y^3 + 3y^5 = 0 \tag{55a}$$

subject to the boundary conditions

$$y(0) = 1, \qquad y(1) = \frac{1}{\sqrt{2}}. \tag{55b}$$

The exact solution is $y(x) = \frac{1}{\sqrt{x^2+1}}$. (56)

Table 1(a)
Numerical solutions for Example 1.

x	y_1 solution (present method)	y_1 solution in [24]	Exact solution	Error estimate for the present method	Error estimate for the solution in [24]
0	0.99479166666667	0.86458333333333	1	5.208333×10^{-3}	1.3541666×10^{-1}
0.1	0.99543007812500	0.86651093750000	1.000625	5.194921×10^{-3}	1.3411406×10^{-1}
0.2	0.99735000000000	0.87230000000000	1.002500	5.150000×10^{-3}	1.3020000×10^{-1}
0.3	1.00056549479167	0.88196927083333	1.0056250	5.059505×10^{-3}	1.2365572×10^{-1}
0.4	1.00510000000000	0.89555000000000	1.0100000	4.900000×10^{-3}	1.1445000×10^{-1}
0.5	1.01098632812500	0.91308593750000	1.0156250	4.638671×10^{-3}	1.0253906×10^{-1}
0.6	1.01826666666667	0.93463333333333	1.0225000	4.233333×10^{-3}	8.786666×10^{-2}
0.7	1.02699257812500	0.96026093750000	1.0306250	3.632421×10^{-3}	7.036406×10^{-2}
0.8	1.03722500000000	0.99005000000000	1.0400000	2.775000×10^{-3}	4.995000×10^{-2}
0.9	1.04903424479167	1.02409427083333	1.0506250	1.590755×10^{-3}	2.653072×10^{-2}
1.0	1.06250000000000	1.06250000000000	1.0625000	0	0

Table 1(b)
Numerical solutions for Example 1.

x	y_2 solution for the present method	y_2 solution in [24]	Error estimate for y_2 (present method)	Error estimate for y_2 (solution in [24])	y_8 solution for the present method	Exact solution
0	1.00014172335601	1.06683333333333	1.42×10^{-4}	6.68×10^{-2}	1	1
0.1	1.00076636916055	1.06683291718750	1.41×10^{-4}	6.62×10^{-2}	1.000625	1.000625
0.2	1.00264030272109	1.06682666666666	1.40×10^{-4}	6.43×10^{-2}	1.002500	1.002500
0.3	1.00576347341580	1.06679951302083	1.39×10^{-4}	6.12×10^{-2}	1.005625	1.005625
0.4	1.01013566666667	1.06672613333333	1.36×10^{-4}	5.67×10^{-2}	1.010000	1.010000
0.5	1.01575630862697	1.06657063802083	1.31×10^{-4}	5.09×10^{-2}	1.015625	1.015625
0.6	1.02262419274376	1.06628613333333	1.24×10^{-4}	4.38×10^{-2}	1.022500	1.022500
0.7	1.03073712819542	1.06581415885416	1.12×10^{-4}	3.52×10^{-2}	1.030625	1.030625
0.8	1.04009151020408	1.06508400000000	9.15×10^{-5}	2.51×10^{-2}	1.040000	1.040000
0.9	1.05068181222320	1.06401187552083	5.68×10^{-5}	1.34×10^{-2}	1.050625	1.050625
1.0	1.06250000000000	1.06250000000000	0	0	1.062500	1.062500

Table 2
Numerical solutions for Example 2.

x	y_2 solution	Exact solution	Error estimate
0	1	1	0
0.1	0.99503641555580	0.99503719020999	7.747×10^{-7}
0.2	0.98056071288077	0.98058067569092	1.996×10^{-5}
0.3	0.95770971424905	0.95782628522115	1.165×10^{-4}
0.4	0.92811663912754	0.92847669088526	3.600×10^{-4}
0.5	0.89366278641727	0.89442719099992	7.644×10^{-4}
0.6	0.85624870175000	0.85749292571254	1.244×10^{-3}
0.7	0.81761162385095	0.81923192051904	1.615×10^{-3}
0.8	0.77920204316445	0.78086880944303	1.666×10^{-3}
0.9	0.74212043532876	0.74329414624717	1.173×10^{-3}
1.0	0.70710678118655	0.70710678118655	0

According to (5), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[s \log\left(\frac{s}{x}\right) \left[(y_n)_{ss} + \frac{1}{s}(y_n)_s - (y_n)^3 + 3(y_n)^5 \right] \right] ds, \quad n \geq 0. \tag{57}$$

We start with the initial approximation $y_0 = 1 + ax$. By the iteration formula (57), we have the following first iteration:

$$y_1 = 1 - \frac{1}{2}x^2 - \frac{4}{3}ax^3 - \frac{27}{16}a^2x^4 - \frac{29}{25}a^3x^5 - \frac{5}{12}a^4x^6 - \frac{3}{49}a^5x^7. \tag{58}$$

Incorporating the boundary condition at $x = 1$ in (58), we get

$$\begin{aligned} y_1 &= 1 - 0.5x^2 + 0.26549088946503x^3 - 0.06690607504106x^4 + 0.00915778929466x^5 \\ &\quad - 0.0006549864292491067x^6 + 0.00001916371061877311x^7. \end{aligned} \tag{59}$$

Errors obtained by using the proposed method and the exact solution are presented in Table 2.

Table 3
Numerical solutions for Example 3.

x	y_2 solution	y_3 solution	Exact solution	Error estimate for y_2	Error estimate for y_3
0	0.99367798994631	1.00039235823213	1	6.32×10^{-3}	-3.92×10^{-4}
0.1	0.99206727417884	0.99872658914718	0.99833748845958	6.27×10^{-3}	-3.89×10^{-4}
0.2	0.98728198889363	0.99377876835073	0.99339926779878	6.12×10^{-3}	-3.79×10^{-4}
0.3	0.97946055655100	0.98569331719336	0.98532927816429	5.87×10^{-3}	-3.64×10^{-4}
0.4	0.96882658381330	0.97469805007759	0.97435470369245	5.53×10^{-3}	-3.43×10^{-4}
0.5	0.95567861075264	0.96108672604832	0.96076892283052	5.09×10^{-3}	-3.18×10^{-4}
0.6	0.94037653084024	0.94519799113548	0.94491118252307	4.53×10^{-3}	-2.87×10^{-4}
0.7	0.92332540762778	0.92739326693012	0.92714554082312	3.82×10^{-3}	-2.48×10^{-4}
0.8	0.90495755846766	0.90803595337261	0.90784129900320	2.88×10^{-3}	-1.94×10^{-4}
0.9	0.88571387561130	0.88747377843500	0.88735650941611	1.64×10^{-3}	-1.17×10^{-4}
1.0	0.86602540378444	0.86602540378444	0.86602540378444	0	0

Example 3. Consider a nonlinear singular two-point boundary value problem arising in astronomy; the equilibrium of isothermal gas spheres can be described by

$$y'' + \frac{2}{x}y' + y^5 = 0 \tag{60a}$$

subject to the boundary conditions

$$y'(0) = 0, \qquad y(1) = \frac{\sqrt{3}}{2}. \tag{60b}$$

This problem has been discussed in [1]; it has an exact solution: $y(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}$.

According to (5), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[\left(\frac{s^2}{x} - s\right)\left[(y_n)_{ss} + \frac{2}{s}(y_n)_s + (y_n)^5\right]\right] \mathrm{d}s, \quad n \geq 0. \tag{61}$$

We start with the initial approximation $y_0 = a$. The next iterates $y_1, y_2, \ldots$ are given below:

$$y_1 = a - \frac{1}{6}x^2a^5 \tag{62}$$

$$y_2 = a - \frac{1}{6}x^2a^5 + \frac{1}{24}x^4a^9 - \frac{5}{756}x^6a^{13} + \frac{5}{7776}x^8a^{17} - \frac{1}{28\,512}x^{10}a^{21} + \frac{1}{1213\,056}x^{12}a^{25}. \tag{63}$$

Incorporating the boundary condition at $x = 1$ in (63), we get

$$\begin{aligned} y_2 = {} & 0.99367798994631 - 0.16146451817512x^2 + 0.03935498857774x^4 \\ & - 0.00609034538522x^6 + 0.0005772848323487203x^8 \\ & - 0.00003069950643358666x^{10} + 0.0000007034948239090159x^{12}. \end{aligned} \tag{64}$$

Table 3 shows the comparison between the exact solution and the proposed method solution. It also shows the errors obtained by using the proposed method solution.

Example 4. Consider the nonlinear singular boundary value problem arising in oxygen tension in a cell with Michaelis–Menten oxygen uptake kinetics (cf. [6,31–33]):

$$y'' + \frac{2}{x}y' + \frac{ny}{y+k} = 0 \tag{65a}$$

with $n = 0.76129$, $k = 0.03119$, subject to the boundary conditions

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5. \tag{65b}$$

According to (5), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[\left(\frac{s^2}{x} - s\right)\left[(y_n)_{ss} + \frac{2}{s}(y_n)_s - \frac{ny}{y+k}\right]\right] \mathrm{d}s, \quad n \geq 0. \tag{66}$$

Table 4
Numerical solutions for Example 4.

x	y_1 solution	y_2 solution	Solution in [6]
0	0.82880802432336	0.82848355162932	0.82848327295802
0.1	0.83003082414962	0.82970635371727	0.82970607521884
0.2	0.83369922362841	0.83337499490687	0.83337471691089
0.3	0.83981322275972	0.83949017524076	0.83948989814383
0.4	0.84837282154356	0.84805304589079	0.84805277036165
0.5	0.85937801997992	0.85906518654929	0.85906491397434
0.6	0.87282881806880	0.87252857519543	0.87252830841853
0.7	0.88872521581021	0.88844555152002	0.88844529589927
0.8	0.90706721320415	0.90681877548439	0.90681854026297
0.9	0.92785481025061	0.92765118257926	0.92765098252660
1.0	0.95108800694959	0.95094593734191	0.95094579461056

Table 5
Numerical solutions for Example 5.

x	y_1 solution	y_2 solution	y_4 solution
0	0.95165060747995	0.95215096881481	0.95214843208264
0.1	0.95213410140515	0.95263426595262	0.95263172997768
0.2	0.95358458318075	0.95408358172916	0.95408104816938
0.3	0.95600205280675	0.95649718704352	0.95649465884735
0.4	0.95938651028316	0.95987219276016	0.95986967791053
0.5	0.96373795560996	0.96420453875787	0.96420205840048
0.6	0.96905638878717	0.96948897859825	0.96948658142561
0.7	0.97534180981477	0.97571905981385	0.97571684469400
0.8	0.98259421869278	0.98288709981585	0.98288524880584
0.9	0.99081361542119	0.99098415742131	0.99098298148966
1.0	1.00000000000000	1.00000000000000	1.00000000000000

We start with the initial approximation $y_0 = a$. The next iterate y_1 is given below:

$$y_1 = a + \frac{0.12688166666a}{0.03119 + a}x^2. \tag{67}$$

Incorporating the boundary condition at $x = 1$ in (67), we get

$$y_1 = 0.82880802432336 + 0.12227998262623x^2. \tag{68}$$

Table 4 shows the comparison between the solution obtained by using the proposed method and the solution in [6].

Example 5. Next we consider the radial stress on a rotationally symmetric shallow membrane cap (cf. [34,35]):

$$y'' + \frac{3}{x}y' - \frac{1}{2} + \frac{1}{8y^2} = 0 \tag{69a}$$

subject to the boundary conditions

$$y'(0) = 0, \qquad y(1) = 1. \tag{69b}$$

According to (5), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[\left(\frac{s^3}{2x^2} - \frac{s}{2}\right)\left[(y_n)_{ss} + \frac{3}{s}(y_n)_s - \frac{1}{2} + \frac{1}{8y_n^2}\right]\right] \mathrm{d}s, \quad n \geq 0. \tag{70}$$

We start with the initial approximation $y_0 = a$. The next iterate y_1 is given below:

$$y_1 = a + \frac{1}{8}\left(\frac{a^2}{8} - \frac{1}{2}\right)x^2 + \frac{1}{2}\left(\frac{1}{4} - \frac{a^2}{16}\right)x^2. \tag{71}$$

Incorporating the boundary condition at $x = 1$ in (71), we get

$$y_1 = 0.95165060747995 + 0.04834939252005x^2. \tag{72}$$

Table 5 exhibits the numerical results obtained by using the proposed method.

5. Conclusion

In this paper, He's variational iteration method has been successfully employed to obtain the approximate solutions of various linear and nonlinear singular boundary value problems. He's variational iteration method yields solutions in the form of convergent series with easily calculable terms. It is shown that the He's variational iteration method is a promising tool for treating linear and nonlinear singular boundary value problems and in some cases yields exact solutions in a few iterations.

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