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Observer Design and Stabilization of the Dominant State of Discrete-Time Linear Systems

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Abstract—In this paper, we present necessary and sufficient conditions for both observer design and stabilization of the dominant state of discrete-time linear systems. Our method essentially uses the model reduction of the original linear control system. We also establish a separation principle so that when the reduced-order plant is stabilizable, the state of the reduced order observer may be used in lieu of the state of the reduced order plant for implementing the stabilizing state feedback control law. The model reduction and the observer design detailed in this paper are discrete-time analogs of the results of Aldeen and Trinh (1994) for continuous-time linear systems. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Model reduction, reduced-order observers, Stabilization, Dominant state, Discrete-time linear systems, Separation principle.

1. INTRODUCTION

During the past four decades, significant attention has been paid to the construction of reduced-order observers and stabilization using reduced-order controllers for linear control systems [1–7]. The standard observer design for linear systems either estimates the full state vector or a linear functional of the state vector as originally proposed by Luenberger [1]. As far as the stabilization of linear control systems is concerned, the state vector may not be available for measurement and so when the linear control system is both stabilizable and observable, we use the separation principle for linear control systems and use an estimate of the state in lieu of the state vector. This approach works well with small-scale linear systems. However, for large-scale systems, the observer design problem for a full-state vector or a linear functional of the state vector leads to a design problem with high dimensionality. For such large-scale systems, the order of the observer is comparable with the order of the observed state dynamics. As a consequence, the observer design problem for large-scale systems involves potential numerical and practical difficulties, and so the state feedback laws using an estimate of the state in lieu of the state vector may not yield the desired stabilization results.

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In this paper, we first derive the reduced-order model for a given discrete-time linear system keeping only the dominant state of the given linear plant. The *dominant state* of a linear dynamical system corresponds to the *slow modes* of the system, while the *nondominant state* of a linear dynamical system corresponds to the *fast modes* of the system [4,5,7–9]. As an application of our recent work [9], we first derive the reduced-order model of the given linear discrete-time plant. Using the reduced-order model thus obtained, we characterize the existence of a reduced-order exponential observer that tracks the state of the reduced-order model, i.e., the dominant state of the original linear plant. We note that the model reduction and the reduced-order observer design detailed in this paper are discrete-time analogs of the results of Aldeen and Trinh [7].

Using the reduced-order model of the original plant, we also characterize the existence of a stabilizing state feedback control law that uses only the dominant state of the original plant. Also, when the plant is stabilizable by a state feedback control law, the full information of the dominant state is not always available. For this reason, we establish a separation principle so that the state of the observer may be used in lieu of the dominant state of the original plant, which facilitates the implementation of the stabilizing feedback control law derived. The model-reduction, observer design, stabilization, and separation principle derived in this paper for discrete-time linear control systems have important applications in practice.

This paper is organized as follows. In Section 2, we derive the reduced-order model of a given discrete-time linear system. In Section 3, we derive necessary and sufficient conditions for exponential observer for the reduced-order plant. In Section 4, we derive necessary and sufficient conditions for the reduced-order plant to be stabilizable by a state feedback control law, and we also present a separation principle.

2. REDUCED-ORDER MODEL OF THE LINEAR SYSTEM

Consider a discrete-time linear time-invariant system given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state*, $y \in \mathbb{R}^p$, the *output*, and $u \in \mathbb{R}^m$, the *input* of linear plant (1).

First, we suppose that we have performed an identification of the *dominant* and *nondominant states* of the given linear system using the *modal approach* as in [9].

Without loss of generality, we may assume that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $x_1 \in \mathbb{R}^r$ represents the dominant state and $x_2 \in \mathbb{R}^{n-r}$ represents the nondominant state.

Then, system (1) takes the form,

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}. \end{aligned} \tag{2}$$

From (2), we have

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k), \\ x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k), \\ y(k) &= C_1x_1(k) + C_2x_2(k). \end{aligned} \tag{3}$$

For the sake of simplicity, we will assume that matrix A has distinct eigenvalues. We note that this condition is usually satisfied in most practical situations. Then, A is diagonalizable.

Thus, we can find a nonsingular matrix P consisting of the eigenvectors of A so that

$$P^{-1}AP = \Lambda,$$

where Λ is a diagonal matrix consisting of the n eigenvalues of A.

We introduce new coordinates on the state space given by

$$\xi = P^{-1}x. \tag{4}$$

Then, the plant takes the form,

$$\begin{aligned} \xi(k+1) &= \Lambda\xi(k) + P^{-1}Bu(k), \\ y(k) &= CP\xi(k). \end{aligned}$$

Thus, we have

$$\begin{aligned} \begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} &= \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} + P^{-1}Bu(k), \\ y(k) &= CP \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix}, \end{aligned} \tag{5}$$

where Λ_1 and Λ_2 are $r \times r$ and $(n-r) \times (n-r)$ diagonal matrices, respectively.

Define matrices Γ_1, Γ_2, Ψ_1, and Ψ_2 by

$$P^{-1}B = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \qquad \text{and} \qquad CP = [\Psi_1 \quad \Psi_2],$$

where $\Gamma_1 \in \mathbb{R}^{r\times m}$, $\Gamma_2 \in \mathbb{R}^{(n-r)\times m}$, $\Psi_1 \in \mathbb{R}^{p\times r}$, and $\Psi_2 \in \mathbb{R}^{p\times(n-r)}$.

From (5) and (6), we see that plant (3) has the following simple form in the new coordinates (4),

$$\begin{aligned} \xi_1(k+1) &= \Lambda_1\xi_1(k) + \Gamma_1 u(k), \\ \xi_2(k+1) &= \Lambda_2\xi_2(k) + \Gamma_2 u(k), \\ y(k) &= \Psi_1\xi_1(k) + \Psi_2\xi_2(k). \end{aligned} \tag{7}$$

Next, we make the following assumptions.

(H1) As $k \to \infty$, $\xi_2(k+1) \approx \xi_2(k)$, i.e., $\xi_2(k)$ takes a constant value in the steady-state.

(H2) Matrix $I - \Lambda_2$ is invertible.

Then, it follows from (7) that, for large values of k, we have

$$\xi_2(k) \approx \Lambda_2\xi_2(k) + \Gamma_2 u(k),$$

i.e.,

$$\xi_2(k) \approx (I - \Lambda_2)^{-1}\Gamma_2 u(k). \tag{8}$$

Substituting (8) into (9), we obtain the reduced-order model in the ξ-coordinates as

$$\begin{aligned} \xi_1(k+1) &= \Lambda_1\xi_1(k) + \Gamma_1 u(k), \\ y(k) &= \Psi_1\xi_1(k) + \Psi_2(I - \Lambda_2)^{-1}\Gamma_2 u(k). \end{aligned} \tag{9}$$

We note that (9) represents the reduced-order model of the plant in the new coordinates. To obtain the reduced-order model of the plant in its original coordinates, we proceed as follows.

Set

$$P^{-1} = Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

where $Q_{11} \in \mathbb{R}^{r\times r}$, $Q_{12} \in \mathbb{R}^{r\times(n-r)}$, $Q_{21} \in \mathbb{R}^{(n-r)\times r}$, and $Q_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$.

By the coordinates transformation (4), it follows that

$$\xi = P^{-1}x = Qx.$$

Thus, we have

$$\begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} = Q \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

Hence, it follows that

$$\begin{aligned} \xi_1(k) &= Q_{11}x_1(k) + Q_{12}x_2(k), \\ \xi_2(k) &= Q_{21}x_1(k) + Q_{22}x_2(k). \end{aligned} \tag{10}$$

Using (10) and (8), we get

$$\xi_2(k) = Q_{21}x_1(k) + Q_{22}x_2(k) \approx (I - \Lambda_2)^{-1}\Gamma_2 u(k). \tag{11}$$

Next, we make the following assumption.

(H3) Matrix Q_{22} is invertible.

Using (H3), equation (11) may be simplified as

$$x_2(k) \approx -Q_{22}^{-1}Q_{21}x_1(k) + Q_{22}^{-1}(I - \Lambda_2)^{-1}\Gamma_2 u(k). \tag{12}$$

To simplify the notations, we define matrices R and S by

$$R = -Q_{22}^{-1}Q_{21} \qquad \text{and} \qquad S = Q_{22}^{-1}(I - \Lambda_2)^{-1}\Gamma_2. \tag{13}$$

Using (13), equation (12) may be simplified as

$$x_2(k) \approx R\,x_1(k) + S\,u(k).$$

Substituting (14) into (3), we obtain the reduced-order model of the given plant in its original coordinates as

$$\begin{aligned} x_1(k+1) &= A_1^\star x_1(k) + B_1^\star u(k), \\ y(k) &= C_1^\star x_1(k) + D_1^\star u(k), \end{aligned} \tag{15}$$

where matrices $A_1^\star$, $B_1^\star$, $C_1^\star$, and $D_1^\star$ are defined by

$$\begin{aligned} A_1^\star &= A_{11} + A_{12}R, \\ B_1^\star &= B_1 + A_{12}S, \\ C_1^\star &= C_1 + C_2R, \\ D_1^\star &= C_2S. \end{aligned} \tag{16}$$

3. REDUCED-ORDER OBSERVER DESIGN

In this section, we establish the following main result.

THEOREM 1. *Let S_1 be the linear system described by*

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k). \end{aligned} \tag{17}$$

Under Assumptions (H1)–(H3), the reduced-order model S_2 of the plant S_1 can be obtained (see Section 2) as

$$\begin{aligned} x_1(k+1) &= A_1^\star x_1(k) + B_1^\star u(k), \\ y(k) &= C_1^\star x_1(k) + D_1^\star u(k), \end{aligned} \tag{18}$$

where $A_1^\star$, $B_1^\star$, $C_1^\star$, and $D_1^\star$ are as defined in (16).

Consider the *candidate observer* S_3 described by

$$\hat{x}_1(k+1) = A_1^\star \hat{x}_1(k) + B_1^\star u(k) + K_1^\star \left[C_1 x_1(k) + C_2 x_2(k) - C_1^\star \hat{x}_1(k) - D_1^\star u(k)\right]. \qquad (19)$$

Define the estimation error as

$$e(k) = \hat{x}_1(k) - x_1(k).$$

Then, $e(k) \to 0$ as $k \to \infty$ if and only if matrix $K_1^\star$ is such that $A_1^\star - K_1^\star C_1^\star$ is *convergent*, i.e., all the eigenvalues of $A_1^\star - K_1^\star C_1^\star$ lie inside the open unit disc of the complex plane. If $(C_1^\star, A_1^\star)$ is observable, then, we can construct an observer of the form (19) with any desired speed of convergence.

PROOF. From (3), we have

$$x_1(k+1) = A_{11} x_1(k) + A_{12} x_2(k) + B_1 u(k). \qquad (20)$$

Adding and subtracting the term $(A_{12} - K_1^\star C_2) R x_1(k)$ in the right-hand side of (20), we get

$$x_1(k+1) = (A_{11} + A_{12} R - K_1^\star C_2 R) x_1(k) - (A_{12} - K_1^\star C_2) R x_1(k) + A_{12} x_2(k) + B_1 u(k). \quad (21)$$

Subtracting (21) from (19), and simplifying using the definitions (16), we get

$$e(k+1) = (A_1^\star - K_1^\star C_1^\star) e(k) - (A_{12} - K_1^\star C_2)\left[x_2(k) - R x_1(k) - S u(k)\right]. \qquad (22)$$

As proved in Section 2, Assumptions (H1)–(H3) yield

$$x_2(k) \approx R x_1(k) + S u(k).$$

Therefore,

$$e(k+1) \approx (A_1^\star - K_1^\star C_1^\star) e(k).$$

Therefore,

$$e(k) \approx (A_1^\star - K_1^\star C_1^\star)^k e(0). \qquad (23)$$

From (23), it is immediate that $e(k) \to 0$ as $k \to \infty$, for all values of $e(0)$ if and only if all the eigenvalues of matrix $A_1^\star - K_1^\star C_1^\star$ lie inside the open unit disc of the complex plane.

This completes the proof. ∎

4. REDUCED-ORDER CONTROLLER DESIGN

In this section, we first establish the following result.

THEOREM 2. *Suppose that Assumptions (H1)–(H3) hold. Let S_1 and S_2 be as in Theorem 1. For the reduced-order model S_2, the feedback control law,*

$$u(k) = F_1^\star x_1(k), \qquad (24)$$

stabilizes the state x_1 if and only if matrix $F_1^\star$ is such that $A_1^\star + B_1^\star F_1^\star$ is convergent, i.e., all the eigenvalues of $A_1^\star + B_1^\star F_1^\star$ lie inside the open unit disc of the complex plane. If $(A_1^\star, B_1^\star)$ is controllable, then, we can construct a feedback control law (24) that stabilizes the state x_1 of the reduced-order model S_2 with any desired speed of convergence.

PROOF. It is easy to see that for the reduced-order model S_2, the feedback control law (24) yields the closed-loop dynamics,

$$x_1(k+1) = (A_1^\star + B_1^\star F_1^\star) x_1(k),$$

so that

$$x_1(k) = (A_1^\star + B_1^\star F_1^\star)^k x_1(0).$$

From (25), it is immediate that $x_1(k) \to 0$ for all values of $x_1(0)$ if and only if matrix $A_1^\star + B_1^\star F_1^\star$ is convergent.

This completes the proof. ∎

Next, we establish a *separation principle*.

THEOREM 3. *Suppose that Assumptions (H1)–(H3) hold. Let S_1, S_2, and S_3 be as in Theorem 1. Suppose that there exist matrices $F_1^\star$ and $K_1^\star$, such that $A_1^\star + B_1^\star F_1^\star$ and $A_1^\star - K_1^\star C_1^\star$ are both convergent matrices. By Theorem 1, we know that system S_3 defined by (19) is an exponential observer for the dominant state x_1 of S_1 and that the feedback control law $u(k) = F_1^\star x_1(k)$ is a stabilizing control law for the dominant state x_1 of S_1. Then, the control law,*

$$u(k) = F_1^\star \hat{x}_1(k),$$

also stabilizes the dominant state x_1 of the system S_1.

PROOF. Under the feedback law (26), the observer dynamics (19) becomes

$$\hat{x}_1(k+1) = [A_1^\star + B_1^\star F_1^\star - K_1^\star C_1^\star - K_1^\star D_1^\star F_1^\star]\,\hat{x}_1(k) + K_1^\star [C_1 x_1(k) + C_2 x_2(k)].$$

By (14), we know that

$$x_2(k) \approx R x_1(k) + S u(k) = R x_1(k) + S F_1^\star \hat{x}_1(k). \tag{28}$$

Substituting (28) into (27), and simplifying using the definitions (16), we get

$$\hat{x}_1(k+1) = [A_1^\star + B_1^\star F_1^\star - K_1^\star C_1^\star]\,\hat{x}_1(k) + K_1^\star C_1^\star x_1(k).$$

Also,

$$x_1(k+1) = A_1^\star x_1(k) + B_1^\star F_1^\star \hat{x}_1(k).$$

In matrix form, we have

$$\begin{bmatrix} x_1(k+1) \\ \hat{x}_1(k+1) \end{bmatrix} = \begin{bmatrix} A_1^\star & B_1^\star F_1^\star \\ K_1^\star C_1^\star & A_1^\star + B_1^\star F_1^\star - K_1^\star C_1^\star \end{bmatrix} \begin{bmatrix} x_1(k) \\ \hat{x}_1(k) \end{bmatrix}. \tag{29}$$

The estimation error is given by $e(k) = \hat{x}_1(k) - x_1(k)$. It is easy to see that the error dynamics is governed by the equation

$$e(k+1) = [A_1^\star - K_1^\star C_1^\star]\, e(k). \tag{30}$$

Under the (x, e) coordinates, the composite system (29) simplifies to

$$\begin{bmatrix} x_1(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A_1^\star + B_1^\star F_1^\star & B_1^\star F_1^\star \\ 0 & A_1^\star - K_1^\star C_1^\star \end{bmatrix} \begin{bmatrix} x_1(k) \\ e(k) \end{bmatrix}. \tag{31}$$

Note that

$$\operatorname{eig} \begin{bmatrix} A_1^\star + B_1^\star F_1^\star & B_1^\star F_1^\star \\ 0 & A_1^\star - K_1^\star C_1^\star \end{bmatrix} = \operatorname{eig}[A_1^\star + B_1^\star F_1^\star] \cup \operatorname{eig}[A_1^\star - K_1^\star C_1^\star].$$

Since $A_1^\star + B_1^\star F_1^\star$ and $A_1^\star - K_1^\star C_1^\star$ are both convergent matrices, it is immediate that the system matrix in (31) is also a convergent matrix. Hence, it is immediate that for all values of $x_1(0)$ and $e(0)$, $x_1(k) \to 0$ and $e(k) \to 0$ as $k \to \infty$.

This completes the proof. ∎

REFERENCES

1. D.G. Luenberger, Observers for multivariable systems, *IEEE Trans. Auto. Control* **11**, 190–197, (1966).
2. S.D. Cumming, Design of observers of reduced dynamics, *Electronics Letters* **5**, 213–214, (1969).
3. T.E. Fortman and D. Williamson, Design of low-order observers for linear feedback control laws, *IEEE Trans. Auto. Control* **17**, 301–308, (1972).
4. G.J. Lastman, N.K. Sinha and P. Rozsa, On the selection of states to be retained in a reduced-order model, *IEE Proceedings-Control Theory* **131**, 15–24, (1984).
5. L. Litz and H. Roth, State decomposition for singular perturbation order reduction—A modal approach, *International J. Control* **34**, 937–954, (1981).
6. P. Murdoch, Observer design of a linear functional of the state vector, *IEEE Trans. Auto. Control* **18**, 308–310, (1973).
7. M. Aldeen and H. Trinh, Observing a subset of the states of linear systems, *IEE Proceedings-Control Theory* **141**, 137–144, (1994).
8. M. Aldeen, Interaction modelling approach to distributed control with application to power systems, *International J. Control* **53**, 1035–1054, (1991).
9. V. Sundarapandian, Distributed control schemes for large-scale interconnected discrete-time linear systems, *Mathl. Comput. Modelling* (to appear).