Computers and Mathematics with Applications 57 (2009) 356–366

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Subclasses of analytic functions associated with the generalized hypergeometric function

M.K. Aouf, H.E. Darwish *

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

ARTICLE INFO

Article history:
Received 29 July 2007
Received in revised form 31 October 2007
Accepted 17 November 2007

Keywords:
Analytic
Distortion theorem
Hypergeometric function
Modified Hadamard product

ABSTRACT

Using the generalized hypergeometric function, we study a class $\Phi_k^p(q, s; A, B, \lambda)$ of analytic functions with negative coefficients. Coefficient estimates, distortion theorem, extreme points and the radii of close-to-convexity and convexity for this class are given. We also derive many results for the modified Hadamard product of functions belonging to the class $\Phi_k^p(q, s; A, B, \lambda)$.

1. Introduction

Let $A(p, k)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; p, k \in N = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic in $U = U(1)$, where $U(r) = \{z : z \in C \text{ and } |z| < r\}$. Also let us put $A(p) = A(p, p+1)$ and $A = A(1)$. Let the functions $f(z)$ and $g(z)$ be analytic in U. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. We denote this subordination by $f(z) \prec g(z)$.

A function $f(z)$ belonging to the class $A(p)$ is said to be p-valent starlike of order α in $U(r)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U(r);\ 0 < r \le 1;\ 0 \le \alpha < p), \tag{1.2}$$

and a function $f(z)$ belonging to the class $A(p)$ is said to be p-valent convex of order α in $U(r)$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U(r);\ 0 < r \le 1;\ 0 \le \alpha < p). \tag{1.3}$$

Also a function belonging to the class $A(p)$ is said to be p-valent close-to-convex of order α in $U(r)$ if and only if

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (z \in U(r);\ 0 < r \le 1;\ 0 \le \alpha < p). \tag{1.4}$$

* Corresponding author.
E-mail addresses: mkaouf127@yahoo.com (M.K. Aouf), darwish333@yahoo.com (H.E. Darwish).

0898-1221/$ – see front matter © 2009 Published by Elsevier Ltd
doi:10.1016/j.camwa.2007.11.054

We denote by $S_p^*(\alpha)$ the class of all functions in $A(p)$ which are p-valent starlike of order α in U, by $S_p^c(\alpha)$ the class of all functions in $A(p)$ which are p-valent convex of order α in U and by $S_p^k(\alpha)$ the class of all functions in $A(p)$ which are p-valent close-to-convex functions of order α in U. We also set

$$S_p^* = S_p^*(0), \qquad S^*(\alpha) = S_1^*(\alpha), \qquad S_p^c = S_p^c(0), \qquad C(\alpha) = S_1^c(\alpha),$$
$$S_p^k = S_p^k(0) \quad \text{and} \quad K(\alpha) = S_1^k(\alpha).$$

Let G be a subclass of the class A. We define the radius of starlikeness $R^*(G)$, the radius of convexity $R^c(G)$ and the radius of close-to-convexity $R^k(G)$ for the class G by

$$R^*(G) = \inf_{f\in G}(\sup\{r \in (0, 1] : f \text{ is starlike in } U(r)\}),$$
$$R^c(G) = \inf_{f\in G}(\sup\{r \in (0, 1] : f \text{ is convex in } U(r)\}),$$

and

$$R^k(G) = \inf_{f\in G}(\sup\{r \in (0, 1] : f \text{ is close-to-convex in } U(r)\}).$$

For analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, by $(f * g)(z)$ we denote the Hadamard product (or convolution) of $f(z)$ and $g(z)$, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n .$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \neq 0, -1, -2, \ldots; j = 1, \ldots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n} \cdot \frac{z^n}{n!} \quad (q \le s + 1; q, s \in N_0 = N \cup \{0\}; z \in U), \tag{1.5}$$

where $(\theta)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_n = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} = \begin{cases} 1 & (n = 0) \\ \theta(\theta + 1)\ldots(\theta + n - 1) & (n \in N). \end{cases} \tag{1.6}$$

Corresponding to a function $h_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$h_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z) = z^p \, {}_qF_s(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z),$$

we consider a linear operator $H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s) : A(p) \to A(p)$, defined by the convolution

$$H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s)f(z) = h_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{1.7}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s)f(z) = z^p + \sum_{n=k}^{\infty} \Gamma_n a_n z^n, \tag{1.8}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-p} \ldots (\alpha_q)_{n-p}}{(\beta_1)_{n-p} \ldots (\beta_s)_{n-p}(n - p)!}. \tag{1.9}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s), \tag{1.10}$$

then one can easily verify from the definition (1.7) that

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1)f(z) - (\alpha_1 - p)H_{p,q,s}(\alpha_1)f(z). \tag{1.11}$$

The linear operator $H_p(\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s)$ was introduced and studied by Dziok and Srivastava [1].

We note that, for $f(z) \in A(p, p + 1) = A(p)$, we have :

(i) $H_{p,2,1}(a, 1; c) = L_p(a, c)$ $(a > 0; c > 0)$ Saitoh [2];

(ii) $H_{p,2,1}(\nu+p, 1; \nu+p+1)f(z) = J_{\nu,p}(f)$, where $J_{\nu,p}(f)$ is the generalized Bernari–Libera–Livingston operator (see [3–5]) defined by

$$(J_{\nu,p}f)(z) = \frac{\nu + p}{z^\nu} \int_0^z t^{\nu-1} f(t)\mathrm{d}t \quad (\nu > -p; p \in N); \tag{1.12}$$

(iii) $H_{p,2,1}(\mu+p,1;1)f(z)=D^{\mu+p-1}f(z)$ $(\mu>-p)$, where $D^{\mu+p-1}f(z)$ is the $(\mu+p-1)$th order Ruscheweyh derivative of a function $f(z)\in A(p)$ (see Kumar and Shukla [6,7]);

(iv) $H_{p,2,1}(1+p,1;1+p-\mu)=\Omega_z^{(\mu,p)}f(z)$, where the operator $\Omega_z^{(\mu,p)}f(z)$ is defined by (see Srivastava and Aouf [8])

$$\Omega_z^{(\mu,p)}f(z)=\frac{\Gamma(1+p-\mu)}{\Gamma(1+p)}z^{\mu}D_z^{\mu}f(z)\quad(0\le\mu\le1;\,p\in N),\tag{1.13}$$

where D_z^{μ} is the fractional derivative operator (see, for details, [9,10]).

Making use of the operator $H_{p,q,s}(\alpha_1)$, we say that a function $f(z)\in A(p,k)$ is in the class $\Psi_k^p(q,s;A,B,\lambda)$ if it satisfies the following condition :

$$\frac{1}{p-\lambda}\left(\frac{(H_{p,q,s}(\alpha_1)f(z))'}{z^{p-1}}-\lambda\right)\prec\frac{1+Az}{1+Bz}\quad(z\in U)$$
$$(0\le B\le1;\,-B\le A<B;\,0\le\lambda<p;\,p,k,q,s\in N)\tag{1.14}$$

or, equivalently, if

$$\left|\frac{\frac{(H_{p,q,s}(\alpha_1)f(z))'}{z^{p-1}}-p}{B\frac{(H_{p,q,s}(\alpha_1)f(z))'}{z^{p-1}}-[pB+(A-B)(p-\lambda)]}\right|<1.\tag{1.15}$$

Furthermore, we say that a function $f(z)\in\Psi_k^p(q,s;A,B,\lambda)$ is in the subclass $\Phi_k^p(q,s;A,B,\lambda)$ of $\Psi_k^p(q,s;A,B,\lambda)$ if $f(z)$ is of the following form :

$$\left\{f\in T(p,k):f(z)=z^p-\sum_{n=k}^{\infty}a_nz^n\ (a_n\ge0;\,n=k,k+1,k+2,\ldots)\right\}.\tag{1.16}$$

In particular, for $q=s+1$ and $\alpha_{s+1}=1$, we write $\Phi_k^p(s;A,B,\lambda)=\Phi_k^p(s+1,s;A,B,\lambda)$.

We note that :

(i) The subclass $V_k^p(q,s;A,B,\lambda)$ of $T(p,k)$ obtained by replacing $\frac{(H_{p,q,s}(\alpha_1)f(z))'}{z^{p-1}}$ with $\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}$ in (1.15) was studied by Aouf [11];

(ii) The subclass $V_k^p(q,s;A,B)$ of $T(p,k)$ obtained by replacing $\frac{(H_{p,q,s}(\alpha_1)f(z))'}{z^{p-1}}$ by $\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}$ in (1.15) (with $\lambda=0$) was studied by Dziok and Srivastava [1].

We note that for $k=p+1$, $q=2$ and $s=1$, we obtain the following interesting relationships with some of the special classes which were investigated recently :

(i) Taking $\alpha_1=\beta_1$ and $\alpha_2=1$, we obtain :
$\Phi_{p+1}^p(2,1;A,B,\lambda)=P^*(p,A,B,\lambda)$ (Aouf [12]);

(ii) Taking $\alpha_1=\beta_1$, $\alpha_2=1$, $A=-\beta$ and $B=\beta$ $(0<\beta\le1)$, we obtain :
$\Phi_{p+1}^p(2,1;-\beta,\beta,\lambda)=T_p^*(\lambda,\beta)$ (Aouf [13]);

(iii) Taking $\alpha_1=\beta_1$, $\alpha_2=1$, $A=-1$ and $B=1$, we obtain :
$\Phi_{p+1}^p(2,1;-1,1,\lambda)=T_p^*(\lambda)$ (Aouf [13] and Lee et al. [14]);

$$=\left\{f(z)\in T(p):\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>\lambda,\,0\le\lambda<p,\,z\in U\right\};\tag{1.17}$$

(iv) Taking $\alpha_1=\nu+p(\nu>-p)$, $\alpha_2=1$, $\beta_1=1$, $A=-1$ and $B=1$

$\Phi_{p+1}^p(2,1;-1,1,\lambda)=Q_{\nu+p-1}($ Aouf and Darwish [15])

$$=\left\{f(z)\in T(p):\mathrm{Re}\left\{\frac{(D^{\nu+p-1}f(z))'}{z^{p-1}}\right\}>\lambda,\,0\le\lambda<p,\,\nu>-p,\,z\in U\right\}.\tag{1.18}$$

Also we note that :

$\Phi_k^p(q,s;-\rho,\rho,\lambda)=\Phi_k^p(q,s;\lambda,\rho)$

$$=\left\{f\in T(p,k):\left|\frac{\frac{(H_{p,q,s}(\alpha_1)f(z))'}{z^{p-1}}-p}{\frac{(H_{p,q,s}(\alpha_1)f(z))'}{z^{p-1}}+p-2\lambda}\right|<\rho,\ (z\in U;\,0\le\lambda<p;\,0<\rho\le1;\,p\in N)\right\}.\tag{1.19}$$

2. Coefficient estimates

Theorem 1. *A function $f(z)$ of the form (1.16) belongs to the class $\Phi_k^p(q, s; A, B, \lambda)$ if and only if*

$$\sum_{n=k}^{\infty} n(1+B)\Gamma_n a_n \leq (B-A)(p-\lambda), \tag{2.1}$$

where Γ_n is given by (1.9).

Proof. Let $|z| = r$ $(0 \leq r < 1)$. If (2.1) holds, we find from (1.15) and (1.16) that

$$\begin{aligned}
&\left|(H_{p,q,s}(\alpha_1)f(z))' - pz^{p-1}\right| - \left|B(H_{p,q,s}(\alpha_1)f(z))' - [pB + (A-B)(p-\lambda)]z^{p-1}\right| \\
&= \left|-\sum_{n=k}^{\infty} n\Gamma_n a_n z^{n-1}\right| - \left|(B-A)(p-\lambda)z^{p-1} - \sum_{n=k}^{\infty} Bn\Gamma_n a_n z^{n-1}\right| \\
&\leq \sum_{n=k}^{\infty} n\Gamma_n a_n r^{n-1} - \left\{(B-A)(p-\lambda)r^{p-1} - \sum_{n=k}^{\infty} Bn\Gamma_n a_n r^{n-1}\right\} \\
&= r^{p-1}\left(\sum_{n=k}^{\infty} n(1+B)\Gamma_n a_n r^{n-p} - (B-A)(p-\lambda)\right) \\
&< \sum_{n=k}^{\infty} n(1+B)\Gamma_n a_n - (B-A)(p-\lambda) \leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, $f(z) \in \Phi_k^p(q, s; A, B, \lambda)$.

Conversely, let $f(z) \in \Phi_k^p(q, s; A, B, \lambda)$ be given by (1.16). Then, from (1.15) and (1.16), we have

$$\left|\frac{(H_{p,q,s}(\alpha_1)f(z))' - pz^{p-1}}{B(H_{p,q,s}(\alpha_1)f(z))' - [pB + (A-B)(p-\lambda)]z^{p-1}}\right| = \left|\frac{-\sum_{n=k}^{\infty} n\Gamma_n a_n z^{n-p}}{(B-A)(p-\lambda) - \sum_{n=k}^{\infty} Bn\Gamma_n a_n z^{n-p}}\right| < 1 \quad (z \in U), \tag{2.2}$$

where Γ_n is defined by (1.9). Putting $z = r$ $(0 \leq r < 1)$, we obtain

$$\sum_{n=k}^{\infty} n\Gamma_n a_n r^{n-p} < (B-A)(p-\lambda) - \sum_{n=k}^{\infty} Bn\Gamma_n a_n r^{n-p},$$

which, upon letting $r \to 1^-$, readily yields the assertion (2.1). This completes the proof of Theorem 1. □

Since $n\Gamma_n$, where Γ_n is defined by (1.9) is a decreasing function with respect to β_j $(j = 1, \ldots, s)$ and an increasing function with respect to α_ℓ $(\ell = 1, \ldots, q)$, from Theorem 1, we obtain :

Corollary 1. *If $\ell \in \{1, \ldots, q\}, j \in \{1, \ldots, s\}, 0 \leq \alpha'_\ell \leq \alpha_\ell$ and $\beta'_j \geq \beta_j, \beta_j > 0$, then the class $\Phi_k^p(q, s; A, B, \lambda)$ (for the parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$) is included in the class $\Phi_k^p(q, s; A, B, \lambda)$ for the parameters*

$$\alpha_1, \ldots, \alpha_{\ell-1}, \alpha'_\ell, \alpha_{\ell+1}, \ldots, \alpha_q \quad \text{and} \quad \beta_1, \ldots, \beta_{j-1}, \beta'_j, \beta_{j+1}, \ldots, \beta_s.$$

From Theorem 1, we also have the following corollary:

Corollary 2. *If a function $f(z)$ of the form (1.16) belongs to the class $\Phi_k^p(q, s; A, B, \lambda)$, then*

$$a_n \leq \frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} \quad (n = k, k+1, k+2, \ldots), \tag{2.3}$$

where Γ_n is defined by (1.9). The result is sharp, the functions $f_n(z)$ of the form :

$$f_n(z) = z^p - \frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} z^n \quad (n \geq k) \tag{2.4}$$

being the extremal function.

Let $f(z)$ be defined by (1.16) with $k = p+1$, $p \in N$ and for $A = -1$ and $B = 1$, the condition (1.16) is equivalent to :

$$H_p(\alpha_1)f(z) \in T_p^*(\lambda) \quad (0 \leq \lambda < p). \tag{2.5}$$

Thus we have the following lemma:

Lemma 1. *If* $\alpha_j = \beta_j$ $(j = 1, 2, \ldots, s)$, *then*

$$\Phi_k^p(s; -1, 1, \lambda) \subset T_p^*(\lambda) \quad (0 \leq \lambda < p).$$

By the definition of the class $\Phi_k^p(q, s; A, B, \lambda)$, we have the following lemma.

Lemma 2. *If* $A_1 \leq A_2, B_1 \geq B_2$ *and* $0 \leq \lambda_1 < \lambda_2 < p$, *then*

$$\Phi_k^p(q, s; A_1, B_1, \lambda_2) \subset \Phi_k^p(q, s; A_2, B_2, \lambda_1) \subset \Phi_k^p(q, s; -1, 1, 0).$$

3. Distortion theorem and extreme points

Theorem 2. *Let a function* $f(z)$ *of the form* (1.16) *belong to the class* $\Phi_k^p(q, s; A, B, \lambda)$. *If the sequence* $\{n\Gamma_n\}$ *is nondecreasing, then*

$$r^p - \frac{(B-A)(p-\lambda)}{k(1+B)\Gamma_k} r^k \leq |f(z)| \leq r^p + \frac{(B-A)(p-\lambda)}{k(1+B)\Gamma_k} r^k \quad (|z| = r < 1). \tag{3.1}$$

If the sequence $\{\Gamma_n\}$ is nondecreasing, then

$$pr^{p-1} - \frac{(B-A)(p-\lambda)}{(1+B)\Gamma_k} r^{k-1} \leq |f'(z)| \leq pr^{p-1} + \frac{(B-A)(p-\lambda)}{(1+B)\Gamma_k} r^{k-1} \quad (|z| = r < 1), \tag{3.2}$$

where Γ_n is defined by (1.9). The result is sharp, with the extremal function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)(p-\lambda)}{k(1+B)\Gamma_k} z^k \quad (k, p \in N). \tag{3.3}$$

Proof. Let a function $f(z)$ of the form (1.16) belongs to the class $\Phi_k^p(q, s; A, B, \lambda)$. If the sequence $\{n\Gamma_n\}$ is nondecreasing and positive, by Theorem 1, we have

$$\sum_{n=k}^{\infty} a_n \leq \frac{(B-A)(p-\lambda)}{k(1+B)\Gamma_k}, \tag{3.4}$$

and if the sequence $\{\Gamma_n\}$ is nondecreasing and positive, by Theorem 1, we have

$$\sum_{n=k}^{\infty} na_n \leq \frac{(B-A)(p-\lambda)}{(1+B)\Gamma_k}. \quad \square \tag{3.5}$$

Making use of the conditions (3.4) and (3.5), in conjunction with the definition (1.16), we readily obtain the assertions (3.1) and (3.2) of Theorem 2.

Corollary 3. *Let a function* $f(z)$ *of the form* (1.16) *belong to the class* $\Phi_k^p(s; A, B, \lambda)$. *If* $\alpha_1 \geq \beta_1 + 1$, *and* $\alpha_j \geq \beta_j$ $(j = 2, \ldots, s)$, *then the assertion* (3.1) *holds true. Moreover, if* $\alpha_1 \geq \beta_1$, *then the assertion* (3.2) *holds true.*

Proof. If $q = s$, $\alpha_1 \geq \beta_1 + 1$, and $\alpha_j \geq \beta_j$ $(j = 2, \ldots, s)$, then the sequence $\{n\Gamma_n\}$ is nondecreasing. Moreover, if $\alpha_1 \geq \beta_1$, then the sequence $\{\Gamma_n\}$ is nondecreasing. Thus, by Theorem 2, we have Corollary 3. $\square$

Theorem 3. *Let* Γ_n *be defined by* (1.9) *and let us put*

$$f_{k-1}(z) = z^p \tag{3.6}$$

and

$$f_n(z) = z^p - \frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} z^n \quad (n = k, k+1, k+2, \ldots). \tag{3.7}$$

A function $f(z)$ belongs to the class $\Phi_k^p(q, s; A, B, \lambda)$ if and only if it is of the form :

$$f(z) = \sum_{n=k-1}^{\infty} \mu_n f_n(z) \quad (z \in U), \tag{3.8}$$

where

$$\sum_{n=k-1}^{\infty} \mu_n = 1 \quad (\mu_n \geq 0; n = k-1, k, k+1, \ldots). \tag{3.9}$$

Proof. Let a function $f(z)$ of the form (1.16) belong to the class $\Phi_k^p(q, s; A, B, \lambda)$. Setting

$$\mu_n = \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_n \quad (n = k, k+1, k+2, \ldots) \quad \text{and} \quad \mu_{k-1} = 1 - \sum_{n=k}^{\infty} \mu_n,$$

we see that $\mu_n \geq 0$ $(n = k, k+1, k+2, \ldots)$. Since $\mu_{k-1} \geq 0$, by (2.1), we thus have

$$\begin{aligned}
\sum_{n=k-1}^{\infty} \mu_n f_n(z) &= \left(1 - \sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_n\right) z^p + \sum_{n=k}^{\infty} \left(z^p - \frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} z^n\right) \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_n \\
&= z^p - \sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_n z^p + \sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_n z^p - \sum_{n=k}^{\infty} a_n z^n \\
&= z^p - \sum_{n=k}^{\infty} a_n z^n = f(z),
\end{aligned}$$

and the condition holds true.

Next, let a function $f(z)$ satisfy the condition (3.8). Then we have

$$\begin{aligned}
f(z) &= \sum_{n=k-1}^{\infty} \mu_n f_n(z) = \mu_{k-1} f_{k-1}(z) + \sum_{n=k}^{\infty} \mu_n f_n(z) \\
&= \left(1 - \sum_{n=k}^{\infty} \mu_n\right) z^p + \sum_{n=k}^{\infty} \mu_n \left(z^p - \frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} z^n\right) \\
&= z^p - \sum_{n=k}^{\infty} \mu_n \frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} z^n.
\end{aligned}$$

Thus the function $f(z)$ is of the form (1.16), where

$$a_n = \frac{(B-A)(p-\lambda)\mu_n}{n(1+B)\Gamma_n} \quad (n = k, k+1, k+2, \ldots).$$

It is sufficient to prove that the assertion (2.1) holds true. Since

$$\sum_{n=k}^{\infty} n(1+B)\Gamma_n a_n = \sum_{n=k}^{\infty} (B-A)(p-\lambda)\mu_n = (B-A)(p-\lambda)(1-\mu_{k-1}) \leq (B-A)(p-\lambda),$$

the required condition is indeed true. □

From Theorem 3, we obtain the following corollary.

Corollary 4. *The class $\Phi_k^p(q, s; A, B, \lambda)$ is convex. The extremal points are the functions $f_{k-1}(z)$ and $f_n(z) (n \geq k)$ given by (3.6) and (3.7), respectively.*

4. The radii of close-to-convexity and convexity

Theorem 4. *The radius of p-valently close-to-convex for the class $\Phi_k^p(q, s; A, B, \lambda)$ is given by*

$$R^* \left(\Phi_k^p(q, s; A, B, \lambda)\right) = \inf_{n \geq k} \left[\frac{p(1+B)\Gamma_n}{(B-A)(p-\lambda)}\right]^{\frac{1}{(n-p)}}, \tag{4.1}$$

where Γ_n is defined by (1.9). The result is sharp.

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p \quad (z\in U(r);\, 0<r\le 1). \tag{4.2}$$

Since

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|-\sum_{n=k}^{\infty} na_n z^{n-p}\right|,$$

putting $|z|=r$, the condition (4.2) is true if

$$\sum_{n=k}^{\infty}\frac{n}{p}a_n r^{n-p}\le 1. \tag{4.3}$$

By Theorem 1, we have

$$\sum_{n=k}^{\infty}\frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)}a_n\le 1,$$

where Γ_n is defined by (1.9). Thus the condition (4.3) is true if

$$\frac{n}{p}r^{n-p}\le\frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} \quad (n=k,k+1,k+2,\ldots),$$

that is, if

$$r\le\left[\frac{p(1+B)\Gamma_n}{(B-A)(p-\lambda)}\right]^{\frac{1}{(n-p)}} \quad (n=k,k+1,k+2,\ldots). \quad \square$$

It follows that any function $f(z)\in\Phi_k^p(q,s;A,B,\lambda)$ is p-valently close-to-convex in the disc $U\left(R^*(\Phi_k^p(q,s;A,B,\lambda))\right)$, where $R^*\left(\Phi_k^p(q,s;A,B,\lambda)\right)$ is defined by (4.1).

Corollary 5.

$$R^*\left(\Phi_k^p(s;A,B,\lambda)\right)=\begin{cases}1, & (\alpha_j\ge\beta_j, j=1,\ldots,s),\\ \min\limits_{n\ge k}\left[\dfrac{p(1+B)\Gamma_n}{(B-A)(p-\lambda)}\right]^{\frac{1}{(n-p)}}, & (\alpha_j<\beta_j, j=1,\ldots,s),\end{cases} \tag{4.4}$$

where Γ_n is defined by (1.9). The result is sharp.

Proof. By Corollary 1 and Lemmas 1 and 2, we have

$$\Phi_k^p(s;A,B,\lambda)\subset T_p^*(\lambda) \quad (\alpha_j\ge\beta_j, j=1,\ldots,s).$$

By Theorem 4, any function $f(z)\in\Phi_k^p(s;A,B,\lambda)$ is p-valently close-to-convex in the disc $U(r)$, where

$$r=\inf_{n\ge k}(d_n)^{\frac{1}{(n-p)}}\left(d_n=\frac{p(1+B)\Gamma_n}{(B-A)(p-\lambda)}\right).$$

Since, for $\alpha_j<\beta_j$ $(j=1,\ldots,s)$, we have

$$\lim_{n\to\infty}d_n=d<1, \qquad \lim_{n\to\infty}(d_n)^{\frac{1}{n-p}}=1 \quad \text{and} \quad d_n>0 \quad (n=k,k+1,k+2,\ldots),$$

the infimum of the set $\left\{(d_n)^{\frac{1}{(n-p)}}: n\ge k\right\}$ is realized for an element of this set for some $n=n_0$. Moreover, the function

$$f_{n_0}(z)=z^p-\frac{(B-A)(p-\lambda)}{n_0(1+B)\Gamma_{n_0}}z^{n_0},$$

belongs to the class $\Phi_k^p(s;A,B,\lambda)$, and for $z=\left(d_{n_0}\right)^{\frac{1}{(n_0-p)}}$, we have

$$\text{Re}\left\{\frac{f'_{n_0}(z)}{z^{p-1}}\right\}=0.$$

Thus the result is sharp. $\square$

From Theorem 4 we can obtain direct estimation of the radius of p-valently close-to-convex for the class $\Phi_k^p(s; A, B, \lambda)$ with $\alpha_j \le \beta_j$ $(j = 1, \dots, s)$.

Corollary 6. *If a function $f(z)$ belongs to the class $\Phi_k^p(s; A, B, \lambda)$ with $\alpha_j \le \beta_j$ $(j = 1, \dots, s)$, then $f(z)$ is p-valently close-to-convex in the disc $U(r^*)$, where*

$$r^* = \frac{\alpha_1 \dots\dots \alpha_s}{\beta_1 \dots\dots \beta_s}. \tag{4.5}$$

Proof. Since

$$\frac{p(1+B)}{(B-A)(p-\lambda)} > 1 \quad (0 \le B \le 1; -B \le A < B; 0 \le \lambda < p; p \in N) \tag{4.6}$$

and, for $\alpha_j \le \beta_j$ $(j = 1, \dots, s)$,

$$\Gamma_n = \frac{(\alpha_1)_{n-p} \dots\dots (\alpha_s)_{n-p}}{(\beta_1)_{n-p} \dots\dots (\beta_s)_{n-p}} \ge \frac{(\alpha_1)^{n-p} \dots\dots (\alpha_s)^{n-p}}{(\beta_1)^{n-p} \dots\dots (\beta_s)^{n-p}} = \left(\frac{\alpha_1 \dots\dots \alpha_s}{\beta_1 \dots\dots \beta_s} \right)^{n-p},$$

we obtain

$$R^*(\Phi_k^p(s; A, B, \lambda)) = \inf_{n \ge k} \left[\frac{p(1+B)\Gamma_n}{(B-A)(p-\lambda)} \right]^{\frac{1}{(n-p)}} \ge \frac{\alpha_1 \dots\dots \alpha_s}{\beta_1 \dots\dots \beta_s},$$

which completes the proof of Corollary 6. □

Theorem 5. *The radius of p-valently convex for the class $\Phi_k^p(q, s; A, B, \lambda)$ is given by*

$$R^c(\Phi_k^p(q, s; A, B, \lambda)) = \inf_{n \ge k} \left[\frac{p^2(1+B)\Gamma_n}{n(B-A)(p-\lambda)} \right]^{\frac{1}{(n-p)}}, \tag{4.7}$$

where Γ_n is defined by (1.9). The result is sharp.

Proof. It is sufficient to show that $\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| < p$ $(z \in U(r); 0 < r \le 1)$. Using similar arguments as given by Theorem 4, we can get the result. □

From Theorem 5 we can obtain direct estimation of the radius of p-valently convex for the class $\Phi_k^p(s; A, B, \lambda)$.

Corollary 7. *If a function $f(z)$ belongs to the class $\Phi_k^p(s; A, B, \lambda)(\alpha_1 \le \beta_1 + 1)$; $\alpha_1 \le p + 1$; $\alpha_j \le \beta_j$ $(j = 2, \dots, s)$, then $f(z)$ is p-valently convex in the disc $U(r)$, where*

$$r = \frac{(\alpha_1 - 1)(\alpha_2) \dots\dots (\alpha_s)}{(\beta_1)(\beta_2) \dots\dots\dots\dots\dots (\beta_s)}. \tag{4.8}$$

Proof. For $\alpha_1 \le p + 1$, we have

$$\frac{p}{n} \left(\frac{\alpha_1 + n - p - 1}{\alpha_1 - 1} \right) \ge 1.$$

Since

$$\begin{aligned} \frac{p}{n}\Gamma_n &= \frac{p}{n} \left(\frac{(\alpha_1)_{n-p} \dots\dots (\alpha_s)_{n-p}}{(\beta_1)_{n-p} \dots\dots (\beta_s)_{n-p}} \right) \\ &= \frac{p}{n} \left(\frac{\alpha_1 + n - p - 1}{\alpha_1 - 1} \right) \frac{(\alpha_1 - 1)_{n-p} (\alpha_2)_{n-p} \dots (\alpha_s)_{n-p}}{(\beta_1)_{n-p} \dots\dots\dots\dots\dots\dots\dots\dots (\beta_s)_{n-p}} \\ &\ge \frac{p}{n} \left(\frac{\alpha_1 + n - p - 1}{\alpha_1 - 1} \right) \frac{(\alpha_1 - 1)^{n-p} \alpha_2^{n-p} \dots \alpha_s^{n-p}}{\beta_1^{n-p} \dots\dots\dots\dots\dots\dots\dots \beta_s^{n-p}} \\ &\ge \left(\frac{(\alpha_1 - 1)\alpha_2 \dots \alpha_s}{\beta_1 \dots\dots\dots\dots\dots \beta_s} \right)^{n-p} \end{aligned}$$

by (4.6), we have

$$\begin{aligned} R^c(\Phi_k^p(s; A, B, \lambda)) &= \inf_{n\geq k}\left[\frac{p^2(1+B)\Gamma_n}{n(B-A)(p-\lambda)}\right]^{\frac{1}{(n-p)}} \\ &= \inf_{n\geq k}\left[\frac{p(1+B)}{(B-A)(p-\lambda)}\cdot\frac{p\Gamma_n}{n}\right]^{\frac{1}{(n-p)}} \\ &\geq \frac{(\alpha_1-1)(\alpha_2)\ldots(\alpha_s)}{(\beta_1)(\beta_2)\ldots\ldots\ldots\ldots(\beta_s)}, \end{aligned}$$

which completes the proof of Corollary 7. □

5. Modified Hadamard product

For the functions

$$f_j(z) = z^p - \sum_{n=k}^{\infty} a_{n,j}z^n \quad (a_{n,j} \geq 0; j = 1, 2; k, p \in N), \tag{5.1}$$

we denote by $(f_1 \otimes f_2)(z)$ the modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 \otimes f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{n,1}a_{n,2}z^n. \tag{5.2}$$

Theorem 6. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\Phi_k^p(q, s; A, B, \lambda)$. If the sequence $\{n\Gamma_n\}$ is nondecreasing, then $(f_1 \otimes f_2)(z) \in \Phi_k^p(q, s; A, B, \delta)$, where*

$$\delta = p - \frac{(B-A)(p-\lambda)^2}{k(1+B)\Gamma_k}. \tag{5.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [16], we need to find the largest δ such that

$$\sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\delta)} a_{n,1}a_{n,2} \leq 1. \tag{5.4}$$

Since

$$\sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_{n,1} \leq 1 \tag{5.5}$$

and

$$\sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_{n,2} \leq 1, \tag{5.6}$$

by the Cauchy–Schwarz inequality, we have

$$\sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{5.7}$$

Thus it is sufficient to show that

$$\frac{n(1+B)\Gamma_n}{(B-A)(p-\delta)} a_{n,1}a_{n,2} \leq \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} \sqrt{a_{n,1}a_{n,2}} \quad (n \geq k) \tag{5.8}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(p-\delta)}{(p-\lambda)} \quad (n \geq k). \tag{5.9}$$

Note that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} \quad (n \geq k). \tag{5.10}$$

Consequently, we need only to prove that

$$\frac{(B-A)(p-\lambda)}{n(1+B)\Gamma_n} \leq \frac{(p-\delta)}{(p-\lambda)} \quad (n \geq k), \tag{5.11}$$

or, equivalently, that

$$\delta \leq p - \frac{(B-A)(p-\lambda)^2}{n(1+B)\Gamma_n} \quad (n \geq k). \tag{5.12}$$

Since

$$\Phi(n) = p - \frac{(B-A)(p-\lambda)^2}{n(1+B)\Gamma_n} \tag{5.13}$$

is an increasing function of n $(n \geq k)$, letting $n = k$ in (5.13), we obtain

$$\delta \leq \Phi(k) = p - \frac{(B-A)(p-\lambda)^2}{k(1+B)\Gamma_k}, \tag{5.14}$$

which proves the main assertion of Theorem 6.

Finally, by taking the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^p - \frac{(B-A)(p-\lambda)}{k(1+B)\Gamma_k} z^k \quad (j = 1, 2;\ k, p \in N) \tag{5.15}$$

we can see that the result is sharp. □

Theorem 7. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\Phi_k^p(q, s; A, B, \lambda)$. If the sequence $\{n\Gamma_n\}$ is nondecreasing. Then the function*

$$h(z) = z^p - \sum_{n=k}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \tag{5.16}$$

belongs to the class $\Phi_k^p(q, s; A, B, \tau)$, where

$$\tau = p - \frac{2(B-A)(p-\lambda)^2}{k(1+B)\Gamma_k}. \tag{5.17}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.15).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{n=k}^{\infty} \left\{ \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} \right\}^2 a_{n,1}^2 \leq \left\{ \sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_{n,1} \right\}^2 \leq 1 \tag{5.18}$$

and

$$\sum_{n=k}^{\infty} \left\{ \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} \right\}^2 a_{n,2}^2 \leq \left\{ \sum_{n=k}^{\infty} \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} a_{n,2} \right\}^2 \leq 1. \tag{5.19}$$

It follows from (5.18) and (5.19) that

$$\sum_{n=k}^{\infty} \frac{1}{2} \left\{ \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} \right\}^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{5.20}$$

Therefore, we need to find the largest τ such that

$$\frac{n(1+B)\Gamma_n}{(B-A)(p-\tau)} \leq \frac{1}{2} \left\{ \frac{n(1+B)\Gamma_n}{(B-A)(p-\lambda)} \right\}^2 \quad (n \geq k), \tag{5.21}$$

that is,

$$\tau \leq p - \frac{2(B-A)(p-\lambda)^2}{n(1+B)\Gamma_n} \quad (n \geq k). \tag{5.22}$$

Since

$$D(n) = p - \frac{2(B-A)(p-\lambda)^2}{n(1+B)\Gamma_n}, \tag{5.23}$$

is an increasing function of n $(n \geq k)$, we readily have

$$\tau \leq D(k) = p - \frac{2(B-A)(p-\lambda)^2}{k(1+B)\Gamma_k}, \tag{5.24}$$

and Theorem 8 follows at once. □

Remark 1. Taking $A = -\rho$ and $B = \rho$ $(0 < \rho \leq 1)$ in the above results, we obtain the corresponding results for the class $\Phi_k^p(q, s; \lambda, \rho)$.

Acknowledgment

The authors would like to thank the referee of the paper for the helpful suggestions.

References

[1] J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999) 1–13.
[2] H. Saitoh, A linear operator and its applications of first order differential subordinations, Math. Japan. 44 (1996) 31–38.
[3] S.D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1996) 429–446.
[4] R.J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965) 755–758.
[5] A.E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966) 352–357.
[6] Vinod Kumar, S.L. Shukla, Multivalent functions defined by Ruscheweyh derivatives, Indian J. Pure Appl. Math. 15 (11) (1984) 1216–1227.
[7] Vinod Kumar, S.L. Shukla, Multivalent functions defined by Ruscheweyh derivatives, Indian J. Pure Appl. Math. 15 (11) (1984) 1228–1238.
[8] H.M. Srivastava, M.K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II, J. Math. Anal. Appl. 171 (1992) 1–13 ; 192 (1995) 673–688.
[9] S. Owa, On the distortion theorems. I, Kyungpook Math. J. 18 (1978) 53–59.
[10] H.M. Srivastava, S. Owa (Eds.), Univalent Functions, Fractional Calculus, and their Applications, Halsted Press (Ellis Horwood Limited), Chichester, 1989, John Wiley and Sons, New York, Chichester, Brisbane and Toronto.
[11] M.K. Aouf, Certain class of analytic functions associated with the generalized hypergeometric function, J. Math. Appl. 29 (2007) 17–31.
[12] M.K. Aouf, A generalization of multivalent functions with negative coefficients, Bull. Korean Math. Soc. 25 (2) (1988) 221–232.
[13] M.K. Aouf, Certain classes of p-valent functions with negative coefficients. II, Indian J. Pure Appl. Math. 19 (8) (1988) 761–767.
[14] S.K. Lee, S. Owa, H.M. Srivastava, Basic properties and characterizations of a certain class of analytic functions with negative coefficients, Utilitas. Math. 36 (1989) 121–128.
[15] M.K. Aouf, H.E. Darwish, Some classes of multivalent functions with negative coefficients, Honam Math. J. 16 (1) (1994) 119–135.
[16] A. Schild, H. Silverman, Convolution univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska Sect. A 29 (1975) 99–107.