On higher order geometric and renormalization group flows

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ABSTRACT

Renormalization group (RG) flows of the bosonic nonlinear σ-model are governed, perturbatively, at different orders of α', by perturbatively evaluated β-functions. In regions where $\frac{\alpha'}{R_c^2} \ll 1$ ($\frac{1}{R_c^2}$ represents the curvature scale), the flow equations at various orders in α' can be thought of as *approximating* the full, non-perturbative RG flow. On the other hand, taking a different viewpoint, we may consider the above-mentioned RG flow equations as viable *geometric* flows in their own right, without any reference to the RG aspect. Looked at as purely geometric flows where higher order terms appear, we no longer have the perturbative restrictions (small curvatures). In this paper, we perform our analysis from both these perspectives using specific target manifolds such as S^2, H^2, unwarped $S^2 \times H^2$ and simple warped products.

We analyse and solve the higher order RG flow equations within the appropriate perturbative domains and find the *corrections* arising due to the inclusion of higher order terms. Such corrections, within the perturbative regime, are shown to be small and they provide an estimate of the error that arises when higher orders are ignored.

We also investigate higher order geometric flows on the same manifolds and figure out generic features of geometric evolution, the appearance of singularities and solitons. The aim, in this context, is to demonstrate the role of higher order terms in modifying the flow. One interesting aspect of our analysis is that, separable solutions of the higher order flow equations for simple warped spacetimes (of the kind used in bulk-brane models with a single extra dimension), correspond to constant curvature anti de Sitter (AdS) spacetimes, modulo an overall flow parameter dependent scale factor. The functional form of this scale factor (that we obtain) changes on the inclusion of successive higher order terms in the flow.

1. Introduction and overview

In the 1980s, Friedan [1] first analysed renormalization group (henceforth referred to as RG) flows for nonlinear σ-models and showed how one could arrive at the vacuum Einstein equations $R_{ij} = 0$, if one assumed quantum conformal (Weyl) invariance, thereby equating the metric β-functions to zero. This became a crucial result in the context of string theory and a route towards connecting/obtaining Einstein's general relativity with/from string theory. Several authors, subsequently proved important generalizations [2] (particularly in the context of heterotic string theory), which provided further evidence for the above-mentioned connection. The fact that general relativity (modulo some additions) does emerge from string theory, from the requirement of quantum conformal invariance, is more or less an accepted fact, today.

Around the same time in the 1980s, Hamilton [3], in the context of mathematics, proposed Ricci flow equations, with a largely different motivation, primarily related to the geometrization conjecture of Thurston, which involves the topological

classification of three manifolds. More recently, Perelman's work [4] on Ricci flows and its utility in proving the Poincare conjecture have drawn many researchers to revisit such geometric flows [5].

It is well known that the Ricci flow equations are the same (modulo a proper scaling of the flow parameter) as the first order RG flow equations of the nonlinear σ-model. However, one needs to keep in mind the fact that the RG flow equations are obtained from the perturbative evaluation of the β-function and thus, they *approximate* the full, non-perturbative (and as yet unknown) flow in the small curvature regime (i.e. $\frac{\alpha'}{R_c^2} \ll 1$, where $\frac{1}{R_c^2}$ is related to the curvature scale) [6]. On the other hand, from a different perspective, the same equations can be thought of as *geometric flows* where higher order terms are consistently included. We shall perform our analysis from both these points of view, in this article.

Recently, various aspects (eg. irreversibility, gradient flow) of σ-model RG flows (with and without the dilaton and Kalb–Ramond field), at the lowest order have been investigated from a mathematical/geometrical perspective [7]. The connection between RG and geometric flows have been nicely discussed in [8,9]. In particular, Carfora [8] makes a valid point about the connection through the following statement in his article: *'the structure of Ricci flow singularities suggests a natural way for extending, beyond the weak coupling regime, the embedding of the Ricci flow into the renormalization group flow'*.

Going beyond the lowest order, the authors in [10–12] have analysed second order RG flows in some detail by looking at gradient flow aspects, monotonicity etc. Earlier, Tseytlin [13], claimed to have proved the monotonicity of the σ-model RG flow to all orders in α' though some important objections (at the second order level) regarding monotonicity were pointed out in [11].

In our work here, we revisit 2nd order flows for various simple target manifolds and also analyse further higher order flows. We first treat the flow equations as geometric flows in their own right, and figure out the consequences thereof by looking for general solutions, solitons, singularities etc. Subsequently, we point out the behaviour of the flows in the perturbative, RG flow domain and investigate the effect of including the higher order terms.

It is important to mention the classes of manifolds we consider in our discussion. These are essentially of two kinds— unwarped ones and warped product varieties. In the former case, we consider the simplest toy examples, namely, the two-sphere and two dimensional hyperbolic space. We look at the higher order flow equations and find exact solutions, from which we can arrive at useful conclusions. We also investigate an unwarped product briefly. Later, we look at warped product manifolds where the generic line element is assumed to be that of a bulk spacetime in a braneworld model with one extra dimension [14]. For all the manifolds under consideration gradient flow and monotonicity, up to second order in α', holds without any problems [11].

Thus, our main focus in this work is two-fold:

- Obtaining the perturbative corrections that arise due to the inclusion of higher order terms in the RG flow equations.
- Studying the general geometric evolution, solitons, singularities etc. which arise when we treat the RG flow equations at various orders, as genuine geometric flows in their own right, without reference to the perturbative RG aspect.

2. The RG flow equations at various orders

The RG flow equations for the background metric coupling g_{ij}, in bosonic nonlinear *sigma*-models are given by

$$\frac{\partial g_{ij}}{\partial \lambda} = -\beta_{ij} \tag{1}$$

where, β_{ij} are the *beta functions*, usually obtained order by order, perturbatively. In a field theory context, the parameter λ is related to the momentum cut-off scale Λ. Cut-off independence of the theory at length scales larger than inverse Λ gives rise to the above RG flow equation.

As mentioned earlier, the β_{ij} is usually obtained at various orders—i.e. as a series in the parameter α' (proportional to the inverse of the string tension). More specifically, we may write,

$$\beta_{ij} = \alpha' \beta_{ij}^{(1)} + \alpha'^2 \beta_{ij}^{(2)} + \alpha'^3 \beta_{ij}^{(3)} + \alpha'^4 \beta_{ij}^{(4)} + O(\alpha'^5) \tag{2}$$

where the various terms in the R.H.S. of the above equation, for the metric nonlinear σ-model, are given by:

$$\beta_{ij}^{(1)} = R_{ij} \tag{3}$$

$$\beta_{ij}^{(2)} = \frac{1}{2} R_{iklm} R_j{}^{klm} \tag{4}$$

$$\beta_{ij}^{(3)} = \frac{1}{8} \nabla_p R_{iklm} \nabla^p R_j{}^{klm} - \frac{1}{16} \nabla_i R_{klmp} \nabla_j R^{klmp} + \frac{1}{2} R_{klmp} R_i{}^{mlr} R_j{}^{kp}{}_r - \frac{3}{8} R_{iklj} R^{kspr} R^l{}_{spr} \tag{5}$$

$$\beta_{ij}^{(4)} = -\frac{1}{16}R_1 + \frac{1}{48}R_2 - \frac{1}{16}\left[\frac{1}{2} + \zeta(3)\right]R_3 + \frac{1}{4}[1 + \zeta(3)]R_4$$
$$+ \frac{1}{16}\left[\frac{13}{3} - 3\zeta(3)\right]R_5 + \frac{1}{8}\left[\frac{2}{3} - \zeta(3)\right]R_6 + \frac{1}{4}\left[\frac{8}{3} + \zeta(3)\right]R_7 + \frac{1}{4}\left[-\frac{1}{3} + \zeta(3)\right]R_8$$
$$+ \frac{1}{12}R_9 + \frac{1}{12}R_{10} - \frac{1}{6}R_{11} + \frac{1}{16}\left[\frac{4}{3} + \zeta(3)\right]R_{12} - \frac{1}{4}\left[\frac{4}{3} + \zeta(3)\right]R_{13} + \text{higher derivatives} \qquad (6)$$

with R_1 to R_{13} representing various combinations of Riemann tensors (for explicit expressions see [15]). The above expressions are valid, in the RG flow context, only if $\frac{\alpha'}{R_c^2} \ll 1$ (where $\frac{1}{R_c^2}$ is the curvature scale).

We rescale the flow parameter $\lambda \to \lambda\alpha'/2$ to bring the leading order term of Eq. (1) in the form of a Ricci flow. Thus we now have the modified flow equations as

$$\frac{\partial g}{\partial \lambda} = -2Rc - \alpha'\check{R}c^{(2)} - 2\alpha'^2\check{R}c^{(3)} - 2\alpha'^3\check{R}c^{(4)}\ldots \qquad (7)$$

where $\check{R}c^{(2)}$ is a symmetric 2 tensor defined as: $\check{R}c^{(2)} = R_{iklm}R_{jabc}g^{ka}g^{lb}g^{mc}$, $\check{R}c^{(3)} = \frac{1}{2}R_{klmp}R_i{}^{mlr}R_j{}^{kp}{}_r - \frac{3}{8}R_{iklj}R^{kspr}R^l{}_{spr} + \cdots + \cdots$ and $\check{R}c^{(4)} = \beta_{ij}^{(4)}$ or, in component form, we have,

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} - \alpha'R_{iklm}R_j{}^{klm} - \cdots. \qquad (8)$$

The above-mentioned higher order terms are fairly difficult to obtain and involve laborious calculations. We have quoted the above results from articles in [2,15]. As far as we know, the terms beyond 4th order for the metric β-function are not available in the literature. In the following, we will analyse the above flow equations both from the RG and geometric flow perspectives. We also note that knowing the geometric flow does give us useful knowledge about the domain of validity of the RG flow, which is the domain of low curvature.

It may be asked, why we retain the coefficients in the RG flow equations while discussing higher order geometric flows. It is true that these coefficients could be arbitrary when the RG flow aspect is ignored. However, the gradient and monotonicity aspects of the flow is largely governed by the explicit values of these coefficients. Hence, even though flows may be defined with arbitrary coefficients we stick to the RG flow values in our analysis of the higher order geometric flows.

We mention that we have done our analysis for both $\alpha' > 0$ and $\alpha' < 0$. In the σ-model context, $\alpha' < 0$ has no meaning (recall that α' is related to the inverse string tension). However, one may consider $\alpha' < 0$, while treating the equations as those for geometric flows.

3. Higher order flows on surfaces

In this section, we explicitly solve the flow equations for 2nd, 3rd and 4th order flows, using S^2, H^2 and $S^2 \times H^2$ as toy examples. The intention here is to figure out the role of the higher order terms in changing the nature of the geometric flow. Consider (M, g) where M is the two dimensional surface (spherical or hyperbolic) and $g_M(\lambda)$ is the metric on M in the form:

$$g_M(\lambda) = x^2(\lambda)\, g_M^{can}. \qquad (9)$$

Here $x^2(\lambda)$ is the scale factor and g_M^{can} is either the canonical spherical or the hyperbolic metric, depending on the case studied. We again rescale x^2, λ as $\alpha'x'^2$ and $\lambda'\alpha'$ respectively for $\alpha' > 0$. We also follow the same rescaling for $\alpha' < 0$ except that, we change α' to $-|\alpha'|$. In order to distinguish, we will use the scale factor $a(\lambda)$ and $b(\lambda)$ for spherical and hyperbolic surfaces, respectively. After rescaling, the general equation for $x'^2(\lambda')$ is found to be:

$$\frac{dx'^2}{d\lambda'} = -2k - \frac{2}{x'^2} - k\frac{5}{2}\frac{1}{x'^4} - 2\left(\frac{59}{24}\zeta(3) + \frac{29}{24}\right)\frac{1}{x'^6}, \quad \text{for } \alpha' > 0 \qquad (10)$$

$$\frac{dx'^2}{d\lambda'x'} = -2k + \frac{2}{x'^2} - k\frac{5}{2}\frac{1}{x'^4} + 2\left(\frac{59}{24}\zeta(3) + \frac{29}{24}\right)\frac{1}{x'^6}, \quad \text{for } \alpha' < 0. \qquad (11)$$

Here $k = \pm1$ stands for spherical $(+)$ or hyperbolic $(-)$ surfaces. The first term in the R.H.S corresponds to un-normalized Ricci flow whereas the subsequent terms are for 2nd, 3rd and 4th order flows, respectively.

3.1. Geometric flow analysis

3.1.1. 2nd order flow on S^2

Recall that the Ricci flow on the two-sphere gives rise to an ancient solution. Including the second order term leads to the following proposition.
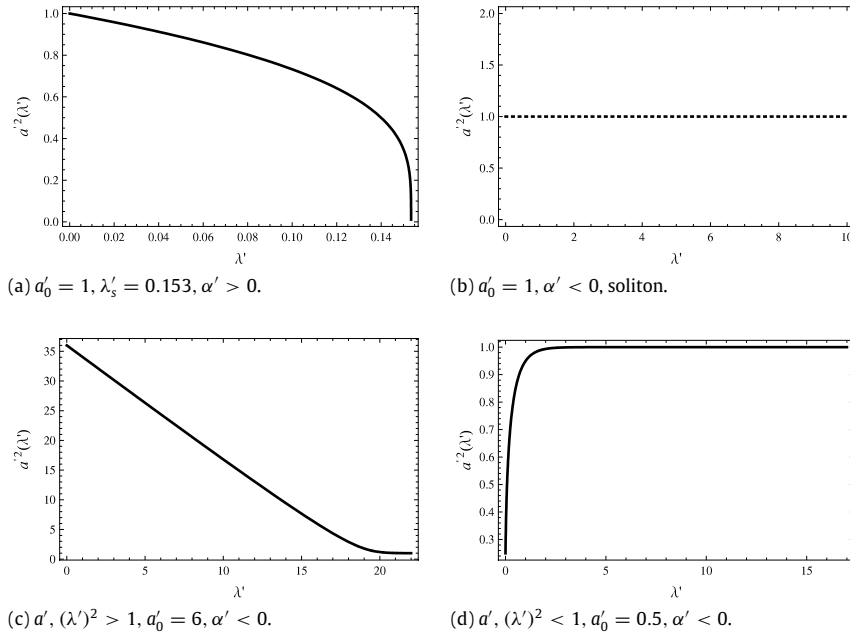
(a) $a_0' = 1$, $\lambda_s' = 0.153$, $\alpha' > 0$.

(b) $a_0' = 1$, $\alpha' < 0$, soliton.

(c) a', $(\lambda')^2 > 1$, $a_0' = 6$, $\alpha' < 0$.

(d) a', $(\lambda')^2 < 1$, $a_0' = 0.5$, $\alpha' < 0$.

Fig. 1. $a'^2(\lambda')$ vs λ' for 2nd order flow.

Proposition 3.1. *For 2nd order flow on the canonical two-sphere, with any choice of initial radius and $\alpha' < 0$ we obtain $a_\infty' = 1$. If $\alpha' < 0$ and $a_0' = 1$ we obtain a soliton, while for $\alpha' > 0$ and with any a_0', we obtain an ancient solution.*

The fact that for $\alpha' > 0$ we get an ancient solution is shown in Fig. 1(top left). If we consider $\alpha' < 0$, which is essentially backward 2nd order flow up to a scaling, we can see that the solution space is divided into two regions namely $a^2 < 1$ and $a^2 > 1$. It is also useful to note that if we choose the initial condition $a_0 = 1$ for $\alpha' < 0$ we get a soliton. The solution of the flow equations for $a^2 > 1$, $a^2 < 1$ with $\alpha' < 0$ are given as:

$$a'(\lambda')^2 + \ln(a'(\lambda')^2 - 1) - a_0'^2 - \ln(a_0'^2 - 1) = -2\lambda', \quad \text{for } a'^2(\lambda') > 1 \tag{12}$$

$$a'(\lambda'^2) + \ln(1 - a'(\lambda')^2) - a_0'^2 - \ln(1 - a_0'^2) = -2\lambda' \quad \text{for } a'^2(\lambda') < 1. \tag{13}$$

The proof of the proposition is evident through the plots (in Fig. 1) and also from the general expressions given above. Here and henceforth, the values chosen while obtaining the graphs are representative. We have also checked things over the full, respective ranges.

3.1.2. 3rd order flow on S^2

Proposition 3.2. *Inclusion of the 3rd order terms for the flow on the canonical two-sphere, with any choice of initial radius ($a_0' \geq 1$) and α' always gives rise to an ancient solution.*

The solution of the flow equation, taken up to 3rd order, will be:

$$a'(\lambda')^2 \mp \frac{1}{2}\ln(4a'(\lambda')^4 \pm 4a'(\lambda')^2 + 5) \mp \frac{3}{4}\tan^{-1}\frac{(2a'(\lambda')^2 \pm 1)}{2} = -2\lambda' + C \tag{14}$$

where $C = a_0'^2 \mp \frac{1}{2}\ln(4a_0'^4 \pm 4a_0'^2 + 5) \mp \frac{3}{4}\tan^{-1}\frac{(2a_0'^2 \pm 1)}{2}$. The $\pm$ signs refer to $\alpha' > 0$ and $\alpha' < 0$ respectively.

From Eq. (14) and the figures the proposition can be proved without much difficulty. Fig. 2 illustrates the conclusions in the proposition. Figures on the left and right are for $\alpha' > 0$ and $\alpha' < 0$ respectively.

3.1.3. 4th order flow on S^2

Proposition 3.3. *Inclusion of the 4th order terms in the flow on a canonical two-sphere, we observe the following behaviour:*

- *$\alpha' > 0$ and any choice of a_0' gives rise to ancient solution.*
- *$\alpha' < 0$, $a_0' = 1.3048$: soliton. For $a_0' \neq 1.3048$ we find, $a_\infty' = const$*
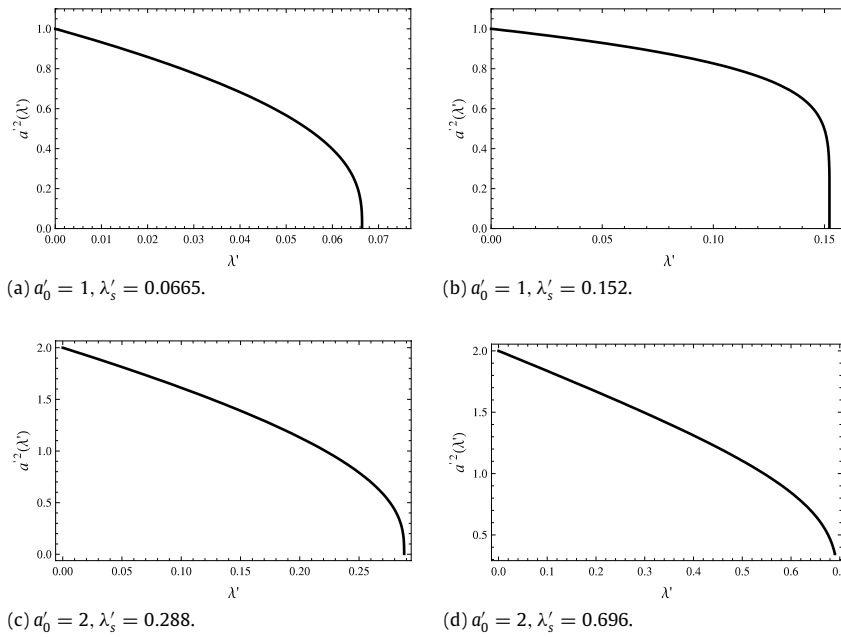- *$\alpha' < 0$; $a_0' < 1.3048$: immortal solution and, $a_0' > 1.3048$: eternal solution.*

(a) $a_0' = 1$, $\lambda_s' = 0.0665$.

(b) $a_0' = 1$, $\lambda_s' = 0.152$.

(c) $a_0' = 2$, $\lambda_s' = 0.288$.

(d) $a_0' = 2$, $\lambda_s' = 0.696$.

Fig. 2. $a'^2(\lambda')$ vs λ' for 3rd order flow.

The solution of the flow equation with the fourth order term included is found to be,

$$a'^2 + c_1 \ln[a'^2 - \eta] - \frac{c_2}{2} \ln[a'^4 + \beta a'^2 + \delta] - \frac{2c_3 - \beta c_2}{2\sqrt{\delta - \frac{\beta^2}{4}}} \tan^{-1}\left[\frac{a'^2 + \frac{\beta}{2}}{\sqrt{\delta - \frac{\beta^2}{4}}}\right] = -2\lambda' + C \tag{15}$$

where the constants are:

$$C = a_0'^2 + c_1 \ln[a_0'^2 - \eta] - \frac{c_2}{2} \ln[a_0'^4 + \beta a_0'^2 + \delta] - \frac{2c_3 - \beta c_2}{2\sqrt{\delta - \frac{\beta^2}{4}}} \tan^{-1}\left[\frac{a_0'^2 + \frac{\beta}{2}}{\sqrt{\delta - \frac{\beta^2}{4}}}\right] \tag{16}$$

$$c_1 \approx 0.7544 \qquad c_2 \approx -0.2456 \qquad c_3 \approx 1.3618 \tag{17a}$$
$$\eta \approx 1.7024 \qquad \beta \approx 0.7024 \qquad \delta \approx 2.4457. \tag{17b}$$

Fig. 3(a) shows the ancient solution, while Fig. 3(b) illustrates the ancient solution for $a_0' < 1.3048$ ($a_\infty' = $ constant). Fig. 3(c) with $a_0' > 1.3048$ ($a_\infty' = $ constant) demonstrates the eternal solution.

3.1.4. 2nd order flow on H^2

Proposition 3.4. *2nd order flow on hyperbolic space with $\alpha' > 0$ generates two kinds of final metrics depending on the initial scale factors. For $b'(\lambda')^2 > 1$ it is expanding and for $b'(\lambda')^2 < 1$ it is converging. In both cases the scale factor asymptotically tends to 1 in backward time ($b'_{-\infty} = 1$). For $\alpha' < 0$, we obtain an immortal solution.*

When $\alpha' > 0$, the flow equation ($k = -1$), has two solutions for $b'^2(\lambda') > 1$ and $b'^2(\lambda') < 1$. These are given as:

$$b'(\lambda')^2 + \ln(b'(\lambda')^2 - 1) - b_0'^2 - \ln(b_0'^2 - 1) = 2\lambda', \quad \text{for } b'^2(\lambda') > 1 \tag{18}$$
$$b'(\lambda')^2 + \ln(1 - b'(\lambda')^2) - b_0'^2 - \ln(1 - b_0'^2) = 2\lambda' \quad \text{for } b'^2(\lambda') < 1. \tag{19}$$

It can be seen that $b_0 = 1$ corresponds to a soliton. If we consider $\alpha' < 0$, which essentially produces an immortal solution, we can see that it expands along the flow. The conclusions mentioned in the above proposition are demonstrated in Fig. 4.

3.1.5. 3rd order flow on H^2

Proposition 3.5. *Inclusion of the 3rd order terms in the flow on hyperbolic space, with any choice of initial radius ($b_0' \geq 1$) and any choice of α' gives rise to an immortal solution.*
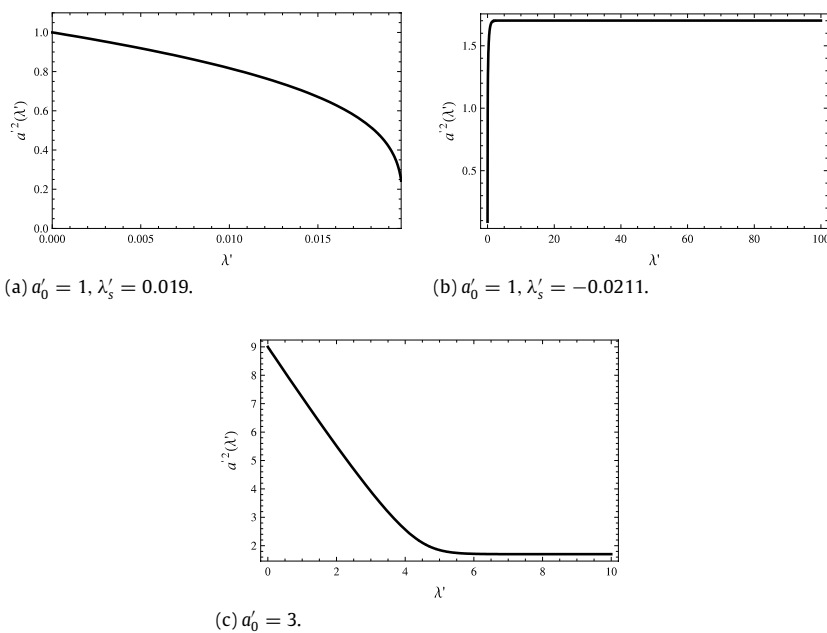
(a) $a_0' = 1$, $\lambda_s' = 0.019$.

(b) $a_0' = 1$, $\lambda_s' = -0.0211$.

(c) $a_0' = 3$.

Fig. 3. $a'^2(\lambda')$ vs λ' for 4th order flow.

(a) $b'(\lambda')^2 > 1$, $b_0' = 8$, $\alpha' > 0$.

(b) $b(\lambda')^2 < 1$, $b_0' = 0.5$, $\alpha' > 0$.

(c) $b_0' = 1$, $\alpha' > 0$, soliton.

(d) $b_0' = 0.4$, $\alpha' < 0$.
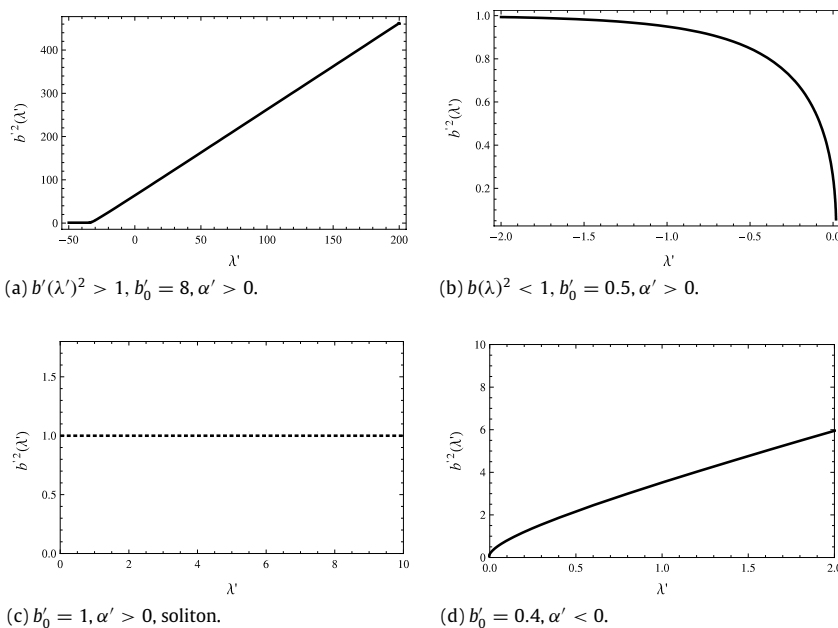
Fig. 4. $b'^2(\lambda')$ vs λ' for 2nd order flow.

For $\alpha' > 0$ the solution turns out to be:

$$b'(\lambda')^2 \pm \frac{1}{2}\ln(4b'(\lambda')^4 \mp 4b'(\lambda')^2 + 5) - \frac{3}{4}\tan^{-1}\left(\frac{2b'(\lambda'^2) \mp 1}{2}\right)$$

$$-b_0'^2 \mp \frac{1}{2}\ln(2b_0'^4 \mp 2b_0'^2 + 5) + \frac{3}{4}\tan^{-1}\left(\frac{2b_0'^2 \mp 1}{2}\right) = 2\lambda'. \tag{20}$$
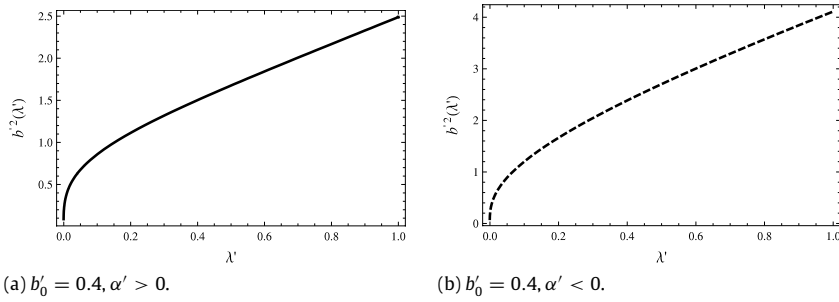
The conclusion in Proposition 3.5 is shown in Fig. 5.

(a) $b'_0 = 0.4, \alpha' > 0.$ (b) $b'_0 = 0.4, \alpha' < 0.$

Fig. 5. $b'^2(\lambda')$ vs λ' for 3rd order flow.

(a) $b'_0 = 2, \alpha' > 0.$ (b) $b'_0 = 1, \alpha' > 0.$
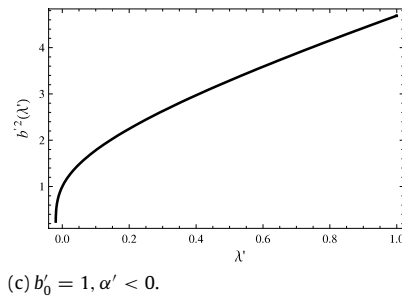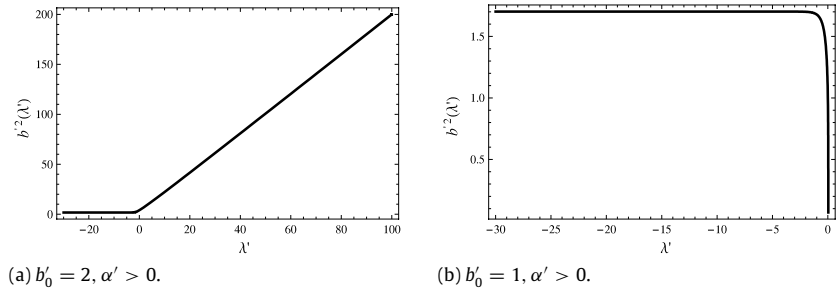
(c) $b'_0 = 1, \alpha' < 0.$

Fig. 6. $b'^2(\lambda')$ vs λ' for 4th order flow.

3.1.6. 4th order flow on $\mathbb{H}^2$

Proposition 3.6. *Inclusion of the 4th order term in the flow on hyperbolic space, yields two cases:*

(1a) *for $\alpha' > 0$ and $b_0'^2 \leq 1$ (except 1.3048, which corresponds to a soliton) we have an ancient solution and $b'_{-\infty} = const$.*
(1b) *for $\alpha' > 0$ and $b_0'^2 > 1$ we have an eternal solution and $b'_{-\infty} = const$.*
 (2) *for $\alpha' < 0$ and any choice of b'_0 we have an expanding immortal solution.*

The solution of this equation would be:

$$b'^2 \pm c_1 \ln[b'^2 \mp \eta] \mp \frac{c_2}{2} \ln\left[b'^4 \pm \beta a'^2 + \delta\right] - \frac{2c_3 - \beta c_2}{2\sqrt{\delta - \frac{\beta^2}{4}}} \tan^{-1}\left[\frac{b'^2 \pm \frac{\beta}{2}}{\sqrt{\delta - \frac{\beta^2}{4}}}\right] = 2\lambda' + C \qquad (21)$$

where the constants are given in Eqs. (16) and (17). The upper and lower sign corresponds to the solution for $\alpha' > 0$ and $\alpha' < 0$ respectively.

All the statements in Proposition 3.6 can be checked, as before, by solving the flow equations. The graphs in Fig. 6 illustrate these facts clearly. Table 1 provides a comparison of the results for $\mathbb{S}^2$ and $\mathbb{H}^2$ at various orders.

3.1.7. Higher order flows on $Ag_{can}^{\mathbb{S}^2} \oplus Bg_{can}^N$

We now briefly discuss flows on an unwarped product manifold before moving on to warped products in the next section. Let N be a manifold (surface) of genus $g \geq 2$ and g_{can}^N is a constant negative curvature (-1). From Eq. (10) we can write the

Table 1
Comparison of flows on $\mathbb{S}^2$ and $\mathbb{H}^2$ at 2nd order and 4th order.

	$\mathbb{S}^2$ $(K > 1)$	$\mathbb{H}^2$ $(K < 1)$
(2nd) $\alpha' > 0$	$a'_0 > 0$, ancient solution shrinking	$b'_0 = 1$, soliton
		$b'^2_0 < 1, b'_{-\infty} = 1$, converging
		$b'^2_0 > 1, b'_{-\infty} = 1$, expanding
(4th) $\alpha' > 0$	$a'_0 > 0$, ancient solution shrinking	$b'_0 \leq 1$, ancient, $b'_{-\infty} = const.$
		$b'_0 > 1$, eternal, expanding
		$b'_0 = 1.3048$, soliton
(2nd) $\alpha' < 0$	$a'_0 = 1$, soliton	$b'_0 > 0$ immortal, expanding
	$a'^2_0 < 1, a'_\infty = 1$	
	$a'^2_0 > 1, a'_\infty = 1$	
(4th) $\alpha' < 0$	$a'_0 = 1.3048$, soliton	any b'_0, immortal solution expanding
	$a'^2_0 > 1, a'_\infty = const.$	
	$a'^2_0 < 1, a'_\infty = const.$	

(a) $(A_0, B_0) = (5, 3), \alpha' > 0, \lambda'_s = 1.6041$.

(b) $(A_0, B_0) = (5, 3), \alpha' < 0$.

(c) $(A_0, B_0) = (5, 3), \alpha' > 0$.

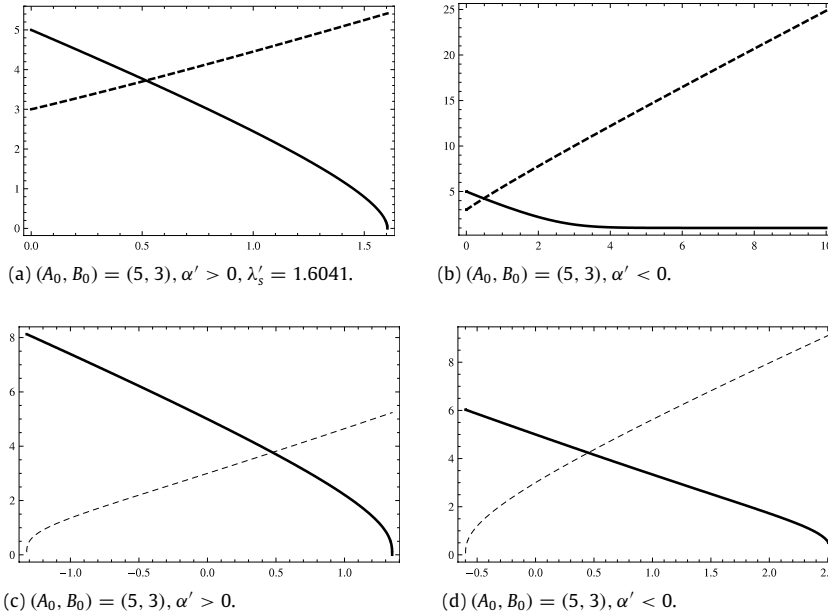(d) $(A_0, B_0) = (5, 3), \alpha' < 0$.

Fig. 7. $(A(\lambda'), B(\lambda'))$ vs λ for 2nd order flow ((a), (b)) and 3rd order flow ((c), (d)).

flow equations up to 4th order (note that they are decoupled equations because the product is unwarped):

$$\frac{dA}{d\lambda'} = -2 - \alpha' \frac{2}{A} - \alpha'^2 \frac{5}{2} \frac{1}{A^2} - 2\alpha'^3 \left(\frac{59}{24} \zeta(3) + \frac{29}{24} \right) \frac{1}{A^3} \tag{22}$$

$$\frac{dB}{d\lambda'} = 2 - \alpha' \frac{2}{B} + \alpha'^2 \frac{5}{2} \frac{1}{B^2} - 2\alpha'^3 \left(\frac{59}{24} \zeta(3) + \frac{29}{24} \right) \frac{1}{B^3}. \tag{23}$$

In this case, for 2nd and 4th order flow we can have ancient as well as immortal solution depending on α' (> 0 or < 0), respectively. Here Fig. 7(a) and (b) show such examples (for 2nd order), of ancient and immortal solutions, respectively. We can also see from Fig. 7(b) that the scale factor corresponding to $\mathbb{S}^2$ approaches a constant for $\alpha' < 0$. The 3rd order flow is neither ancient nor immortal but maintains the same profile with the rest of its class. This may be seen in Fig. 7(c) and (d) for both choices of α'. Here the continuous and dashed lines represent the solutions for $A(\lambda')$ and $B(\lambda')$ respectively.

3.2. Perturbative RG flow domain

Finally, we look at the case where $\frac{1}{x'^2} \ll 1$ or $x'^2 \gg 1$. In this regime, the RG Flow assumption of small curvatures is valid. We look at the behaviour of the flow equations (Eqs. (10) and (11)) at various orders, for spherical and hyperbolic surfaces. To obtain the correction at various orders, we expand the solutions at various orders and try to understand the characteristics of the RG flow within the above-mentioned limit. To achieve our goals we consider a power series in $\frac{1}{x'^2}$.
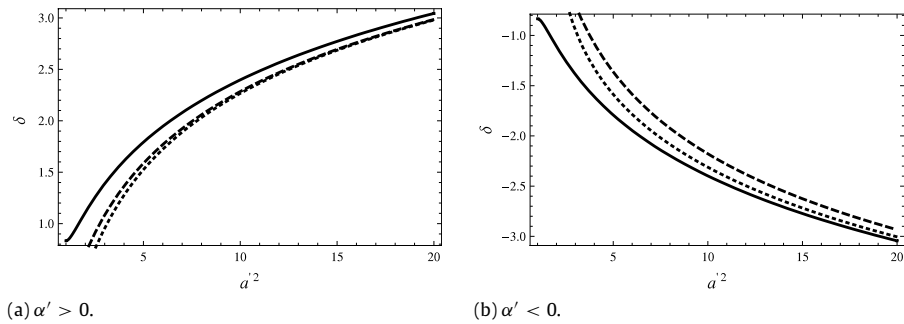
Fig. 8. Spherical surface: the plots for δ at different orders, 2nd (continuous line), 3rd (dashed line), 4th (dotted line).
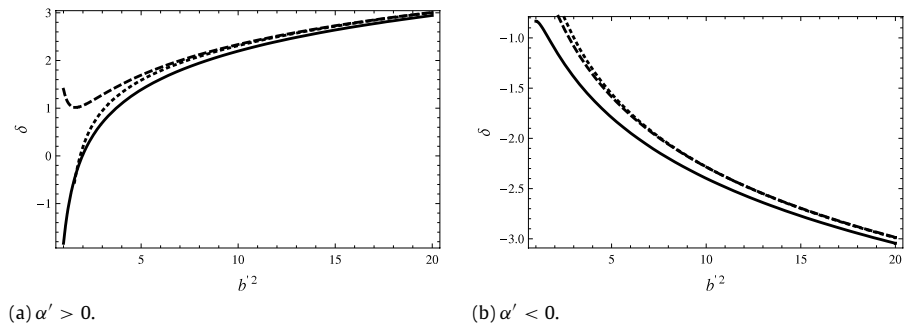

Fig. 9. Hyperbolic surface: the plots for δ at different orders, 2nd (continuous line), 3rd (dashed line), 4th (dotted line).

Below, we show the expansion for $\alpha' > 0$ in the case of a spherical surface at 2nd and 3rd order respectively.

$$-2\lambda' + C_2 = a'^2 + \ln\left(\frac{1}{a'^2}\right) - \frac{1}{a'^2} + \frac{1}{2a'^4} - \frac{1}{3a'^6} + \cdots \tag{24a}$$

$$-2\lambda' + \tilde{C}_3 = a'^2 + \ln\left(\frac{1}{a'^2}\right) + \frac{1}{4a'^2} - \frac{3}{4a'^4} + \frac{19}{48a'^6} + \cdots. \tag{24b}$$

The constant terms are $C_2 = \ln(a_0'^2 - 1) - a_0'^2$ and $\tilde{C}_3 = C + \frac{3\pi}{8} + \frac{\ln 4}{2}$ where C comes from Eq. (14). The expansions for $\alpha' < 0$ (spherical surface) and hyperbolic surface (both $\alpha' < 0$ and $\alpha' > 0$) can be obtained in a similar fashion though the $\alpha' < 0$ case is not important in the σ-model context.

Eq. (24) can be written as $-2\lambda' + C = a'^2 + \delta$ where δ represents the corrections over the Ricci flow due to the higher order terms in the RG flow equation. We plot these δ at various order in Figs. 8 and 9. It is clear that within the RG flow domain the curvature squared terms correspond to corrections which can be quantified order by order in perturbation theory.

4. Higher order flow on warped product manifolds

4.1. Geometric flow analysis

We now consider the behaviour of a five dimensional warped product manifold under the flow equation (1). The metric on the manifold we consider can be written as,

$$ds^2 = e^{2f(\sigma,\lambda)}\eta_{\mu\nu}dx^\mu dx^\nu + r^2(\sigma,\lambda)d\sigma^2 \tag{25}$$

where $\eta_{\mu\nu} = diag[-1, 1, 1, 1]$ is the Minkowski metric.

We will now show that a conformally AdS spacetime, is a solution to the flow up to order $N = 4$ in α'. Also we will prove the following propositions

Proposition 4.1. *The flow equation* (1) *with β taken up to $O(\alpha'^4)$ has a solution of the form*

$$g(\lambda) = \Omega(\lambda)[e^{2k\sigma}\eta_{\mu\nu}dx^\mu dx^\nu + d\sigma^2] \tag{26}$$

with $\Omega(\lambda = 0) = \Omega_0$ and k is a constant. This is a conformally anti de Sitter (AdS) spacetime, similar to the bulk spacetime in the Randall–Sundrum braneworld model [14].

Proposition 4.2. *For the solution in Eq.* (26) *the flow equation up to $O(\alpha'^N)$ where $N \leq 4$ can be reduced to an ODE for $\Omega(\lambda)$ which (after suitable rescaling) is of the form*

$$\frac{d\Omega}{d\lambda} = \sum_{n=0}^{N-1} a_n \Omega^{-n} \tag{27}$$

where a_n are constant coefficients.

Proposition 4.3. *Given $\Omega(\lambda = 0) = \Omega_0 \geq 0$, the ODE equation* (27) *has a unique expanding solution ($\frac{d\Omega}{d\lambda} > 0$), with a finite time singularity in the past at $\lambda = \lambda_s \leq 0$, for each $N = 1, 3$ (odd) and also for $N = 2, 4$ (even) for $\alpha' < 0$.*

Proposition 4.4. *For $N = even$ with $\alpha' > 0$, Eq.* (27) *has 3 distinct solutions.*

- *$\Omega > \alpha' k^2 \xi_N$: An eternal expanding solution without finite time singularities.*
- *$\Omega < \alpha' k^2 \xi_N$: A contracting solution ending in a singularity at finite $\lambda = \lambda_s \geq 0$.*
- *$\Omega = \alpha' k^2 \xi_N$: A fixed point soliton solution.*

with $\xi_2 = 1$ and $\xi_4 \approx 1.5636$.

Next we prove these propositions by reducing the flow equation (1) to an ODE of the form Eq. (27) and then explicitly solving the ODE.

4.1.1. Reduction to ODE

Using the metric of Eq. (25) in the flow equation (7) we get the following dynamical equations for f and r (up to 2nd order)

$$\dot{f} = \frac{1}{r^2}\left[\left(f'' + f'^2 - \frac{f'r'}{r}\right) + 3f'^2\right] - \frac{\alpha'}{r^4}\left[\left(f'' + f'^2 - \frac{f'r'}{r}\right)^2 + 3f'^4\right] \tag{28}$$

$$\dot{r} = \frac{4}{r}\left[f'' + f'^2 - \frac{f'r'}{r}\right] - \frac{4\alpha'}{r^3}\left[f'' + f'^2 - \frac{f'r'}{r}\right]^2 \tag{29}$$

where $\dot{F} = \frac{\partial F}{\partial \lambda}$ and $F' = \frac{\partial F}{\partial \sigma}$, for any function F. To solve the above equations we assume that f and r can be written in a variable separable form as

$$f(\sigma, \lambda) = f_\sigma(\sigma) + f_\lambda(\lambda); \qquad r(\sigma, \lambda) = r_\lambda(\lambda)\, r_\sigma(\sigma). \tag{30}$$

The r_σ part can be absorbed into a redefinition of the σ coordinate and we essentially have $r = r(\lambda)$. These simplify the equations to give (up to 2nd order),

$$\dot{f}_\lambda = \frac{1}{r^2}[(f''_\sigma + f'^2_\sigma) + 3f'^2_\sigma] - \frac{\alpha'}{r^4}[(f''_\sigma + f'^2_\sigma)^2 + 3f'^4_\sigma] \tag{31}$$

$$\dot{r} = \frac{4}{r}[f''_\sigma + f'^2_\sigma] - \frac{4\alpha'}{r^3}[f''_\sigma + f'^2_\sigma]^2. \tag{32}$$

Since the equations are now in a variable separable form, for consistency at the leading order we must have $f''_\sigma + f'^2_\sigma = const_1$ and $(f''_\sigma + f'^2_\sigma) + 3f'^2_\sigma = const_2$; i.e. $f'_\sigma = k = const.$ and $f_\sigma = k\sigma$. Note that this condition comes only from the leading order terms. The equations in λ become $\frac{r^2}{e^{2f_\lambda}}(e^{2f_\lambda})^{\cdot} = (r^2)^{\cdot}$, which gives $e^{2f_\lambda} = r^2 = \Omega(\lambda) \geq 0$.

Thus we have a solution with the metric equation (26) which is conformal to the anti de Sitter metric, as stated in Proposition 4.1. The Ricci and Kretschmann scalars, $R = g^{ij}R_{ij}$ and $K = R_{ijkl}R^{ijkl}$ respectively, for such a metric are given by

$$R = -\frac{4}{r^2}[2f'' + 5f'^2] = -\frac{20k^2}{\Omega} \tag{33}$$

$$K = \frac{8}{r^4}[2(f'' + f'^2)^2 + 3f'^4] = \frac{40k^4}{\Omega^2}. \tag{34}$$

Thus the curvature diverges and the manifolds become singular at $\Omega = 0$.

Using the separable form for the functions in Eq. (5), we get (at third order),

$$\beta_{\mu\mu}^{(3)} = \frac{e^{2f}}{4r^6}(-32f'^6_\sigma - 16f'^4_\sigma f''_\sigma - 12f'^2_\sigma f''^2_\sigma - 14f''^3_\sigma + 4f'^3_\sigma f^{(3)}_\sigma + 8f'_\sigma f''_\sigma f^{(3)}_\sigma + f^{(3)2}_\sigma)$$

$$\beta_{\sigma\sigma}^{(3)} = \frac{1}{r^4}(-8f'^6_\sigma - 12f'^4_\sigma f''_\sigma + 3f'^2_\sigma f''^2_\sigma - 3f''^3_\sigma + 5f'^3_\sigma f^{(3)}_\sigma + 6f'_\sigma f''_\sigma f^{(3)}_\sigma + f^{(3)2}_\sigma + f'^2_\sigma f^{(4)}_\sigma + f''_\sigma f^{(4)}_\sigma) \tag{35}$$

where $F^{(n)} = \frac{d^n F}{d\sigma^n}$. But, from the separability at leading order, we already have $f' = k$ and thus both the terms reduce to $e^{-2f} r^6 \beta_{\mu\mu}^{(3)} = r^4 \beta_{\sigma\sigma}^{(3)} = -8k^6$. Similarly all higher derivative terms vanish at order 4 leaving $e^{-2f} r^8 \beta_{\mu\mu}^{(4)} = r^6 \beta_{\sigma\sigma}^{(4)} = 2(3 + 5\zeta(3))k^8$.

This leads to the following ODE for Ω

$$\frac{1}{8k^2} \frac{d\Omega}{d\lambda} = 1 - \frac{\alpha' k^2}{\Omega} + 2\left(\frac{\alpha' k^2}{\Omega}\right)^2 - \frac{3 + 5\zeta(3)}{2}\left(\frac{\alpha' k^2}{\Omega}\right)^3. \tag{36}$$

We can readily see that $k = 0$ is a fixed point of the flow. This corresponds to a 5 dimensional Minkowski space which is thus, a *soliton* of the flow as is expected. For $k \neq 0$ we rescale the variables using

$$\bar{\Omega} = \frac{\Omega}{|\alpha'| k^2}; \qquad \bar{\lambda} = \frac{8\lambda}{|\alpha'|} \tag{37}$$

with $\bar{\Omega} \geq 0$ and the Eq. (36) becomes

$$\frac{d\bar{\Omega}}{d\bar{\lambda}} = 1 \mp \bar{\Omega}^{-1} + 2\bar{\Omega}^{-2} \mp \frac{3 + 5\zeta(3)}{2}\bar{\Omega}^{-3}. \tag{38}$$

Here, the upper signs are for $\alpha' > 0$ and the lower ones for $\alpha' < 0$, a convention we shall adopt in all that follows. For either case this equation is of the form of Eq. (27) in Proposition 4.2. Note that now a term of the form $\bar{\Omega}^{n-1}$ corresponds to $O(\alpha'^m)$ in the original flow equation (1).

4.1.2. Solitons, solutions and singularities

We can solve Eq. (38) for all orders and obtain explicit solutions.

At $N = 1$ order, the flow is essentially an *un-normalized* Ricci flow independent of α' with the solution

$$\bar{\lambda} + C_1 = \bar{\Omega}. \tag{39}$$

For $N = 2$ there exists a fixed point soliton for $\bar{\Omega} = \xi_2 = 1$ and when $\bar{\Omega} \neq 1$ we get

$$\bar{\lambda} + C_2 = \bar{\Omega} \pm \ln|\bar{\Omega} \mp 1|. \tag{40}$$

Order $N = 3$ has no fixed points and the solution is

$$\bar{\lambda} + C_3 = \bar{\Omega} \pm \frac{1}{2}\ln|\bar{\Omega}^2 \mp \bar{\Omega} + 2| - \frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{2\bar{\Omega} \mp 1}{\sqrt{7}}\right). \tag{41}$$

The ODE at $N = 4$ has one fixed point at $\bar{\Omega} = \xi_4 \approx 1.5636$. To obtain the solutions for $\bar{\Omega} \neq \xi_4$ we use the following factorization and partial fraction splittings

$$\bar{\Omega}^3 - \bar{\Omega}^2 + 2\bar{\Omega} - \frac{3 + 5\zeta(3)}{2} = (\bar{\Omega} - \xi_4)(\bar{\Omega}^2 + \beta\bar{\Omega} + \gamma) \tag{42a}$$

$$\frac{\bar{\Omega}^2 - 2\bar{\Omega} + \frac{3+5\zeta(3)}{2}}{(\bar{\Omega} - \xi_4)(\bar{\Omega}^2 + \beta\bar{\Omega} + \gamma)} = \frac{a}{\bar{\Omega} - \xi_4} - \frac{b\bar{\Omega} + c}{\bar{\Omega}^2 + \beta\bar{\Omega} + \gamma} \tag{42b}$$

where the various parameters can be found approximately as

$$\xi_4 \approx 1.5636 \qquad \beta \approx 0.5636 \qquad \gamma \approx 2.8812 \tag{43a}$$

$$a \approx 0.6158 \qquad b \approx -0.3841 \qquad c \approx 1.7464. \tag{43b}$$

In terms of these parameters the solution for $N = 4$ can be written as

$$\bar{\lambda} + C_4 = \bar{\Omega} \pm a\ln|\bar{\Omega} \mp \xi_4| \mp \frac{b}{2}\ln|\bar{\Omega}^2 \pm \beta\bar{\Omega} + \gamma| - \frac{2c - \beta b}{\sqrt{4\gamma - \beta^2}}\tan^{-1}\left(\frac{2\bar{\Omega} \pm \beta}{\sqrt{4\gamma - \beta^2}}\right). \tag{44}$$

To compare these solutions, we plot all of them with the same initial condition $\bar{\Omega}(\bar{\lambda} = 0) = \bar{\Omega}_0$ in Fig. 10.

It can be seen that the $N = odd$ order solutions and the $N = even$ solution for $\alpha' < 0$ begin at a finite time singularity and continue to expand indefinitely. Whereas for $N = even$ order and $\alpha' > 0$ we have 3 solutions — a soliton ($\bar{\Omega} = \xi_N$), an eternal solution ($\bar{\Omega} > \xi_N$) and a solution with a finite time singularity ($\bar{\Omega} < \xi_N$).

The singularity time $\lambda = \lambda_s$ such that $\Omega(\lambda_s) = 0$ depends on Ω_0. This behaviour can be seen in the Fig. 11.

We see from Fig. 11 that $\bar{\lambda}_s \leq 0$ except for the case of order $N = even$ and $\alpha' > 0$ in which case $\bar{\lambda}_s$ starts from zero and then increases up to infinity as $\bar{\Omega}_0 \to \xi_N$. Furthermore for $\alpha' < 0$ we always have $\bar{\lambda}_s^{(4)} < \bar{\lambda}_s^{(1)} < \bar{\lambda}_s^{(2)} < \bar{\lambda}_s^{(3)}$ for the respective orders of RG flow as seen in Fig. 11(b).
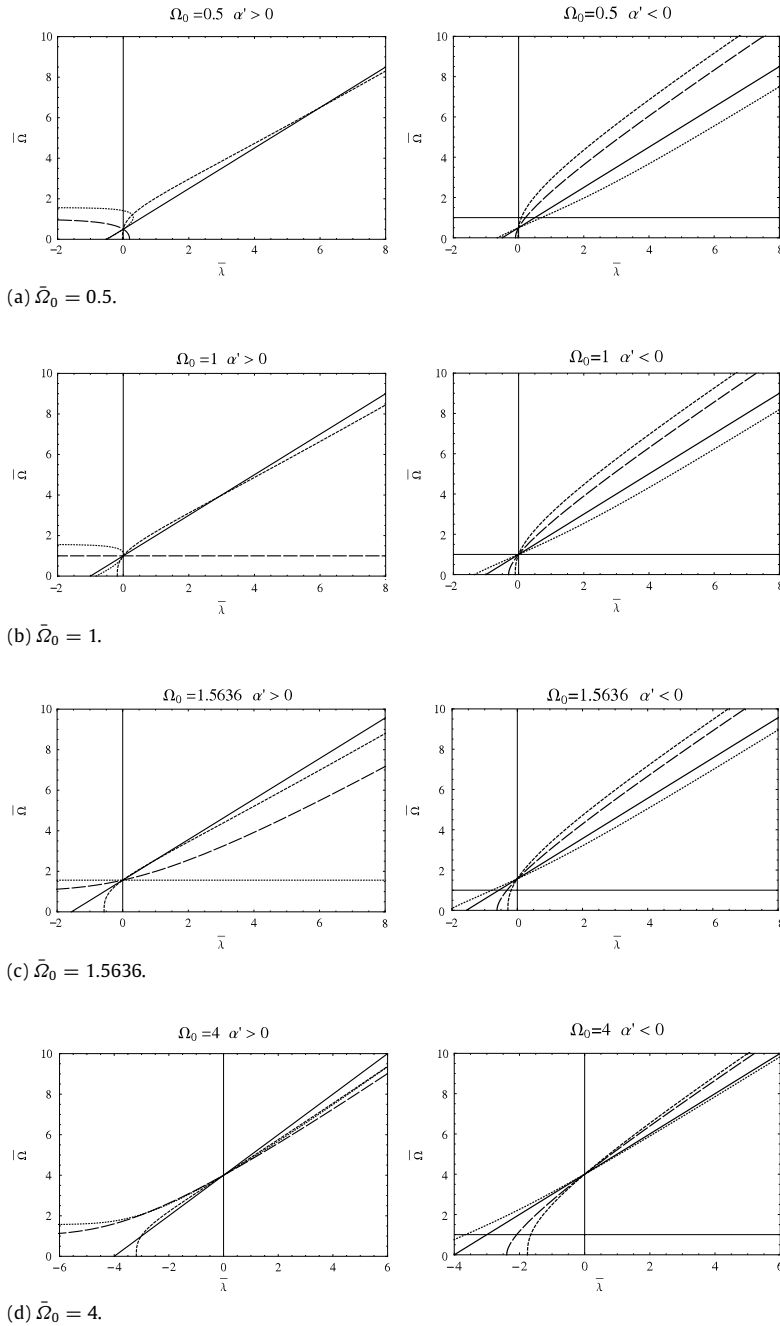
Fig. 10. $\bar{\Omega}$ vs $\bar{\lambda}$ for various initial values of $\bar{\Omega}_0$ at $\bar{\lambda} = 0$. The solutions at different orders are 1st (continuous line), 2nd (long dashed line), 3rd (short dashed line) and 4th (dotted line). The figures on the left are for $\alpha' > 0$ and on the right are $\alpha' < 0$.

4.2. Perturbative RG flow domain

Considered as an RG flow equation, Eq. (38) is valid only in the region where $1/\bar{\Omega} \ll 1$. Note that this regime does not contain any solitons or singularities. To study the flow in that limit, we expand the solutions obtained at various orders Eqs. (39)–(41) and (44) in the powers of $1/\bar{\Omega}$ as follows

$$\lambda + C_1 = \bar{\Omega} \tag{45a}$$

$$\lambda + C_2 = \bar{\Omega} \pm \ln \bar{\Omega} - \left(\frac{1}{\bar{\Omega}} \right) \mp \frac{1}{2} \left(\frac{1}{\bar{\Omega}} \right)^2 - \frac{1}{3} \left(\frac{1}{\bar{\Omega}} \right)^3 + \cdots \tag{45b}$$
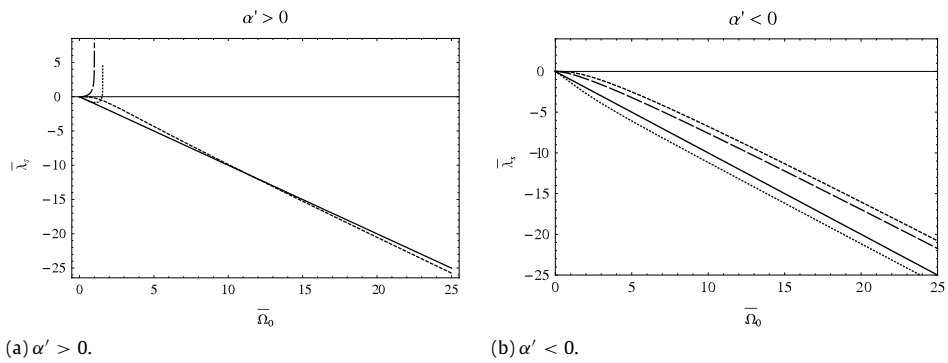
Fig. 11. $\bar{\lambda}_s$ vs $\bar{\Omega}_0$. The plots for different orders are 1st (continuous line), 2nd (long dashed line), 3rd (short dashed line) and 4th (dotted line).
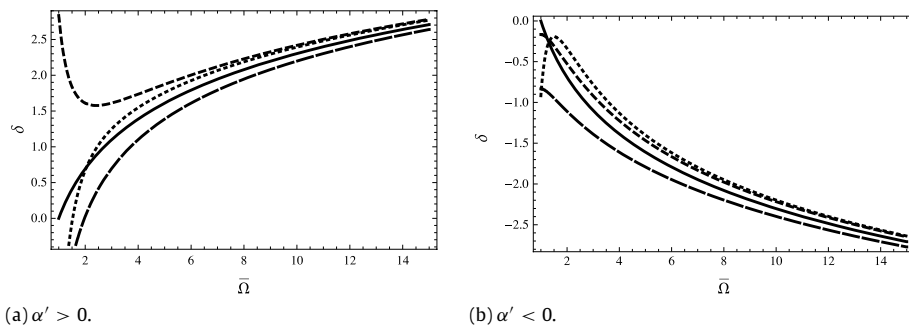

Fig. 12. δ vs $\bar{\Omega}_0$. The plots for $\pm \ln \bar{\Omega}$ (continuous line) and δ at different orders, 2nd (long dashed line), 3rd (short dashed line) and 4th (dotted line).

$$\lambda + \tilde{C}_3 = \bar{\Omega} \pm \ln \bar{\Omega} + \left(\frac{1}{\bar{\Omega}} \right) \pm \frac{3}{2} \left(\frac{1}{\bar{\Omega}} \right)^2 + \frac{1}{3} \left(\frac{1}{\bar{\Omega}} \right)^3 + \cdots \tag{45c}$$

$$\lambda + \tilde{C}_4 = \bar{\Omega} \pm \ln \bar{\Omega} + \left(\frac{1}{\bar{\Omega}} \right) \mp 0.7526 \left(\frac{1}{\bar{\Omega}} \right)^2 - 2.6699 \left(\frac{1}{\bar{\Omega}} \right)^3 + \cdots \tag{45d}$$

where constant terms have been absorbed as $\tilde{C}_3 = C_3 + \frac{3\pi}{2\sqrt{7}}$ and $\tilde{C}_4 = C_4 + 1.74045$.

We can write all of these as $\lambda + C = \bar{\Omega} + \delta$ where δ represents the corrections over the Ricci flow due to the higher order terms in the RG flow equation. We plot these δ over the Ricci flow obtained at various order in Fig. 12. We see that the leading correction over the Ricci flow is $\sim \ln \bar{\Omega}$, and other higher order corrections then vanish in the limit of large $\bar{\Omega}$.

We can see that the 2nd order solution is correct up to $\ln \bar{\Omega}$ term and the 3rd order up to $1/\bar{\Omega}$. We suspect, similarly, that the 4th order solution will be correct till $1/\bar{\Omega}^2$, but this can be verified only if even higher order solutions are available.

5. Summary and conclusion

In this article we have investigated the higher order flows on various manifolds, from uniform homogeneous spaces to warped manifolds. Let us summarize the work done and mention some future possibilities briefly.

We began with higher order geometric flows for the toy examples of the two-sphere and hyperbolic space in two dimensions. It turns out that the results for the 1st order (Ricci) and 3rd order flows are similar in nature while the 2nd and 4th order flows have similarities. There is a marked difference for flows with $\alpha' > 0$ and $\alpha' < 0$, which has been pointed out. The existence of fixed points (solitons) have also been mentioned. We emphasize that these toy examples, though simplistic, do provide us with pointers towards distinguishing between the specific effects that occur because of the inclusion of higher order terms. As expected, the higher order terms do not *resolve* the lower order singularities though there are definite changes in terms of the singularity 'time', initial condition dependences and the appearance of solitons. Further, we discuss the perturbative RG flow domain. Here, we are able to quantify the corrections due to the presence of the higher order terms.

Subsequently, we turned towards higher order flows in the context of warped product manifolds. This analysis is largely inspired by the physics of warped extra dimensions and brane-bulk models. As claimed in Proposition 4.1 we show that the conformally AdS spacetime in Eq. (26) is a solution of the RG flow up to order $N = 4$ in α'. While reducing the RG flow PDEs to ODEs, assuming separability of the functions in the line element, we noted that the conformal AdS nature arises by using

just the leading order terms. The higher order equations are then consistent, giving the conformal factor $e^{2f_\lambda} = r^2 = \Omega$. The behaviour of Ω is then determined by the ODE and the order of the expansion of the beta functions.

It is then natural to conjecture that the conformally AdS metric equation (26) will solve the flow up to any order N. Note that using only the leading order we get $f'_\sigma = k$. Also since curvature is a function of derivatives of the metric, the β-function at any order N will look like $\beta^{(N)}_{\mu\mu} \sim \frac{e^{2f} A^{(N)}(k)}{r^{2N}}$ and $\beta^{(N)}_{\sigma\sigma} \sim \frac{B^{(N)}(k)}{r^{2(N-1)}}$, where A and B are functions of k, just like in Eq. (35). If $A^{(N)}(k) = B^{(N)}(k)$ then we can write $e^{2f} = r^2 = \Omega$ and thus get the conformally AdS solution. Then the ODE for Ω will also be of the form of Eq. (27) for any order N. Thus we conjecture that both Propositions 4.1 and 4.2 hold for any order N. Since explicit forms of the β_{ij} are not known for $N > 4$, we cannot explicitly check these conjectures presently. It will be interesting to look for a proof using general principles.

Apart from the Minkowski spacetime, we have found another soliton solution to the flow at even orders $N = 2, 4$. For this solution $\Omega = \alpha' k^2 \xi_N$ and from Eq. (33) we see that $R = -20/\alpha' \xi_N$. Thus the spacetime has curvature inversely proportional to α'. In the context of RG flows, the existence of a soliton is a non-perturbative effect. Smaller values of α' correspond to larger curvatures and hence the perturbative expansion of the β functions in Eq. (2) fails. It is for this reason that the solutions obtained at various orders show markedly different behaviour close to the fixed point value of $\bar{\Omega} = \xi_N$. In regions of low curvature, we have evaluated the higher order corrections to the RG flows—these corrections are small as long as we are in the perturbative RG flow domain. However, they are useful in quantifying the RG flow behaviour at successive higher orders. Moreover, the singularities and solitons which appear in the geometric flow analysis can serve as pointers to the precise domain in which the σ-model RG flow equations are valid.

In a way, we have shown that the bulk spacetime used in the warped braneworld models naturally arise as solutions of the flow equations (at least up to fourth order). Though this result assumes separability of the metric functions we are able to demonstrate the uniqueness of AdS spacetime as a solution of the flow equations. The task of looking at non-separable situations, which at the lowest order was discussed in our previous article [16] remains and will be taken up in future.

Further, the evolution of geometric quantities such as the Riemann and Ricci scalars in the context of higher order flows will be an exercise worth pursuing, the primary motivation being to distinguish between the lowest order (Ricci) and these higher order geometric flows.

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