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Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces

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ABSTRACT

The purpose of this paper is to present some fixed point theorems for $\mathcal{T}$-weakly isotone increasing mappings which satisfy a generalized nonlinear contractive condition in complete ordered metric spaces. As application, we establish an existence theorem for a solution of some integral equations.

1. Introduction and preliminaries

The literature on Fixed Point Theory presents a lot of generalizations of the Banach contraction mapping principle. One of the most interesting of them is the result of Khan et al. [1], in which the authors addressed a new category of fixed point problems for a single self-mapping with the help of a control function which they called an altering distance function. To be precise, $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if it is continuous, non-decreasing and satisfies $\varphi(0) = 0$.

Khan et al. [1] gave the following result.

Theorem 1. *Let $(\mathcal{X}, d)$ be a complete metric space, φ be an altering distance function and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-mapping which satisfies the following inequality:*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq c\varphi(d(x, y)), \tag{1.1}$$

for all $x, y \in \mathcal{X}$ and for some $0 < c < 1$. Then $\mathcal{T}$ has a unique fixed point.

In recent years, there have appeared many results related to fixed point theorems in complete metric spaces endowed with a partial ordering $\preceq$ [2–14]. In many cases, these results can be viewed as an hybrid of two fundamental results, that are, the Banach contraction principle and the weakly contractive condition. Indeed, these results deal with a monotone (either order-preserving or order-reversing) mapping satisfying, with some restriction, a classical contractive condition, and such

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that for some $x_0 \in \mathcal{X}$, either $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, where $\mathcal{T}$ is a self-mapping on $\mathcal{X}$. The first result in this direction was given by Ran and Reurings [14, Theorem 2.1] who presented its applications to matrix equations. Subsequently, Nieto and Rodŕiguez-López [10] extended the result of Ran and Reurings [14] for non-decreasing mappings and applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions.

Later on, Harjani and Sadarangani [15,16] proved ordered version of results for weakly contractive mappings and Amini-Harandi and Emami [5] proved an ordered version of results for mappings of the Reich type.

Very recently, Jachymski [17] established a very useful geometric lemma giving a list of equivalent conditions for some subsets of the plane. As its application, he get that various contractive conditions using the so-called altering distance functions coincide with classical ones and proved that some fixed point theorems for generalized contractions on ordered metric spaces are indeed equivalent and do follow from an earlier result of O'Regan and Petrusel [13].

Moreover, Agarwal et al. [2] presented some new results for generalized nonlinear contractions in partially ordered metric spaces. The main idea in [2,10,14] involves combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

We recall that if $(\mathcal{X}, \preceq)$ is a partially ordered set and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$, then a mapping $\mathcal{T}$ is said to be non-decreasing.

The main result of Agarwal et al. in [2] is the following fixed point theorem.

Theorem 2 *([2, Theorem 2.2]). Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Assume that there is a non-decreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$ and also suppose that $\mathcal{T}$ is a non-decreasing mapping with*

$$d(\mathcal{T}x, \mathcal{T}y) \le \varphi\left(\max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \left(\frac{d(y, \mathcal{T}x) + d(x, \mathcal{T}y)}{2}\right)\right\}\right) \tag{1.2}$$

for all $x \succeq y$. Also suppose either

(a) *$\mathcal{T}$ is continuous or*
(b) *if $\{x_n\} \subset \mathcal{X}$ is a non-decreasing sequence with $x_n \to x$ in $\mathcal{X}$, then $x_n \preceq x$ for all n holds.*

If there exists an $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point.

Agarwal et al. [2] observed that in certain circumstances it is possible to remove the hypothesis that φ is non-decreasing in Theorem 2. So they proved the following fixed point theorem.

Theorem 3 *([2, Theorem 2.3]). Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there is a metric d on $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Assume that there is a continuous function, $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$ and also suppose that $\mathcal{T}$ is a non-decreasing mapping with*

$$d(\mathcal{T}x, \mathcal{T}y) \le \varphi(\max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y)\}). \tag{1.3}$$

Also suppose either (a) or (b) holds. If there exists an $x_0 \in \mathcal{X}$ with $x_0 \preceq \mathcal{T}x_0$ then $\mathcal{T}$ has a fixed point.

Recently, Ćirić [9] generalized Theorems 2 and 3 by introducing the concept of $\mathcal{S}$-monotone mapping and proved some fixed and common fixed point theorems for pair of mappings satisfying $\mathcal{S}$-non-decreasing generalized nonlinear contractions in partially ordered complete metric spaces. To prove these results, the nature of commutativity was used.

The aim of this paper is to give an improved version of the results of Ćirić [9]. We will do this by relaxing the concept of commutativity of mappings and using the concept of $\mathcal{S}$-weakly isotone increasing mapping introduced in [18]. Our results generalize and complement analogous results in the literature (see [3,4,19,20]). To conclude the paper, we establish an existence theorem for a solution of some integral equations.

2. Main results

We recall the following definitions, which are given, respectively, in [19,20] and in [18].

Definition 1. Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Two mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are said to be weakly increasing if $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preceq \mathcal{S}\mathcal{T}x$ for all $x \in \mathcal{X}$.

Note that two weakly increasing mappings need not be non-decreasing. There exist some examples to illustrate this fact in [4].

Definition 2. Let $(\mathcal{X}, \preceq)$ be a partially ordered set and be $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ two mappings. The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-weakly isotone increasing if for all $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x \preceq \mathcal{S}\mathcal{T}\mathcal{S}x$.

Remark 1. If $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are weakly increasing, then $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing.

The following theorem can be viewed as a generalization of Theorem 2.2 of Ćirić et al. [9].

Theorem 4. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Assume that there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two mappings such that*

$$d(\mathcal{T}x, \mathcal{S}y) \le \max\left\{\varphi(d(x,y)), \varphi(d(x,\mathcal{T}x)), \varphi(d(y,\mathcal{S}y)), \varphi\left(\frac{d(y,\mathcal{T}x)+d(x,\mathcal{S}y)}{2}\right)\right\}, \tag{2.1}$$

for all comparable $x, y \in \mathcal{X}$.

Also suppose that $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing and one of $\mathcal{S}$ and $\mathcal{T}$ is continuous. Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point.

Proof. Let x_0 be an arbitrary point in $\mathcal{X}$. If $x_0 = \mathcal{S}x_0$ or $x_0 = \mathcal{T}x_0$ the proof is finished, so we assume that $x_0 \ne \mathcal{S}x_0$ and $x_0 \ne \mathcal{T}x_0$. We can define a sequence $\{x_n\}$ in $\mathcal{X}$ as follows:

$$x_{2n+1} = \mathcal{S}x_{2n} \quad \text{and} \quad x_{2n+2} = \mathcal{T}x_{2n+1} \quad \text{for } n \in \{0, 1, \ldots\}. \tag{2.2}$$

Without loss of generality we can suppose that the successive terms of $\{x_n\}$ are different. Otherwise we have again finished. Note that, since $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing, we have

$$x_1 = \mathcal{S}x_0 \preceq \mathcal{T}\mathcal{S}x_0 = \mathcal{T}x_1 = x_2 \preceq STSx_0 = STx_1 = Sx_2 = x_3,$$
$$x_3 = \mathcal{S}x_2 \preceq \mathcal{T}\mathcal{S}x_2 = \mathcal{T}x_3 = x_4 \preceq STSx_2 = STx_3 = Sx_4 = x_5,$$

and continuing this process we get

$$x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots. \tag{2.3}$$

Now we claim that for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \tag{2.4}$$

Denote

$$M(x,y) := \max\left\{\varphi(d(x,y)), \varphi(d(x,\mathcal{T}x)), \varphi(d(y,\mathcal{S}y)), \varphi\left(\frac{d(y,\mathcal{T}x)+d(x,\mathcal{S}y)}{2}\right)\right\}$$

for all $x, y \in X$. From (2.3) we have that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Then from (2.1) with $x = x_{2n+1}$ and $y = x_{2n}$, we get

$$d(x_{2n+1}, x_{2n+2}) = d(\mathcal{T}x_{2n+1}, \mathcal{S}x_{2n}) \le M(x_{2n+1}, x_{2n}). \tag{2.5}$$

By (2.2), we have

$$M(x_{2n+1}, x_{2n}) = \max\left\{\varphi(d(x_{2n}, x_{2n+1})), \varphi(d(x_{2n+1}, x_{2n+2})), \varphi\left(\frac{1}{2}d(x_{2n}, x_{2n+2})\right)\right\}.$$

- If $M(x_{2n+1}, x_{2n}) = \varphi(d(x_{2n+1}, x_{2n+2}))$, by (2.5) and using the fact that $\varphi(t) < t$ for all $t > 0$, we have

  $$d(x_{2n+1}, x_{2n+2}) \le \varphi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),$$

  a contradiction.
- If $M(x_{2n+1}, x_{2n}) = \varphi\left(\frac{1}{2}d(x_{2n}, x_{2n+2})\right)$, we get

  $$d(x_{2n+1}, x_{2n+2}) \le \varphi\left(\frac{1}{2}d(x_{2n}, x_{2n+2})\right) < \frac{1}{2}d(x_{2n}, x_{2n+2}).$$

  On the other hand, by the triangular inequality, we have

  $$\frac{1}{2}d(x_{2n}, x_{2n+2}) \le \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{2}d(x_{2n+1}, x_{2n+2}).$$

  Thus, we have

  $$d(x_{2n+1}, x_{2n+2}) < \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{2}d(x_{2n+1}, x_{2n+2}),$$

  which implies that

  $$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}).$$
- If $M(x_{2n+1}, x_{2n}) = \varphi(d(x_{2n}, x_{2n+1}))$, we get

  $$d(x_{2n+1}, x_{2n+2}) \le \varphi(d(x_{2n}, x_{2n+1})) < d(x_{2n}, x_{2n+1}).$$

Then, in all cases, we have $d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$ for all $n \in \mathbb{N}$. Similarly, we can prove that $d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n})$ for all $n \in \mathbb{N}^*$. Therefore, we conclude that (2.4) holds.

Now, from (2.4) it follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = \delta. \tag{2.6}$$

We are able to prove that $\delta = 0$. In fact, by the triangular inequality, we get

$$\frac{1}{2}d(x_n, x_{n+2}) \leq \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}). \tag{2.7}$$

By (2.4), we have

$$\frac{1}{2}d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}). \tag{2.8}$$

From (2.8), taking the upper limit as $n \to +\infty$, we get

$$\limsup_{n\to+\infty} \frac{1}{2}d(x_{2n}, x_{2n+2}) \leq \lim_{n\to+\infty} d(x_{2n}, x_{2n+1}). \tag{2.9}$$

If we set

$$\limsup_{n\to+\infty} \frac{1}{2}d(x_{2n}, x_{2n+2}) = b, \tag{2.10}$$

then clearly $0 \leq b \leq \delta$.

As φ is continuous and taking the upper limit on both the sides of (2.5), we get

$$\begin{aligned}\limsup_{n\to+\infty} d(x_{2n+1}, x_{2n+2}) \leq \max\Big\{&\varphi(\limsup_{n\to+\infty} d(x_{2n+1}, x_{2n+2})), \varphi(\limsup_{n\to+\infty} d(x_{2n+1}, x_{2n})),\\ &\varphi\left(\frac{1}{2}(\limsup_{n\to+\infty} d(x_{2n}, x_{2n+2}))\right)\Big\}.\end{aligned} \tag{2.11}$$

Hence by (2.6) and (2.10), we deduce

$$\delta \leq \max\{\varphi(\delta), \varphi(b)\}. \tag{2.12}$$

If we suppose that $\delta > 0$, then we have

$$\delta \leq \max\{\varphi(\delta), \varphi(b)\} < \max\{\delta, b\} = \delta, \tag{2.13}$$

a contradiction. Thus $\delta = 0$ and consequently

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{2.14}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. To this end, it is sufficient to verify that $\{x_{2n}\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ such that for each even integer $2k$ there are even integers $2n(k)$, $2m(k)$ with $2m(k) > 2n(k) > 2k$ such that

$$r_k = d(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon \quad \text{for } k \in \{1, 2, 3, \ldots.\}. \tag{2.15}$$

For every even integer $2k$, let $2m(k)$ be the smallest number exceeding $2n(k)$ satisfying condition (2.15) for which

$$d(x_{2n(k)}, x_{2m(k)-2}) < \varepsilon. \tag{2.16}$$

From (2.15) and (2.16) and the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq r_k &\leq d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)})\\ &\leq \varepsilon + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}).\end{aligned}$$

Hence by (2.14), it follows that

$$\lim_{k\to+\infty} r_k = \varepsilon. \tag{2.17}$$

Now, from the triangular inequality, we have

$$|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leq d(x_{2m(k)-1}, x_{2m(k)}).$$

Letting $k \to +\infty$ and using (2.14) and (2.17), we get

$$\lim_{k\to+\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon. \tag{2.18}$$

On the other hand, we have

$$\begin{aligned} d(x_{2n(k)}, x_{2m(k)}) &\le d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}) \\ &\le d(x_{2n(k)}, x_{2n(k)+1}) + d(Sx_{2n(k)}, Tx_{2m(k)-1}) \\ &\le d(x_{2n(k)}, x_{2n(k)+1}) + M(x_{2m(k)-1}, x_{2n(k)}), \end{aligned} \tag{2.19}$$

where

$$M(x_{2m(k)-1}, x_{2n(k)}) = \max\Bigg\{\varphi(d(x_{2m(k)-1}, x_{2n(k)})), \varphi(d(x_{2m(k)-1}, x_{2m(k)})), \varphi(d(x_{2n(k)}, x_{2n(k)+1})), \varphi\left(\frac{d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)-1}, x_{2n(k)+1})}{2}\right)\Bigg\}.$$

From

$$d(x_{2m(k)-1}, x_{2n(k)+1}) \le d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2n(k)}) + d(x_{2n(k)}, x_{2n(k)+1}),$$

taking the upper limit as $k \to +\infty$, using (2.14) and (2.17), we get

$$\limsup_{k\to+\infty} d(x_{2m(k)-1}, x_{2n(k)+1}) \le \varepsilon.$$

On the other hand, we have

$$\varepsilon \le d(x_{2m(k)}, x_{2n(k)}) \le d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)})$$

and taking the lower limit as $k \to +\infty$, we get

$$\varepsilon \le \liminf_{k\to+\infty} d(x_{2m(k)}, x_{2n(k)}) \le \liminf_{k\to+\infty} d(x_{2m(k)-1}, x_{2n(k)+1}).$$

It follows that

$$\varepsilon \le \liminf_{k\to+\infty} d(x_{2m(k)-1}, x_{2n(k)+1}),$$

and so,

$$\lim_{k\to+\infty} d(x_{2m(k)-1}, x_{2n(k)+1}) = \varepsilon. \tag{2.20}$$

Now, using (2.18), (2.14), (2.17) and (2.20) and the continuity of φ, we get

$$\lim_{k\to+\infty} M(x_{2m(k)-1}, x_{2n(k)}) = \max\{\varphi(\varepsilon), 0, 0, \varphi(\varepsilon)\} = \varphi(\varepsilon). \tag{2.21}$$

Letting $k \to +\infty$ in (2.19), we obtain

$$\varepsilon \le \varphi(\varepsilon) < \varepsilon,$$

a contradiction. Thus, assumption (2.15) is wrong. Therefore, $\{x_n\}$ is a Cauchy sequence.

From the completeness of $\mathcal{X}$, there exists $z \in \mathcal{X}$ such that $x_n \to z$ as $n \to +\infty$. Now we show that z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Clearly, if $\mathcal{S}$ or $\mathcal{T}$ is continuous then $z = \mathcal{S}z$ or $z = \mathcal{T}z$. Thus it is immediate to conclude that $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. □

Now, referring to the paper of Jachymski [17], we give some remarks on the contractive condition (2.1).

Remark 2. The following condition

$$d(\mathcal{T}x, \mathcal{S}y) \le \varphi\left(\max\left\{(d(x, y)), (d(x, \mathcal{T}x)), (d(y, \mathcal{S}y)), \left(\frac{d(y, \mathcal{T}x) + d(x, \mathcal{S}y)}{2}\right)\right\}\right), \tag{2.22}$$

implies condition (2.1). We observe also that condition (2.22) is equivalent to condition (2.1) if we suppose that φ is a non-decreasing function.

Remark 3. Clearly, from our Theorem 4 we can derive a corollary involving condition (2.22). Moreover, under the hypothesis that φ is a non-decreasing function, we can state many others corollaries using the equivalences established in Jachymski [17]. To avoid repetition, these results are omitted.

From Theorem 4 and Remark 1, we deduce the following corollary.

Corollary 1. *The same conclusion of Theorem 4 holds if we suppose that*

$\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ *are two weakly increasing mappings*,

instead of

$\mathcal{S}$ *is* $\mathcal{T}$*-weakly isotone increasing*.

In the following theorem we prove the existence of a common fixed point of two mappings without using the continuity of $\mathcal{S}$ or $\mathcal{T}$.

Theorem 5. *Let* $(\mathcal{X}, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *in* $\mathcal{X}$ *such that* $(\mathcal{X}, d)$ *is a complete metric space. Assume that there is a continuous function* $\varphi : [0, +\infty) \to [0, +\infty)$ *with* $\varphi(t) < t$ *for each* $t > 0$, $\varphi(0) = 0$ *and that* $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ *are two mappings such that*

$$d(\mathcal{T}x, \mathcal{S}y) \leq \max\left\{\varphi(d(x, y)), \varphi(d(x, \mathcal{T}x)), \varphi(d(y, \mathcal{S}y)), \varphi\left(\frac{d(y, \mathcal{T}x) + d(x, \mathcal{S}y)}{2}\right)\right\}, \tag{2.23}$$

for all comparable $x, y \in \mathcal{X}$. *Also suppose that* $\mathcal{S}$ *is* $\mathcal{T}$*-weakly isotone increasing. If the condition*

$$\begin{cases} \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases} \tag{2.24}$$

holds, then S *and* T *have a common fixed point.*

Proof. Using the same arguments in the proof of Theorem 4, we deduce that $\{x_n\}$ is a Cauchy sequence. From (2.2) and the completeness of $\mathcal{X}$, there exists $z \in \mathcal{X}$ such that

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} Sx_{2n} = \lim_{n\to+\infty} Tx_{2n+1} = z.$$

Now we show that z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. By the triangular inequality and the property of the sequence $\{x_n\}$, for $x = x_{2n+1}$ and $y = z$, we have

$$\begin{aligned} d(\mathcal{T}x_{2n+1}, \mathcal{S}z) \leq \max\Bigg\{ & \varphi(d(x_{2n+1}, z)), \varphi(d(x_{2n+1}, \mathcal{T}x_{2n+1})), \varphi(d(z, \mathcal{S}z)), \\ & \varphi\left(\frac{d(z, \mathcal{T}x_{2n+1}) + d(x_{2n+1}, \mathcal{S}z)}{2}\right)\Bigg\}. \end{aligned} \tag{2.25}$$

Letting $n \to \infty$, we have

$$d(z, \mathcal{S}z) \leq \max\{\varphi(d(z, \mathcal{S}z)), \varphi(d(z, \mathcal{S}z)/2)\}.$$

Hence $d(z, \mathcal{S}z) = 0$ and so $\mathcal{S}z = z$. Analogously, for $x = z$ and $y = x_{2n}$, one can prove that $\mathcal{T}z = z$. It follows that $z = \mathcal{S}z = \mathcal{T}z$, that is, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. □

Corollary 2. *The same conclusion of Theorem 5 holds if we suppose that*

$\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ *are two weakly increasing mappings*,

instead of

$\mathcal{S}$ *is* $\mathcal{T}$*-weakly isotone increasing*.

Putting $S = T$ in Corollary 2, we obtain immediately the following result.

Corollary 3. *Let* $(\mathcal{X}, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *in* $\mathcal{X}$ *such that* $(\mathcal{X}, d)$ *is a complete metric space. Assume that there is a continuous function* $\varphi : [0, +\infty) \to [0, +\infty)$ *with* $\varphi(t) < t$ *for each* $t > 0$, $\varphi(0) = 0$ *and that* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *is a mapping such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \max\left\{\varphi(d(x, y)), \varphi(d(x, \mathcal{T}x)), \varphi(d(y, \mathcal{T}y)), \varphi\left(\frac{d(y, \mathcal{T}x) + d(x, \mathcal{T}y)}{2}\right)\right\},$$

for all comparable $x, y \in \mathcal{X}$. Also suppose that $\mathcal{T}x \preceq \mathcal{T}(\mathcal{T}x)$ for all $x \in X$. If the condition

$$\begin{cases} \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases}$$

holds, then T has a fixed point.

Now, we are ready to give a sufficient condition to obtain the uniqueness of the fixed point in the above Theorems 4 and 5. We use the following notion.

Definition 3. Let $(\mathcal{X}, d)$ be a metric space. For any subset $\mathcal{A}$ of $\mathcal{X}$, we define the diameter of $\mathcal{A}$ as

$$\text{diam}(\mathcal{A}) := \sup\{d(x, y) : x, y \in \mathcal{A}\}.$$

Then, we state the following theorem.

Theorem 6. *Adding to the hypotheses of Theorem 4 (resp. Theorem 5) the following condition:*

$$\lim_{n\to+\infty} \text{diam}((\mathcal{T} \circ \mathcal{S})^n(\mathcal{X})) = 0,$$

where $\circ$ denotes the composition of mappings, we obtain the uniqueness of the fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Proof. Let z and z' be two common fixed points of $\mathcal{S}$ and $\mathcal{T}$, that is,

$$z = \mathcal{T}z = \mathcal{S}z$$

and

$$z' = \mathcal{T}z' = \mathcal{S}z'.$$

It is immediate to show that for all $n \in \mathbb{N}$, we have:

$$(\mathcal{T} \circ \mathcal{S})^n x = x, \quad \text{for all } x \in \{z, z'\}.$$

Then

$$\begin{aligned} d(z, z') &= d((\mathcal{T} \circ \mathcal{S})^n z, (\mathcal{T} \circ \mathcal{S})^n z') \\ &\leq \text{diam}((\mathcal{T} \circ \mathcal{S})^n(\mathcal{X})) \\ &\to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

Hence $z = z'$ and the proof is completed. □

3. Application

In this section, we establish an existence theorem for a solution of an integral equation.

Consider the integral equation

$$u(t) = \int_0^T K(t, s, u(s))\, ds + g(t), \quad t \in [0, T] \tag{3.1}$$

where $T > 0$. The purpose of this section is to give an existence theorem for a solution of (3.1) using Corollary 3.

Previously, we consider the space $C(I; \mathbb{R})$ ($I = [0, T]$) of real continuous functions defined on I. Obviously, this space with the metric given by

$$d(x, y) = \max_{t\in I} |x(t) - y(t)|, \quad \forall x, y \in C(I; \mathbb{R}),$$

is a complete metric space. $C(I; \mathbb{R})$ can also be equipped with the partial order $\preceq$ given by

$$x, y \in C(I; \mathbb{R}), \qquad x \preceq y \Leftrightarrow x(t) \leq y(t), \quad \forall t \in I.$$

We suppose that $K : I \times I \times \mathbb{R} \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous.

Now, we define $\mathcal{T} : C(I; \mathbb{R}) \to C(I; \mathbb{R})$ by

$$\mathcal{T}x(t) = \int_0^T K(t, s, x(s))\, ds + g(t), \quad t \in [0, T]$$

for all $x \in C(I; \mathbb{R})$. Then, a solution of (3.1) is a fixed point of $\mathcal{T}$.

We will prove the following result.

Theorem 7. *Suppose that the following hypotheses hold:*

(i) *for all $t, s \in I$ and $u \in C(I; \mathbb{R})$, we have*

$$K(t, s, u(t)) \le K\left(t, s, \int_0^T K(s, \tau, u(\tau))\, d\tau + g(s)\right);$$

(ii) *there exist a continuous function $p : I \times I \to [0, +\infty)$ and a non-decreasing continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(r) < r$ for all $r > 0$ and $\varphi(0) = 0$ such that*

$$|K(t, s, a) - K(t, s, b)| \le p(t, s)\varphi(|a - b|),$$

for all $t, s \in I$ and $a, b \in \mathbb{R}$ such that $a \ge b$;

(iii) $\sup_{t\in I} \int_0^T p(t, s)\, ds \le 1$.

Then, the integral equation (3.1) has a solution $u^* \in C(I; \mathbb{R})$.

Proof. From (i), for all $t \in I$, we have

$$\begin{aligned}
\mathcal{T}x(t) &= \int_0^T K(t, s, x(s))\, ds + g(t) \\
&\le \int_0^T K\left(t, s, \int_0^T K(s, \tau, x(\tau))\, d\tau + g(s)\right) ds + g(t) \\
&= \int_0^T K(t, s, \mathcal{T}x(s))\, ds + g(t) \\
&= \mathcal{T}(\mathcal{T}x)(t).
\end{aligned}$$

Then, we have $\mathcal{T}x \preceq \mathcal{T}(\mathcal{T}x)$ for all $x \in C(I; \mathbb{R})$.

Now, for all $x, y \in C(I; \mathbb{R})$ such that $y \preceq x$, by (ii) and (iii), we have

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &\le \int_0^T |K(t, s, x(s)) - K(t, s, y(s))|\, ds \\
&\le \int_0^T p(t, s)\varphi(|x(s) - y(s)|)\, ds \\
&\le \int_0^T p(t, s)\varphi(d(x, y))\, ds \\
&\le \varphi(d(x, y)).
\end{aligned}$$

Then

$$d(\mathcal{T}x, \mathcal{T}y) \le \varphi(d(x, y))$$

for all $x, y \in C(I; \mathbb{R})$ such that $y \preceq x$.

On the other hand, it is proved in [10] that condition (2.24) is satisfied for $\mathcal{X} = C(I; \mathbb{R})$.

As all hypotheses of Corollary 3 are satisfied, then $\mathcal{T}$ has a fixed point $u^* \in C(I; \mathbb{R})$, that is, u^* is a solution to the integral equation (3.1). □

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