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Short Communication

# T-subnorms with strong associated negation: Some properties

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**Abstract**

In this work we investigate t-subnorms $M$ that have strong associated negation. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, nilpotent elements, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations lead us to two open problems which are also presented.



*Keywords:* T-norms; T-subnorms; Archimedeanness; Conditional cancellativity; Left-continuity; $R$-implications; Residual implications

## 1. Introduction

The theory of triangular norms and triangular subnorms has been well studied and its applications well-established. Many algebraic and analytical properties of these operations, viz., Archimedeanness, conditional cancellativity, left-continuity, etc., have been studied and their inter-relationships shown (see for instance, [6]).

Yet another way of categorizing t-subnorms is as follows: Given a t-subnorm $M$, one can obtain its associated negation $n_M$ (see Definitions 2.2 and 2.4 below). Note that $n_M$ is usually not a fuzzy negation, i.e., $n_M(1) \geq 0$. However, we can broadly consider two sub-classes of t-subnorms based on whether their associated negation $n_M$ is strong or not.

In this work, we study the class of t-subnorms whose associated negation $n_M$ is strong. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate some particular classes of these and study the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations have led us to two open problems, which are also presented.

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## 2. Preliminaries

To make this short note self-contained, we present some important definitions and properties, which can be found in [6,1].

**Definition 2.1.** A fuzzy negation is a function $N: [0,1] \to [0,1]$ that is non-increasing and such that $N(1) = 0$ and $N(0) = 1$. Further, it is said to be strong or involutive, if $N \circ N = id_{[0,1]}$.

**Definition 2.2.** A t-subnorm is a function $M: [0,1]^2 \to [0,1]$ such that it is monotonic non-decreasing, associative, commutative and $M(x, y) \le \min(x, y)$ for all $x, y \in [0,1]$, i.e., 1 need not be the neutral element.

**Definition 2.3.** Let $M$ be a t-subnorm.

(i) If 1 is the neutral element of $M$, then it becomes a t-norm. We denote a t-norm by $T$ in the sequel.
(ii) $M$ is said to satisfy the Conditional Cancellation Law if, for any $x, y, z \in (0,1]$,

$$M(x, y) = M(x, z) > 0 \text{ implies } y = z\,. \tag{CCL}$$

Alternately, (CCL) implies that on the positive domain of $M$, i.e., on the set $\{(x, y) \in (0,1]^2 \mid M(x, y) > 0\}$, $M$ is strictly increasing.
(iii) $M$ is said to be *Archimedean*, if for all $x, y \in (0,1)$ there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} < y$.
(iv) An element $x \in (0,1)$ is a *nilpotent* element of $M$ if there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} = 0$.
(v) A t-norm $T$ is said to be *nilpotent*, if it is continuous and if each $x \in (0,1)$ is a nilpotent element of $T$.

**Definition 2.4.** Let $M$ be any t-subnorm and $x, y \in [0,1]$.

- The $R$-implication $I_M$ of $M$ is given by

$$I_M(x, y) = \sup\{t \in [0,1] \mid M(x, t) \le y\}\,. \tag{1}$$

- The associated negation $n_M$ of $M$ is given by

$$n_M(x) = \sup\{t \in [0,1] \mid M(x, t) = 0\}. \tag{2}$$

A brief note on nomenclature is perhaps warranted here. Firstly, the $R$-implication $I_M$ will be termed a *residual* implication only if the underlying t-subnorm $M$ is left-continuous.

Secondly, while $n_M$ is clearly a non-increasing function and $n_M(0) = 1$, note that it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. Hence, only in the case $n_M$ is a fuzzy negation we call $n_M$ the *natural negation* of $M$ in this work. However, many results hold even if $n_M(1) > 0$, see for instance [3,9], and hence to preserve this generality in such situations we term $n_M$ as the *associated negation*.

For instance, the following result is true even when $n_M(1) > 0$.

**Proposition 2.5** *(cf. [1], Proposition 2.3.4). Let $M$ be any t-subnorm and $n_M$ its associated negation. Then we have the following:*

(i) $M(x, y) = 0 \Longrightarrow y \le n_M(x)$.
(ii) $y < n_M(x) \Longrightarrow M(x, y) = 0$.
(iii) *If $M$ is left-continuous then $y = n_M(x) \Longrightarrow M(x, y) = 0$, i.e., the reverse implication of (i) also holds.*

**Proposition 2.6.** *Let $M$ be any t-subnorm with $n_M$ being a natural negation with $e$ as its fixed point, i.e., $n_M(e) = e$. Then*

(i) *Every $x \in (0, e)$ is a nilpotent element; in fact, $x_M^{[2]} = 0$ for all $x \in [0, e)$.*
(ii) *In addition, if $M$ is either conditionally cancellative or left-continuous, then $e$ is also a nilpotent element.*

**Proof.** (i) By definition,

$$n_M(e) = \sup\{t \in [0,1] \mid M(e,t) = 0\} = e,$$

implies that $M(e, e^-) = 0$, from whence we get $M(x,x) \le M(e,e^-) = 0$ for all $x \in [0, e)$. In other words, $x_M^{[2]} = 0$ for all $x \in [0, e)$.

(ii) Let $M$ be conditionally cancellative. If $e_M^{[2]} = 0$ then clearly $e$ is a nilpotent element. If not, then we have $M(e,e) = x < M(1,e) \le e$ and from (ii) above we have $M(x,x) = 0$. Now,

$$e_M^{[4]} = M(M(e,e), M(e,e)) = M(x,x) = 0.$$

If $M$ is left-continuous, then $n_M(e) = \max\{t \in [0,1] \mid M(e,t) = 0\} = e$, i.e., $e \in \{t \in [0,1] \mid M(e,t) = 0\}$ and hence $M(e,e) = 0$, i.e., $e$ is also a nilpotent element. □

**Remark 2.7.**

(i) In the case $n_M$ is a strong natural negation we can show that if $M$ is conditionally cancellative then every $x \in (0,1)$ is also a nilpotent element, see Remark 5.9(ii).

(ii) Note that without any further assumptions, the set of nilpotent elements need not be the whole of $(0,1)$. For instance, for the nilpotent minimum t-norm

$$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x + y \le 1, \\ \min(x,y), & \text{otherwise,} \end{cases} \quad x, y \in [0,1],$$

which is left-continuous but not conditionally cancellative, its set of nilpotent elements is $(0, .5]$, while its set of zero divisors is $(0,1)$.

However, Theorem 6.1 gives an equivalence condition for the whole of $(0,1)$ to be the set of nilpotent elements under a suitable condition on $n_M$.

## 3. T-subnorms with strong associated negation = t-norms

There are works showing that some classes of t-subnorms $M$ whose associated negations $n_M$ are involutive do become t-norms. Jenei [4], also see [5], showed it for the class of left-continuous $M$, while Jayaram [2] did the same for conditionally cancellative $M$. The main result of this section shows that the above results are true in general, i.e., any t-subnorm with a strong natural negation is a t-norm.

The following result was firstly proven by Jenei in [4]. However, we give a very simple proof of this result without resorting to the rotation-invariance property.

**Theorem 3.1** *(Jenei, [4], Theorem 3). If $M$ is a left-continuous t-subnorm with $n_M$ being strong, then $M$ is a t-norm.*

**Proof.** Firstly, note that if $M$ is a left-continuous t-subnorm, then its residual implication satisfies the exchange principle, i.e.,

$$I_M(x, I_M(y,z)) = I_M(y, I_M(x,z)).$$

It follows from the fact that the neutral element of $M$ does not play any role in the proof, see, for instance the proof given for Theorem 2.5.7 in [1].

If $n_M$ is strong, then for every $y \in [0,1]$ there exists $y' \in [0,1]$ such that $n_M(y) = y'$. Now,

$$I_M(1, y') = I_M(1, I_M(y, 0)) = I_M(y, I_M(1, 0)) = I_M(y, 0) = y'.$$

Thus, for all $y' \in [0,1]$,

$$I_M(1, y') = \max\{t \mid M(1,t) \le y'\} = y' \Longrightarrow M(1, y') = y' . \quad □$$

**Theorem 3.2** *(Jayaram [2], Theorem 4.4). Let $M$ be any conditionally cancellative t-subnorm. If $n_M$ is a strong natural negation then $M$ is a t-norm.*

Now, we prove the main result of this section which shows that the above results are true in general.

**Theorem 3.3.** *Let $M$ be any t-subnorm with $n_M$ being a strong natural negation. $M$ is a t-norm.*

**Proof.** Note, firstly, that since $n_M(x) = \sup\{t \in [0,1] \mid M(x,t) = 0\}$, is a strong negation, we have that $n_M(z) = 1 \Longleftrightarrow z = 0$ and $n_M(z) = 0 \Longleftrightarrow z = 1$. Equivalently, $M(1,z) = 0 \Longleftrightarrow z = 0$.

On the contrary, let us assume that $M(1,x) = x' \lneq x$ for some $x \in (0,1]$. Since $n_M$ is strong, the following are true:

(i) $n_M(x') > n_M(x)$,
(ii) if $p > n_M(x)$ then $M(x,p) > 0$,
(iii) there exists a $y \in (0,1)$ such that $n_M(x') > y > n_M(x)$ and $M(y,x) = q > 0$ while $M(y,x') = 0$.

Now, by associativity we have

$$\left.\begin{array}{l} M(y, M(x,1)) = M(y,x') = 0 \\ M(M(y,x),1) = M(q,1) \end{array}\right\} \Longrightarrow M(q,1) = 0,$$

a contradiction. Thus $M(1,x) = x$ for **all** $x \in [0,1]$ and hence we have the result. □

In the following sections, we deal with t-subnorms whose associated negations are strong, or equivalently t-norms whose associated negations are strong. We discuss the inter-relationships between the different algebraic and analytical properties for this subclass of t-norms; in particular, Archimedeanness, Conditional Cancellativity, (Left-)continuity and Nilpotence that are relevant to our context. We begin with listing out some established results and go on to present some new ones.

## 4. Continuity and nilpotence

Let $T$ be a t-norm and $n_T$ a strong negation. The following result, whose proof is straight-forward, shows the equivalence between continuity and nilpotence:

**Theorem 4.1** *(Klement et al. [6]). Let $T$ be a t-norm with $n_T$ being strong. Then the following are equivalent:*

*(i) $T$ is continuous.*
*(ii) $T$ is a nilpotent t-norm.*

Further, we know that every nilpotent t-norm is both Archimedean and Conditionally cancellative, since every nilpotent t-norm is isomorphic to the Łukasiewicz t-norm and the Archimedeanness and Conditionally cancellativity of $T$ are preserved under isomorphism, see [6], Examples 2.14(iv) and 2.15(v). Trivially, every nilpotent t-norm is also left-continuous.

## 5. Conditional cancellativity, left continuity and nilpotence

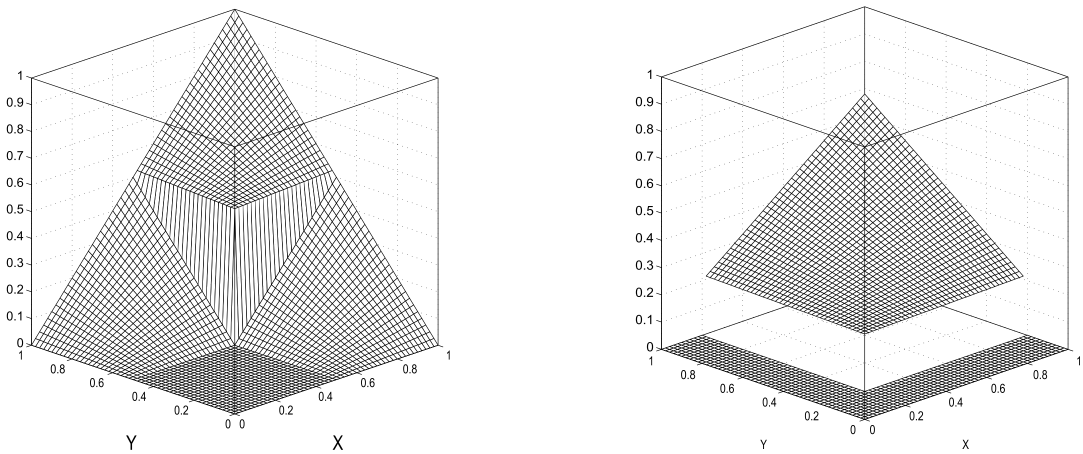
Recently, in Jayaram [2], the following problem of U. Höhle, given in KLEMENT et al. [7] has been solved. Further it was shown that it characterizes the set of all conditionally cancellative t-subnorms.

**Problem 5.1** *(U. Höhle, [7], Problem 11).* Characterize all left-continuous t-norms $T$ which satisfy

$$I_T(x, T(x,y)) = \max(n_T(x), y), \quad x, y \in [0,1], \tag{3}$$

where $I_T, n_T$ are as given in (1) and (2) with $M = T$.

**Theorem 5.2** *(cf. Jayaram [2], Theorem 3.1). Let $M$ be any t-subnorm, not necessarily left-continuous. Then the following are equivalent:*




(a) $T_{\mathbf{OY}}$ (b) $M_{\mathbf{P_f}}$

Fig. 1. A t-norm and a t-subnorm that are conditionally cancellative.

(i) *The pair* $(I_M, M)$ *satisfies* (3).
(ii) $M$ *is a Conditionally Cancellative t-subnorm.*

**Remark 5.3.** The following statements follow from Theorem 5.2 with $M = T$, a t-norm:

(i) If a (right) continuous $T$ satisfies (3) along with its $R$-implication then $T$ is necessarily Archimedean, see [6], Proposition 2.15(ii).
(ii) However, if a left-continuous $T$ satisfies (3) along with its residual implication then $T$ need not be Archimedean and hence not continuous. An example is Hajék's t-norm or the following t-norm $T_{\mathbf{OY}}$ of Ouyang et al. [11], Example 3.4, which is a (CCL) t-norm (and hence a t-subnorm too) that is left-continuous but not continuous at $(0.5, 0.5)$ and hence is not Archimedean (see Fig. 1(a)):

$$T_{\mathbf{OY}}(x, y) = \begin{cases} 2(x-0.5)(y-0.5)+0.5, & \text{if } (x, y) \in (0.5, 1]^2 \\ 2y(x-0.5), & \text{if } (x, y) \in (0.5, 1] \times [0, 0.5] \\ 2x(y-0.5), & \text{if } (x, y) \in [0, 0.5] \times (0.5, 1] \\ 0, & \text{otherwise} \end{cases}.$$

**Theorem 5.4** *(Jenei, [4], Theorem 2). Let $T$ be a left-continuous t-norm with $n_T$ being strong. Then the following are equivalent:*

(i) $T$ *is a conditionally cancellative t-norm.*
(ii) $T$ *is a nilpotent t-norm.*

In fact, for a conditionally cancellative t-subnorm $M$ we can give an equivalent condition for it to be left-continuous.

**Theorem 5.5.** *Let $M$ be a* (CCL) *t-subnorm. Then the following are equivalent:*

(i) $M(x, n_M(x)) = 0$, $x \in [0, 1]$.
(ii) $M$ *is left-continuous.*

**Proof.** (i) $\Longrightarrow$ (ii): Let $M(x, n_M(x)) = 0$, for all $x \in [0, 1]$. On the contrary, let us assume that $M$ is not left-continuous. Then there exist $x_0 \in (0, 1]$, $y_0 \in (0, 1]$ and an increasing sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in [0, 1)$, such that $\lim_{n\to\infty} x_n = x_0$, but

$$\lim_{n\to\infty} M(x_n, y_0) = M(x_0^-, y_0) = z' < z_0 = M(x_0, y_0).$$

Observe that

$$I_M(y_0, z') = \sup\{t \in [0,1] \mid M(y_0, t) \le z'\} = x_0, \tag{4}$$

since from the monotonicity of $M$ we have $M(y_0, x_n) \le z'$ for every $n \in \mathbb{N}$ and $M(y_0, x_0) = z_0 > z'$. Since $M$ is (CCL), we have from (3)

$$I_M(y_0, z') = I_M(y_0, M(y_0, x_0^-)) = \max(n(y_0), x_0^-).$$

Now, we have two cases. On the one hand, if $I_M(y_0, z') = x_0^- \lneq x_0$, then it is a contradiction to (4). On the other hand, if $I_M(y_0, z') = n(y_0)$, then this implies that $n(y_0) = x_0$ from (4) and hence

$$M(x_0, y_0) = M(n(y_0), y_0) = z_0 = 0,$$

by the hypothesis and hence there does not exist any $z' < z_0$ and hence $M$ is left-continuous.

(ii) $\Longrightarrow$ (i): Follows from Proposition 2.5(iii). □

In other words, Theorem 5.5 states that for a (CCL) t-subnorm $M$, the only points at which $M$ may not be left-continuous is the boundary of the zero region $Z_M = \{(x, y) \in [0,1]^2 | M(x, y) = 0\}$ which does not contain the origin.

**Remark 5.6.** In Theorem 5.5 we do not need $n_M$ to be a negation, i.e., $n_M(1) \ge 0$. Consider the following t-subnorm $M_{\mathbf{P_f}}$ (cf. Example 3.15 of [6], see Fig. 1(b)),

$$M_{\mathbf{P_f}} = \begin{cases} 0.2 + \dfrac{3(x-0.2)(y-0.2)}{4}, & \text{if } (x,y) \in (0.2, 1]^2 \\ 0, & \text{otherwise} \end{cases}$$

which is a left-continuous (CCL) t-subnorm but $n_{M_{\mathbf{P_f}}}$ is not a negation since $n_{M_{\mathbf{P_f}}}(1) = 0.2$.

**Theorem 5.7.** *Let $M$ be a (CCL) t-subnorm whose $n_M$ is strong. Then $M$ is left-continuous.*

**Proof.** If possible, let $M(x_0, n(x_0)) = p > 0$ for some $x_0 \in (0,1)$. Since $M$ is (CCL), we have $M(1^-, x_0) < x_0$ and hence by associativity we have

$$M(1^-, M(x_0, n(x_0))) = M(1^-, p)$$
$$M(M(1^-, x_0), n(x_0)) = 0$$

from whence it follows $M(1^-, p) = 0$, i.e., $n(p) = 1$, a contradiction to the fact that $n_M$ is strong. Thus $p = 0$ and the result follows from Theorem 5.5. □

**Theorem 5.8.** *Let $M$ be a t-subnorm such that $n_M$ is strong. Then the following are equivalent:*

(i) *$M$ is conditionally cancellative.*
(ii) *$M$ is a nilpotent t-norm.*

**Proof.** If $M$ satisfies (CCL) then $M$ is left-continuous, from Theorem 5.7 and now, using Theorem 5.4 we have the result. □

**Remark 5.9.**

(i) The nilpotent minimum t-norm $T_{\mathbf{nM}}$ is an example of a t-subnorm $M$ whose $n_M$ is involutive and $M$ satisfies (LEM) with $n_M$ but is not conditionally cancellative and hence is not a nilpotent t-norm.
(ii) In the case $n_M$ is a strong natural negation, from Theorem 5.7 we see that conditionally cancellativity implies left-continuity and from Theorem 5.8 that every $x \in (0,1)$ is a nilpotent element.

## 6. Archimedeanness, left continuity and nilpotence

We begin with a result that shows that if $n_M$ is strong, then the Archimedeanness is equivalent to every element $x \in (0, 1)$ being nilpotent. However, unless $M$ is also left-continuous, $M$ is not a nilpotent t-norm.

**Theorem 6.1.** *Let $M$ be any t-subnorm such that $n_M$ is not completely vanishing, i.e., there exists $z \in (0, 1)$ such that $n_M(z) > 0$. The following are equivalent:*

(i) *Every $x \in (0, 1)$ is a nilpotent element.*
(ii) *$M$ is Archimedean.*

**Proof.** (i) $\Longrightarrow$ (ii): Follows from Proposition 2.15 (iv) in [6].
(ii) $\Longrightarrow$ (i): Let $M$ be any Archimedean t-subnorm such that $n_M$ is not completely vanishing, i.e., there exists $z \in (0, 1)$ such that $n_M(z) > 0$. By Proposition 2.5(ii) we see that for any $0 < z' < n_M(z)$ we have $M(z', z) = 0$. For any $x \in [0, 1)$, by the Archimedeanness of $M$, there exists an $n, p \in \mathbb{N}$ such that $x_M^{[n]} < z'$ and $x_M^{[p]} < z$ from whence we have that

$$x_M^{[n+p]} = M\left(x_M^{[n]}, x_M^{[p]}\right) \leq M(z', z) = 0. \quad \square$$

**Corollary 6.2.** *Let $M$ be any t-subnorm such that $n_M$ is a strong negation. Then the following are equivalent:*

(i) *Every $x \in (0, 1)$ is a nilpotent element.*
(ii) *$M$ is Archimedean.*

The following result is due to Kolesárová [8]:

**Theorem 6.3.** *Let $T$ be any Archimedean t-norm. Then the following are equivalent:*

(i) *$T$ is left-continuous.*
(ii) *$T$ is continuous.*

**Corollary 6.4.** *A left-continuous Archimedean t-subnorm $M$ whose $n_M$ is involutive is a nilpotent t-norm.*

**Proof.** From Theorem 3.1 we see that $M$ is a left-continuous t-norm. From Theorem 6.3, since $M$ is Archimedean it is continuous. Also by Theorem 6.1, we have that every $x \in (0, 1)$ is a nilpotent element. Thus $T$ is nilpotent, i.e., isomorphic to $T_{\mathbf{LK}}(x, y) = \max(0, x + y - 1)$. $\square$

**Remark 6.5.**

(i) Note that there exist left-continuous Archimedean t-subnorms $M$ that are not continuous and hence their $n_M$ is not involutive. For instance, consider the t-subnorm

$$M(x, y) = \begin{cases} x + y - 1, & \text{if } x + y > \frac{3}{2}, \\ 0, & \text{otherwise}, \end{cases} \quad x, y \in [0, 1].$$

(ii) The nilpotent minimum t-norm $T_{\mathbf{nM}}$ is an example of a left-continuous t-subnorm $M$ whose $n_M$ is involutive but is not Archimedean and hence is not a nilpotent t-norm.
(iii) However, it is not clear whether there exists any non-nilpotent Archimedean t-subnorm $M$ whose $n_M$ is involutive. Clearly such t-(sub)norms are not left-continuous.

**Problem 1.** Does there exist any non-nilpotent Archimedean t-subnorm $M$ whose $n_M$ is involutive. In other words, is an Archimedean t-subnorm $M$ whose $n_M$ is involutive necessarily left-continuous?

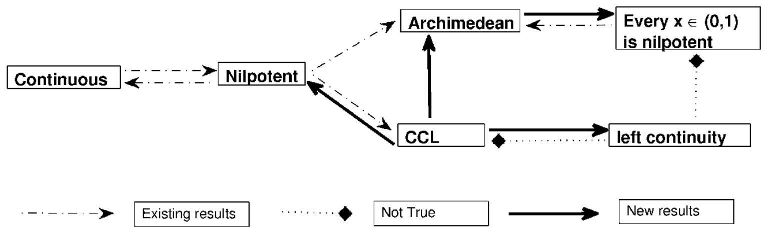



Fig. 2. A summary of the results available so far when $n_T$ is strong.

## 7. Archimedeanness and conditional cancellativity

In general, there does not exist any inter-relationships between Archimedeanness and conditional cancellativity, as the following examples show.

**Example 7.1.**

(i) The Ouyang t-norm $T_{\mathbf{OY}}$ is an example of a t-(sub)norm which is not Archimedean but is both left-continuous and conditionally cancellative.

(ii) The following t-norm is neither Archimedean nor left-continuous but is conditionally cancellative:

$$T(x, y) = \begin{cases} 0, & \text{if } xy \le \frac{1}{2} \,\&\, \max(x, y) < 1 \\ xy, & \text{if } xy > \frac{1}{2} \\ \min(x, y), & \text{otherwise} \end{cases} .$$

(iii) The following t-subnorm is Archimedean and continuous, but not conditionally cancellative:

$$M(x, y) = \max(0, \min(x + y - 1, x - a, y - a, 1 - 2a)),$$

where $a \in (0, 0.5)$. For instance, with $a = 0.25$ we have $M(0.75, 0.75) = M(0.75, 0.8) = 0.5$.

(iv) The nilpotent minimum $T_{\mathbf{nM}}$, whose $n_M$ is strong, is neither Archimedean nor conditionally cancellative, but is left-continuous.

(v) The Lukasiewicz t-norm $T_{\mathbf{LK}}(x, y) = \max(0, x + y - 1)$ is both Archimedean and conditionally cancellative. Further, $n_{T_{\mathbf{LK}}}$ is strong.

In fact, in the case when $n_M$ is strong we have the following partial implication.

**Lemma 7.2.** *Let $M$ be any t-subnorm whose $n_M$ is strong. If $M$ is conditionally cancellative then $M$ is Archimedean.*

**Proof.** From Theorem 5.8, we have that if $M$ is conditionally cancellative then $M$ is a nilpotent t-norm from whence it follows that $M$ is Archimedean. □

**Problem 2.** Does there exist any Archimedean t-subnorm $M$ whose $n_M$ is involutive but is not conditionally cancellative? In other words, is an Archimedean t-subnorm $M$ whose $n_M$ is involutive necessarily conditionally cancellative?

In fact, from Theorem 3.3, it can be easily seen that the above two problems are an alternate formulation of Problem 2.1 in [10].

## 8. Concluding remarks

In this work, we have shown that t-subnorms whose associated negations are strong are necessarily t-norms. Further, we have studied the inter-relationships between some algebraic and analytical properties of such t-(sub)norms. Fig. 2 gives a pictorial summary of the results that exist so far. Our study has also opened up two interesting open problems.

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