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Transient solution of a multiserver Poisson queue with N-policy

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Abstract

We consider an $M/M/c$ queueing system, where the server idles until a fixed number N of customers accumulates in a queue and following the arrival of the N-th customer, the server serves exhaustively the queue. We obtain the exact transient solution for the state probabilities of this N-policy queue by a direct approach. Further we obtain the time-dependent mean, variance of this system and its busy period distribution.

Keywords: System size probabilities; Tridiagonal matrices; Generating functions; Busy period; Modified Bessel functions

1. Introduction

In the area of optimal design and control of queues, the N-policy has received great attention (e.g. [1,2]). According to this policy, the server idles until a *fixed* number N of customers arrives in the queue at the moment the server is "switched on" and serves exhaustively the queue until it empties. The server is then "switched off" and remains idle until N customers accumulate again in the queue. Given the costs of turning the server on and having customers waiting in the queue, an optimal value of N can be determined that minimizes the expected cost of operating the queue. This model is found to be applicable in analysing numerous real world queueing situations such as flexible manufacturing systems, service systems, computer and telecommunication systems (e.g. [3,4]). In many production systems it is assumed that when all the jobs are served, the machine stays idle until the next job arrives. If there is a cost associated with operating the machine, it is plausible that a rational way to operate the system is to shut down the machine when the queue length is zero and bring it up again as the queue length grows to a predetermined level of, say N (≥ 1), jobs. Such a control mechanism is usually good when the machine start-up and shut-down costs are high.

This model is an interesting special case of server vacation models and Doshi [5] has given a large number of examples. Vacation models have received considerable attention for their interesting theoretical properties as well as for their applicability in polling models. Machines producing certain items may need periodic checking and maintenance. The periods of random lengths of preventive maintenance may be considered as periods of server

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vacation when the server is unavailable (turned off) [6], and different arrival rates may be considered when the server is turned off.

Much of the earlier work done in controllable queueing systems has been concerned with the optimality of the operation policy under certain conditions (see, [7,8]). In this paper, we obtain in closed form the transient probabilities of the number in a multiserver queue with N-policy, mean, variance and the busy period distribution.

2. Model description

Consider an $M/M/c$ queue with customers arriving according to a Poisson process of intensity λ_0 if the server is turned off and λ if the server is turned on, each server has an independently and identically distributed exponential service-time distribution with mean $1/\mu$. We consider the case in which there are c servers in the system and when the number of waiting customers reaches N, the servers start to serve the customers until the queue becomes empty. The server is then "switched off" and remains idle until N customers accumulate again in the queue. Let $\{X(t), t \geq 0\}$ denote the number of customers in the system at time t. Let $Y(t)$ be the state of the server at time t. When the the server is turned on $Y(t)$ takes the value 1 and the server is turned off $Y(t)$ takes the value 0. Then $\{X(t), Y(t), t \geq 0\}$ is a continuous time Markov process on the state space $\mathbf{S} = \{(k, n), k : 0, 1; n : 0, 1, 2, \ldots\}$. (See Fig. 1.)

Let $p_{0n}(t)$ = Probability that there are n customers in the system at time t and the server is turned off, $n = 0, 1, 2, \ldots, N-1$, and $p_{1n}(t)$ = Probability is that there are n customers in the system at time t and the server is turned on, $n = 1, 2, 3, \ldots, \infty$.

The forward Chapman Kolmogorov equations for the system are

$$p'_{00}(t) = -\lambda_0 p_{00}(t) + \mu_1 p_{11}(t), \tag{2.1}$$

$$p'_{0n}(t) = -\lambda_0 p_{0n}(t) + \lambda_0 p_{0n-1}(t), \quad 1 \leq n \leq N-1, \tag{2.2}$$

$$p'_{11}(t) = -(\lambda + \mu_1) p_{11}(t) + \mu_2 p_{12}(t), \tag{2.3}$$

$$p'_{1n}(t) = -(\lambda + \mu_n) p_{1n}(t) + \lambda p_{1n-1}(t) + \mu_{n+1} p_{1n+1}(t), \quad n \geqslant 2,\ n \neq N, \tag{2.4}$$

$$p'_{1N}(t) = -(\lambda + \mu_N) p_{1N}(t) + [\lambda_0 p_{0N-1}(t) + \lambda p_{1N-1}(t)] + \mu_{N+1} p_{1N+1}(t), \quad n = N, \tag{2.5}$$

where

$$\mu_n = \min(n, c)\mu. \tag{2.6}$$

If $N = 1$ and $\lambda_0 = \lambda$ then $M/M/c$ with N-policy queue becomes the classical $M/M/c$ queue [9].

2.1. Steady-state probabilities

Let $p_{0n}, n = 0, 1, 2, \ldots, N-1$ and $p_{1n}, n = 1, 2, 3, \ldots,$ be the steady-state probabilities.

It is well known that for steady state

$$\rho = \frac{\lambda}{c\mu} < 1,$$

and

$$\lim_{t \to \infty} p_{kn}(t) = p_{kn}, \quad \text{if } k = 0,\ n = 0, 1, \ldots, N-1;\ k = 1,\ n = 1, 2, \ldots.$$

The steady-state probabilities can be obtained by replacing zero to the left-hand side of the system of equations (2.1)–(2.5).

From these equations, we observe that,

$$p_{00} = p_{01} = p_{02} = \cdots = p_{0N-1}, \qquad p_{11} = \frac{\lambda_0}{\mu} p_{00},$$

and

$$p_{1n} = \begin{cases} \dfrac{\pi_n}{\rho \pi_{N-1}} \dfrac{\sum\limits_{i=0}^{n-1} \pi_i^{-1}}{\sum\limits_{i=0}^{N-1} \pi_i^{-1}} \, p_{1N}, & \text{for } n = 1, 2, \ldots, N-1 \\ \rho^n \, p_{1N}, & \text{for } n = N+1, N+2, \ldots \end{cases}$$

where $\pi_i = \frac{\lambda^i}{\mu_1 \mu_2 \mu_3 \cdots \mu_i}$ with $\pi_0 = 1$.

Using $\sum_{n=0}^{N-1} p_{0n} + \sum_{n=1}^{\infty} p_{1n} = 1$, one gets

$$p_{1N} = \left[\frac{1}{1-\rho} + \frac{Nc\mu}{\lambda_0 \pi_{N-1} \sum\limits_{i=0}^{N-1} \pi_i^{-1}} + \frac{\sum\limits_{n=1}^{N-1} \pi_n \sum\limits_{i=0}^{n-1} \pi_i^{-1}}{\rho \pi_{N-1} \sum\limits_{i=0}^{N-1} \pi_i^{-1}} \right]^{-1}.$$

2.2. *Transient probabilities*

For the sake of simplicity, we first assume that the server is *turned on* initially.

It is natural to assume that $N \geq c$ and initially there are i customers in the system. Therefore,

$$p_{1n}(0) = \delta_{in}, \quad n \geq 1.$$

Define

$$Q(z,t) = q(t) + \sum_{n=1}^{\infty} p_{1N+n}(t) z^n, \qquad Q(z,0) = z^{\tau(i)}$$

with

$$q(t) = \sum_{n=0}^{N-1} p_{0n}(t) + \sum_{n=1}^{N} p_{1n}(t), \qquad \tau(i) = (i-N)\left[1 - \sum_{n=1}^{N} \delta_{in}\right].$$

The system of equations (2.1)–(2.5) gives

$$\frac{\partial Q}{\partial t} = \left[-(\lambda + c\mu) + \lambda z + \frac{c\mu}{z}\right][Q(z,t) - q(t)] + \lambda(z-1) p_{1N}(t).$$

Integrating,

$$Q(z,t) = \left[\lambda + c\mu - \lambda z - \frac{c\mu}{z}\right] \int_0^t e^{-(\lambda+c\mu)(t-y)} e^{(\lambda z + \frac{c\mu}{z})(t-y)} q(y) dy$$
$$+ \lambda(z-1) \int_0^t e^{-(\lambda+c\mu)(t-y)} e^{(\lambda z + \frac{c\mu}{z})(t-y)} p_{1N}(y) dy + z^{\tau(i)} e^{-(\lambda+c\mu)t} e^{(\lambda z + \frac{c\mu}{z})t}. \tag{2.7}$$

It is well known that if $\alpha = 2\sqrt{c\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{c\mu}}$ then

$$e^{(\lambda z + \frac{c\mu}{z})(t-y)} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n[\alpha(t-y)], \tag{2.8}$$

where $I_n(.)$ is the modified Bessel function of first kind.

2.3. *Evaluation of $p_{1N+n}(t), n \geq 1$*

Comparing the coefficients of z^n on both sides of (2.7), for $n = 1, 2, 3, \ldots,$

$$\beta^{-n} p_{1N+n}(t) = \int_0^t e^{-(\lambda+c\mu)(t-y)}[(\lambda + c\mu)I_n(.) - \lambda\beta^{-1}I_{n-1}(.) - c\mu\beta I_{n+1}(.)]q(y)dy$$
$$+ \lambda \int_0^t e^{-(\lambda+c\mu)(t-y)}\{\beta^{-1}I_{n-1}(.) - I_n(.)\}p_{1N}(y)dy + e^{-(\lambda+c\mu)t}\beta^{-\tau(i)}I_{n-\tau(i)}(\alpha t). \quad (2.9)$$

The above holds for $n = -1, -2, -3, \ldots$, with the left-hand side replaced by zero. Using $I_{-n}(.) = I_n(.)$, for $n = 1, 2, 3, \ldots$,

$$0 = \int_0^t e^{-(\lambda+c\mu)(t-y)}[(\lambda + c\mu)I_n(.) - \lambda\beta^{-1}I_{n+1}(.) - c\mu\beta I_{n-1}(.)]q(y)dy$$
$$+ \lambda \int_0^t e^{-(\lambda+c\mu)(t-y)}[\beta^{-1}I_{n+1}(.) - I_n(.)]p_{1N}(y)dy + e^{-(\lambda+c\mu)t}\beta^{-\tau(i)}I_{n+\tau(i)}(\alpha t). \quad (2.10)$$

Subtracting (2.10) from (2.9), for $n = 1, 2, 3, \ldots$,

$$p_{1N+n}(t) = \lambda\beta^{n-1} \int_0^t e^{-(\lambda+c\mu)(t-y)}[I_{n-1}(.) - I_{n+1}(.)]p_{1N}(y)dy$$
$$+ \beta^{n-\tau(i)}e^{-(\lambda+c\mu)t}[I_{n-\tau(i)}(\alpha t) - I_{n+\tau(i)}(\alpha t)], \quad (2.11)$$

where $I_n(.) = I_n[\alpha(t - y)]$. In (2.7), for $n = 0$,

$$q(t) = \int_0^t e^{-(\lambda+c\mu)(t-y)}\,[(\lambda + c\mu)I_0(.) - \alpha\, I_1(.)]\,q(y)dy$$
$$+ \lambda \int_0^t e^{-(\lambda+c\mu)(t-y)}(\beta^{-1}\, I_1(.) - I_0(.))p_{1N}dy + e^{-(\lambda+c\mu)t}\beta^{-\tau(i)}\, I_{\tau(i)}(\alpha t). \quad (2.12)$$

Thus we have expressed $p_{1N+n}(t)$ in terms of $p_{1N}(t)$ and $p_{1N}(t)$ can be evaluated using the expression (2.23).

2.4. *Evaluation of* $p_{1n}(t)$, $1 \le n \le N - 1$

We will now evaluate the probabilities $p_{1n}(t)$, $n = 1, 2, 3, \ldots, N - 1$, and $p_{0n}(t)$, $n = 0, 1, 2, \ldots, N - 1$. From (2.3) and first $N - 2$ equations in (2.4),

$$\frac{d}{dt}\mathbf{R}(t) = B\mathbf{R}(t) + c\mu p_{1N}(t)\mathbf{e}_{N-1}, \quad (2.13)$$

where

$$\mathbf{R}(t) = (p_{11}(t), p_{12}(t), p_{13}(t), \ldots, p_{1N-1}(t))^{\mathrm{T}},$$

and $\mathbf{e}_{N-1}$ is a column vector of order $N - 1$ with 1 in the last place and zero in the remaining places, $B = (c_{mj})_{(N-1)\times(N-1)}$ with

$$c_{mj} = \begin{cases} \lambda, & j = m - 1,\ m = 2, 3, 4, \ldots, N - 1 \\ -(\lambda + \mu_m), & j = m,\ m = 1, 2, 3, \ldots, N - 1 \\ \mu_{m+1}, & j = m + 1,\ m = 1, 2, 3, \ldots, N - 2, \end{cases}$$

and $c_{mj} = 0$ if $|m - j| > 1$.

In the sequel, $\hat{f}(s)$ denotes the Laplace transform of $f(t)$. Now, by taking Laplace transform on (2.13), we get

$$\hat{\mathbf{R}}(s) = (sI - B)^{-1}\mathbf{R}(0) + c\mu p_{1N}(s)(sI - B)^{-1}\mathbf{e}_{N-1} \quad (2.14)$$

with $\mathbf{R}(0) = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \ldots, \delta_{iN-1})$ and $sI - B = (c_{mj}(s))_{(N-1)\times(N-1)}$, a tridiagonal matrix.

Let $(sI-B)^{-1} = (\hat{b}_{mj}(s))_{(N-1)\times(N-1)}$. There are several methods to find $\hat{b}_{mj}(s)$. For example, from Usmani [10], for $m, j = 1, 2, 3, \ldots, N-1$,

$$\hat{b}_{mj}(s) = \begin{cases} (-1)^{m+j} c_{m,m+1}(s)c_{m+1,m+2}(s)\cdots c_{j-1,j}(s)\theta_{m-1}(s)\phi_{j+1}(s)/\theta_{N-1}(s), & m < j \\ \theta_{m-1}(s)\phi_{m+1}(s)/\theta_{N-1}(s), & m = j \\ (-1)^{m+j} c_{j+1,j}(s)c_{j+2,j+1}(s)\cdots c_{m,m-1}(s)\theta_{j-1}(s)\phi_{m+1}(s)/\theta_{N-1}(s), & m > j \end{cases} \tag{2.15}$$

with

$$\begin{aligned} \theta_m(s) &= c_{m,m}(s)\theta_{m-1}(s) - c_{m,m-1}(s)c_{m-1,m}(s)\theta_{m-2}(s), \qquad \theta_0(s) = 1, \qquad \theta_{-1}(s) = 0, \\ &\quad m = 1, 2, 3, \ldots, N-1, \\ \phi_m(s) &= c_{m,m}(s)\phi_{m+1}(s) - c_{m,m+1}(s)c_{m+1,m}(s)\phi_{m+2}(s), \qquad \phi_N(s) = 1, \qquad \phi_{N+1}(s) = 0, \\ &\quad m = N-1, N-2, N-3\ldots, 2, 1, \end{aligned}$$

and $|sI - B| = \theta_{N-1}(s)$.

We observe that $\hat{b}_{mj}(s)$ are rational algebraic functions in s. The cofactor of the (m, j)th element of $(sI - B)$ is a polynomial of degree $N - 2 - |m - j|$. In particular, the cofactor of the diagonal elements are polynomials in s of degree $N - 2$ with leading coefficient equal to 1.

It is also known that the characteristic roots of B are all distinct and negative [11]. Hence the inverse transform $b_{m,j}(t)$ of $\hat{b}_{m,j}(s)$ can be obtained by partial fractions decomposition.

From (2.14), for $n = 1, 2, 3, \ldots, N-1$,

$$\hat{p}_{1n}(s) = \sum_{j=1}^{N-1} \delta_{ij}\, \hat{b}_{nj}(s) + c\mu\hat{b}_{n,N-1}(s)\hat{p}_{1N}(s). \tag{2.16}$$

2.5. Evaluation of $p_{0n}(t)$, $0 \le n \le N-1$

Also from the system of equations (2.1) and (2.2), we obtain for $n = 0, 1, 2, \ldots, N-1$,

$$\hat{p}_{0n}(s) = \mu\hat{a}_n(s)\hat{p}_{11}(s), \tag{2.17}$$

where $\hat{a}_n(s) = \frac{\lambda_0^n}{(s+\lambda_0)^{n+1}}$, the Laplace transform of $a_n(t) = \frac{(\lambda_0 t)^n}{n!}\mathrm{e}^{-\lambda_0 t}$.

2.6. Evaluation of $p_{1N}(t)$

Now, only $p_{1N}(t)$ remains to be found. We observe that if $\mathbf{e} = (1, 1, 1, \ldots, 1)_{1\times N-1}$ then

$$\hat{q}(s) = \sum_{n=0}^{N-1} \hat{p}_{0n}(s) + \mathbf{e}\hat{\mathbf{R}}(s) + \hat{p}_{1N}(s). \tag{2.18}$$

Using (2.14) and (2.17) in the above equation, we obtain

$$s\hat{q}(s) = \sum_{n=0}^{N-1} s\mu\hat{a}_n(s)\hat{p}_{11}(s) + s\mathbf{e}(sI-B)^{-1}\mathbf{R}(0) + c\mu s\hat{p}_{1N}(s)\mathbf{e}(sI-B)^{-1}\mathbf{e}_{N-1} + s\hat{p}_{1N}(s). \tag{2.19}$$

From (2.16) and (2.19),

$$s\hat{q}(s) = \sum_{j=1}^{N-1} \delta_{ij}\left[1 + c^{-1}\hat{\beta}_j(s) + \mu\hat{\gamma}(s)\hat{b}_{1,j}(s)\right] + [s + c\mu + c\mu^2\hat{\gamma}(s)\hat{b}_{1N-1}(s) + \mu\hat{\beta}_{N-1}(s)]\hat{p}_{1N}(s) \tag{2.20}$$

with

$$\hat{\gamma}(s) = \sum_{n=0}^{N-1} s\hat{a}_n(s) = 1 - \left(\frac{\lambda_0}{s+\lambda_0}\right)^N,$$

$$c + \hat{\beta}_j(s) = \sum_{n=1}^{N-1} cs\, \hat{b}_{n,j}(s), \quad j = 1, 2, 3, \ldots, N-1.$$

Further,

$$\gamma(t) = \delta(t) - \frac{\lambda_0(\lambda_0 t)^{N-1} e^{-\lambda_0 t}}{(N-1)!}.$$

Also, the Laplace transform of (2.12) is

$$s\hat{q}(s) = \left(\frac{p - \sqrt{p^2-\alpha^2} - 2\lambda}{2}\right)\hat{p}_{1N}(s) + \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^{\tau(i)}, \tag{2.21}$$

where $p = s + \lambda + \mu$.

From (2.20) and (2.21),

$$\left[s + c\mu + c\mu^2\hat{\gamma}(s)\hat{b}_{1N-1}(s) + \mu\hat{\beta}_{N-1}(s) - \frac{p-\sqrt{p^2-\alpha^2}\ -2\lambda}{2}\right]\hat{p}_{1N}(s)$$
$$= \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^{\tau(i)} - \sum_{j=1}^{N-1}\delta_{ij}\left[1 + c^{-1}\hat{\beta}_j(s) + \mu\hat{\gamma}(s)\hat{b}_{1j}(s)\right].$$

Simplifying this equation,

$$\left[\frac{p+\sqrt{p^2-\alpha^2}}{2} + \mu\hat{f}_N(s)\right]p_{1N}(s) = \hat{G}(s),$$

where

$$\hat{f}_N(s) = \hat{\beta}_{N-1}(s) + c\mu\hat{\gamma}(s)\hat{b}_{1N-1}(s),$$
$$\hat{G}(s) = \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^{\tau(i)} - \sum_{j=1}^{N-1}\delta_{ij}\ \left[1 + c^{-1}\hat{\beta}_j(s) + \mu\hat{\gamma}(s)\hat{b}_{1j}(s)\right].$$

Thus,

$$\hat{p}_{1N}(s) = \frac{\hat{G}(s)}{\frac{p+\sqrt{p^2-\alpha^2}}{2} + \mu\hat{f}_N(s)}$$
$$= \frac{2}{\alpha}\hat{G}(s)\sum_{r=0}^{\infty}(-1)^r\left(\frac{\mu}{c\lambda}\right)^{r/2}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^{r+1}[\hat{f}_N(s)]^r. \tag{2.22}$$

Inverting (2.22),

$$p_{1N}(t) = G(t) * \sum_{r=0}^{\infty}(-1)^r\left(\frac{\mu}{c\lambda}\right)^{r/2} e^{-(\lambda+c\mu)t}[I_r(\alpha t) - I_{r+2}(\alpha t)] * [f_N(t)]^{*r}, \tag{2.23}$$

where

$$f_N(t) = \beta_{N-1}(t) + c\mu\ \gamma(t) * b_{1,N-1}(t),$$
$$G(t) = \frac{\alpha}{2\beta^{\tau(i)}} e^{-(\lambda+c\mu)t}[I_{\tau(i)-1}(\alpha t) - I_{\tau(i)+1}(\alpha t)] - \sum_{j=1}^{N-1}\delta_{ij}\ \left[\delta(t) + c^{-1}\beta_j(t) + \mu\gamma(t) * b_{1,j}(t)\right],$$

and $[f_N(t)]^{*r}$ is the r-fold convolution of $f_N(t)$ with itself. We note that $[f_N(t)]^{*0} = \delta(t)$.

Also, for $n = 0, 1, 2, \ldots, N-1$,

$$p_{0n}(t) = \mu \int_0^t a_n(y) p_{11}(t-y)\mathrm{d}y, \tag{2.24}$$

and for $n = 1, 2, 3, \ldots, N-1$,

$$p_{1n}(t) = \sum_{j=1}^{N-1} \delta_{ij} b_{n,j}(t) + c\mu \int_0^t b_{n,N-1}(y) p_{1N}(t-y)\mathrm{d}y. \tag{2.25}$$

Thus (2.11) and (2.23)–(2.25) completely determine all the state probabilities.

If $c = 1$, then the characteristic roots of $(sI - B)$ can be obtained explicitly. After considerable simplification, for any value of $N > 1$ and for $n = 1, 2, 3, \ldots,$

$$\begin{aligned} p_{1n}(t) = {} & \lambda_0 \beta^{n-N} \int_0^t \mathrm{e}^{-(\lambda+\mu)(t-y)} [I_{n-N}(\alpha(t-y)) - I_{n+N}(\alpha(t-y))]\, p_{0N-1}(y)\mathrm{d}y \\ & + \beta^{n-\tau(i)} \mathrm{e}^{-(\lambda+\mu)t} [I_{n-\tau(i)}(\alpha t) - I_{n+\tau(i)}(\alpha t)], \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} p_{0,N-1}(t) = {} & \left[\frac{\mu}{\beta^{\tau(i)-1}} \frac{\mathrm{e}^{-\lambda_0 t} (\lambda_0 t)^{N-1}}{(N-1)!} * \left(I_{\tau(i)-1}(\alpha t) - I_{\tau(i)+1}(\alpha t) \right) \mathrm{e}^{-(\lambda+\mu)t} \right] \\ & * \sum_{r=0}^{\infty} \left(\frac{1}{\beta^N} \right)^r \left[\frac{\alpha}{2} \frac{\lambda_0 \mathrm{e}^{-\lambda_0 t} (\lambda_0 t)^{N-1}}{(N-1)!} * (I_{N-1}(\alpha t) - I_{N+1}(\alpha t))\, \mathrm{e}^{-(\lambda+\mu)t} \right]^{*r}. \end{aligned} \tag{2.27}$$

Hence (2.24), (2.26) and (2.27) completely determine all the state probabilities.

Remark. We have obtained the transient solution for the system size probabilities with the assumption that the server is turned on initially and i customers in the system. If there are i customers in the system at $t = 0$ and the server is *turned off*, we obtain the system size probabilities, as follows:

$$p_{1N}(t) = G_1(t) * \sum_{r=0}^{\infty} (-1)^r \left(\frac{\mu}{c\lambda} \right)^{r/2} \mathrm{e}^{-(\lambda+c\mu)t} [I_r(\alpha t) - I_{r+2}(\alpha t)] * [f_N(t)]^{*r}, \tag{2.28}$$

where

$$G_1(t) = \delta(t) - \sum_{j=0}^{N-1} \delta_{ij} \alpha_j(t),$$

$$f_N(t) = \beta_{N-1}(t) + c\mu\, \alpha_0(t) * b_{1,N-1}(t),$$

and

$$a_{n,j}(t) = \frac{(\lambda_0 t)^{n-j}\, \mathrm{e}^{-\lambda_0 t}}{(n-j)!}, \qquad \alpha_j(t) = \delta(t) - \frac{\lambda_0 (\lambda_0 t)^{N-1-j} \mathrm{e}^{-\lambda_0 t}}{(N-1-j)!}, \quad j = 0, 1, 2, \ldots, N-1.$$

Also, for $n = 0, 1, 2, \ldots, N-1$,

$$p_{0n}(t) = \sum_{j=0}^{n} \delta_{ij} a_{n,j}(t) + \mu \int_0^t a_{n,0}(y) p_{11}(t-y)\mathrm{d}y, \tag{2.29}$$

and for $n = 1, 2, 3, \ldots, N-1$,

$$p_{1n}(t) = c\mu \int_0^t b_{n,N-1}(y) p_{1N}(t-y)\mathrm{d}y, \tag{2.30}$$

and $p_{1N+n}(t)$ is given by (2.11).

2.7. Mean

We know that

$$m(t) = E(X(t)) = \sum_{n=1}^{N-1} n[p_{0n}(t) + p_{1n}(t)] + \sum_{n=N}^{\infty} np_{1n}(t)$$

$$m'(t) = \sum_{n=1}^{N-1} n[p'_{0n}(t) + p'_{1n}(t)] + \sum_{n=N}^{\infty} np'_{1n}(t).$$

From Eqs. (2.1)–(2.5),

$$m'(t) = \lambda - \sum_{n=1}^{\infty} \mu_n p_{1n}(t) + (\lambda_0 - \lambda) \sum_{n=0}^{N-1} p_{0n}(t).$$

Using (2.6),

$$m'(t) = (\lambda - c\mu) + \mu \sum_{n=1}^{c-1}(c-n) p_{1n}(t) + (c\mu + \lambda_0 - \lambda) \sum_{n=0}^{N-1} p_{0n}(t).$$

Therefore,

$$m(t) = (\lambda - c\mu)t + \mu \sum_{n=1}^{c-1}(c-n) \int_0^t p_{1n}(y)\mathrm{d}y + (c\mu + \lambda_0 - \lambda) \sum_{n=0}^{N-1} \int_0^t p_{0n}(y)\mathrm{d}y + i, \tag{2.31}$$

where $p_{0n}(t)$ and $p_{1n}(t)$ are given in (2.24) and (2.25).

For $N = c = 1$ and $\lambda_0 = \lambda$, the mean $m(t)$ of $M/M/1$ queue is given by,

$$m(t) = (\lambda - \mu)t + \mu \int_0^t p_{00}(y)\mathrm{d}y + i.$$ This agrees with Cohen [12, p. 178].

2.8. Variance

$$\begin{aligned}\mathrm{Var}(X(t)) &= E(X^2(t)) - [E(X(t))]^2 \\ &= \sum_{n=1}^{N-1} n^2[p_{0n}(t) + p_{1n}(t)] + \sum_{n=N}^{\infty} n^2 p_{1n}(t) - [m(t)]^2,\end{aligned}$$

$$[\mathrm{Var}(X(t))]' = \sum_{n=1}^{N-1} n^2[p'_{0n}(t) + p'_{1n}(t)] + \sum_{n=N}^{\infty} n^2 p'_{1n}(t) - 2m(t)m'(t).$$

From the Eqs. (2.1)–(2.5), we obtain

$$\begin{aligned}[\mathrm{Var}(X(t))]' = {} & 2(\lambda_0 - \lambda) \sum_{n=1}^{N-1} np_{0n}(t) + (\lambda_0 - \lambda) \sum_{n=0}^{N-1} p_{0n}(t) + \lambda + 2\lambda m(t) + \sum_{n=1}^{\infty} n^2 \mu_{n+1} p_{1n+1}(t) \\ & - \sum_{n=1}^{\infty} n^2 \mu_n p_{1n}(t) - 2m(t)m'(t).\end{aligned}$$

Using (2.6),

$$\begin{aligned}[\mathrm{Var}(X(t))]' = {} & 2(\lambda_0 - \lambda) \sum_{n=1}^{N-1} np_{0n}(t) + (\lambda_0 - \lambda) \sum_{n=0}^{N-1} p_{0n}(t) + (\lambda + c\mu) + 2(\lambda - c\mu)m(t) \\ & + \mu \sum_{n=1}^{c-1}(2n-1)(c-n) p_{1n}(t) + c\mu \sum_{n=0}^{N-1}(2n-1) p_{0n}(t) - 2m(t)m'(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Var}(X(t)) &= 2(\lambda_0-\lambda)\sum_{n=1}^{N-1} n\int_0^t p_{0n}(y)\mathrm{d}y + (\lambda_0-\lambda)\sum_{n=0}^{N-1}\int_0^t p_{0n}(y)\mathrm{d}y + (\lambda+c\mu)t\\ &\quad + 2(\lambda-c\mu)\int_0^t m(y)\mathrm{d}y + \mu\sum_{n=1}^{c-1}(2n-1)(c-n)\int_0^t p_{1n}(y)\mathrm{d}y\\ &\quad + c\mu\sum_{n=0}^{N-1}(2n-1)\int_0^t p_{0n}(y)\mathrm{d}y - m^2(t) + i^2.\end{aligned}$$

For $N = c = 1$ and $\lambda_0 = \lambda$, the variance of the $M/M/1$ queue is given by

$$\mathrm{Var}(X(t)) = (\lambda+\mu)t + 2(\lambda-\mu)\int_0^t m(y)\mathrm{d}y - \mu\int_0^t p_{00}(y)\mathrm{d}y - m^2(t) + i^2.$$

3. Busy period

Busy period analysis plays a vital role in understanding various operations taking place in any queueing system. This analysis helps to improve the management of these systems to a great extent. In N-policy queueing system, the busy period corresponds to the usual queueing terminology. The busy period starts with the arrival of the N-th customer in the system; however, the idle period includes the additional time when customers are present and the server is *dormant* (time intervals when the server is turned off). A busy cycle is a consecutive busy and idle period. In the busy period the state $c-1$ is an absorbing state and therefore $p'_{1c-1}(t)$ is the busy period density function.

The forward equations for the system are

$$\begin{aligned} &p'_{1c-1}(t) = c\mu p_{1c}(t),\\ &p'_{1c}(t) = -(\lambda+c\mu)p_{1c}(t) + c\mu p_{1c+1}(t),\\ &p'_{1n}(t) = \lambda p_{1n-1}(t) - (\lambda+c\mu)p_{1n}(t) + c\mu p_{1n+1}(t), \quad n \ge c,\end{aligned} \tag{3.32}$$

with $p_{1N}(0) = 1$.

Define $P(z,t) = \sum_{n=c}^{\infty} z^{n-c} p_{1n}(t)$ and $P(z,0) = z^{N-c}$.

The system of equations (3.32) gives,

$$\frac{\partial P(z,t)}{\partial t} = \left[\lambda z + \frac{c\mu}{z} - (\lambda+c\mu)\right] P(z,t) - \frac{c\mu}{z}\, p_{1c}(t).$$

The solution of this differential equation is

$$P(z,t) = -\frac{c\mu}{z}\int_0^t \mathrm{e}^{-(\lambda+c\mu)(t-y)}\mathrm{e}^{(\lambda z+\frac{c\mu}{z})(t-y)} p_{1c}(y)\,\mathrm{d}y + z^{N-c}\mathrm{e}^{-(\lambda+c\mu)t}\mathrm{e}^{(\lambda z+\frac{c\mu}{z})t}.$$

Using (2.8) in the above equation and comparing the coefficients of z^{n-c} on either side, for $n = c, c+1, c+2, \ldots,$

$$p_{1n}(t) = -c\mu\int_0^t \mathrm{e}^{-(\lambda+c\mu)(t-y)}\beta^{n-c+1} I_{n-c+1}(\alpha(t-y))p_{1c}(y)\mathrm{d}y + \beta^{n-N}\mathrm{e}^{-(\lambda+c\mu)t} I_{n-N}(\alpha t).$$

Replace n by c and $c-2$, we get

$$p_{1c}(t) = -c\mu\int_0^t \mathrm{e}^{-(\lambda+c\mu)(t-y)}\beta I_1(\alpha(t-y))p_{1c}(y)\mathrm{d}y + \beta^{c-N}\mathrm{e}^{-(\lambda+c\mu)t} I_{N-c}(\alpha t), \tag{3.33}$$

and

$$0 = -c\mu\int_0^t \mathrm{e}^{-(\lambda+c\mu)(t-y)}\beta^{-1} I_1(\alpha(t-y))p_{1c}(y)\mathrm{d}y + \beta^{c-2-N}\mathrm{e}^{-(\lambda+c\mu)t} I_{N-c+2}(\alpha t). \tag{3.34}$$

Table 1
Probability values, mean and variance at different time points

t	$p_{00}(t)$	$p_{04}(t)$	$p_{1,4}(t)$	$p_{1,7}(t)$	$p_{1,18}(t)$	Mean	Variance
0	0.00000	0.00000	1.00000	0.00000	0.00000	4.0000	7.00(−14)
10	1.96543(−1)	3.20665(−2)	3.45660(−2)	3.91055(−3)	5.421(−9)	1.72175	1.89800
20	7.53193(−2)	1.34080(−1)	1.05341 (−2)	2.12276(−3)	2.81(−8)	3.39933	4.93120
50	8.14158(−2)	4.53385(−2)	4.27867(−2)	2.19605(−2)	1.36612(−6)	4.30291	9.15213
100	7.27222(−2)	6.50333(−2)	3.40096 (−2)	1.74416(−2)	1.03194(−6)	4.34504	8.45406
150	7.04266(−2)	6.88449(−2)	3.22764 (−2)	1.66314(−2)	9.68808(−7)	4.37086	8.32489
200	6.98738(−2)	6.95556(−2)	3.19387 (−2)	1.64892(−2)	9.57004(−7)	4.37937	8.30178
500	6.97101(−2)	6.97102(−2)	3.18597(−2)	1.64608(−2)	9.54401(−7)	4.38244	8.29719
1400	6.97078 (−2)	6.97078(−2)	3.18586(−2)	1.64603(−2)	9.54370(−7)	4.38228	8.29760
1450	6.97078(−2)	6.97078(−2)	3.18586(−2)	1.64603(−2)	9.54370(−7)	4.38228	8.29760

Where (k) denotes $10^{(k)}$.

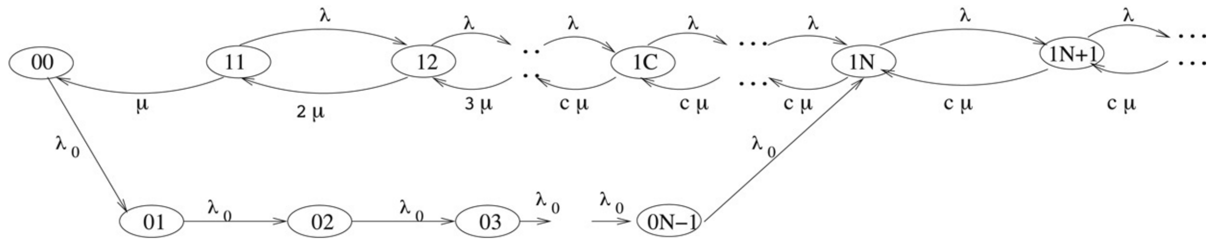

Fig. 1. Transition diagram.

From (3.33) and (3.34),

$$p_{1c}(t) = \frac{e^{-(\lambda+c\mu)t}}{\beta^{N-c}}[I_{N-c}(\alpha t) - I_{N-c+2}(\alpha t)].$$

Therefore, the busy period density function

$$b(t) = p'_{1c-1}(t)$$

$$b(t) = c\mu \frac{e^{-(\lambda+c\mu)t}}{\beta^{N-c}}[I_{N-c}(\alpha t) - I_{N-c+2}(\alpha t)].$$

For $N = c = 1$ then

$$b(t) = \mu e^{-(\lambda+\mu)t}[I_0(\alpha t) - I_2(\alpha t)].$$

This agrees with the busy period distribution of $M/M/1$ queue.

4. Numerical illustration

In Table 1, some of the system size probabilities are presented when $\lambda = 0.6, \mu = 0.4, \lambda_0 = 0.3, c = 5, N = 10, n = 20$ and initially the server is turned on with four customers in the system. We note from the last two columns that the steady-state is reached around 1450 time units. Also we observe that, the probabilities oscillate in the beginning because of N-policy. Obviously, the steady-state probabilities are same when the server is idle.

In Fig. 2, the system size probabilities $p_{0n}(t), n = 0, 1, \ldots, 9$ and $p_{1n}(t), n = 1, 2, \ldots, 7$ are drawn for the parameters $\lambda = 0.6, \mu = 0.4, \lambda_0 = 0.3, c = 5, N = 10, n = 20$. Note that, initially the server is turned on with four customers in the system. We observe that except for the probability curve corresponding to state four, all curves increase initially and decrease gradually up to some time interval. Also, these curves oscillate until they reach the stable value.

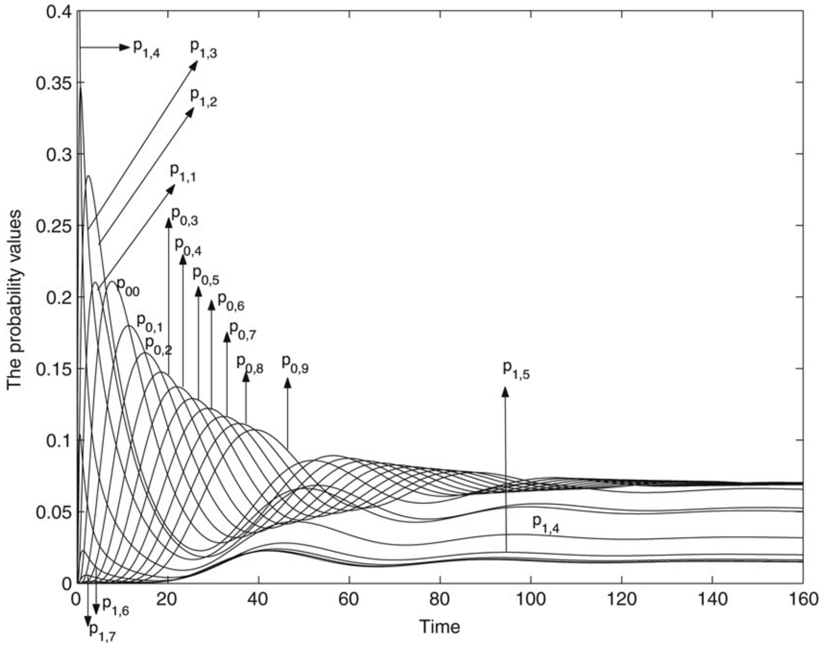

Fig. 2. The probability values for different time points.

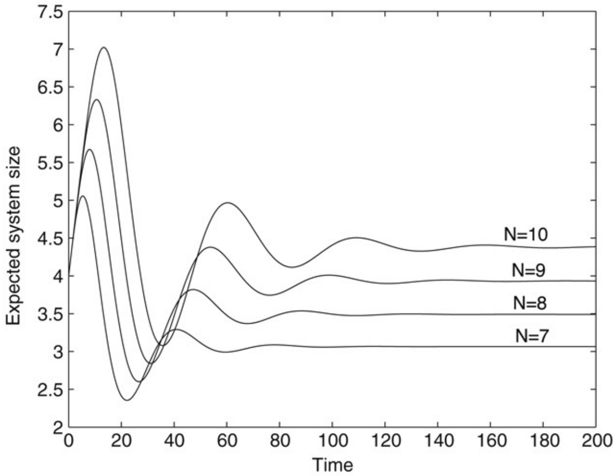

Fig. 3. The expected system size for different values of $N = 7, 8, 9, 10$.

In Fig. 3, the time-dependent expected system size for different values of N are plotted with the assumption $p_{04}(0) = 1$, $\lambda_0 = 0.3$, $\lambda = 0.6$, $\mu = 0.4$, $c = 5$ and $n = 20$. For different values of N, the means oscillate before they reach the stable value due to the N-policy of the system.

Fig. 4 is also a plot for expected system size with the same set of parameters as in Fig. 3 and the assumption that initially the server is on with four customers in the system.

Figs. 5 and 6 represent the time-dependent variances for different values of N with the assumption that $\lambda_0 = 0.3$, $\lambda = 0.6$, $\mu = 0.4$, $c = 5$, when the server is idle with four customers and the server is on with four customers in the system initially.

In Fig. 7, the density function of the busy period with parameter values $\lambda = 0.6$, $\mu = 0.4$, $N = 10$ is represented for different choices of the number of servers in the system.

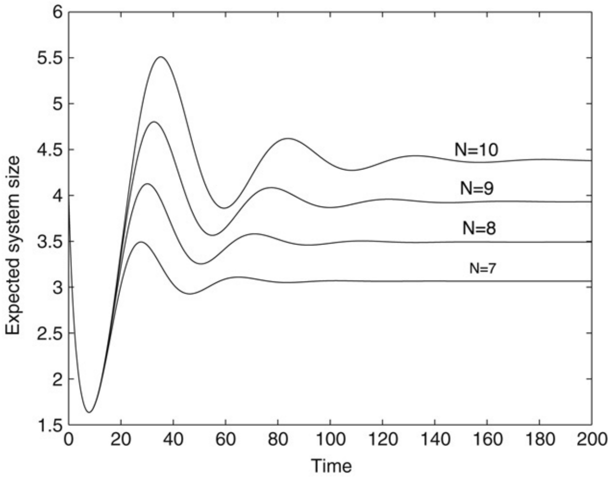

Fig. 4. The expected system size for different values of $N = 7, 8, 9, 10$.

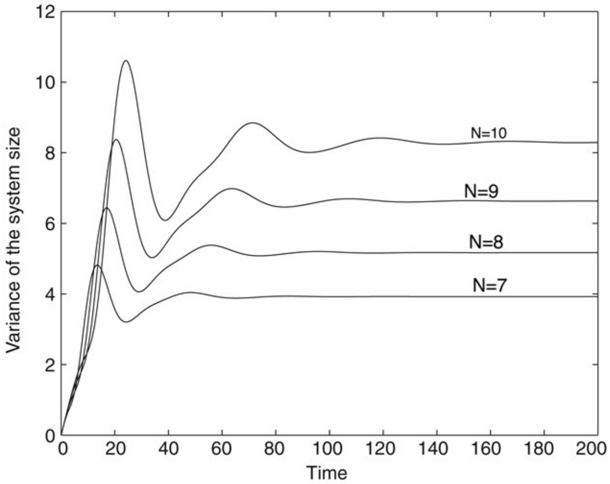

Fig. 5. The variance of the system size for different values of $N = 7, 8, 9, 10$.

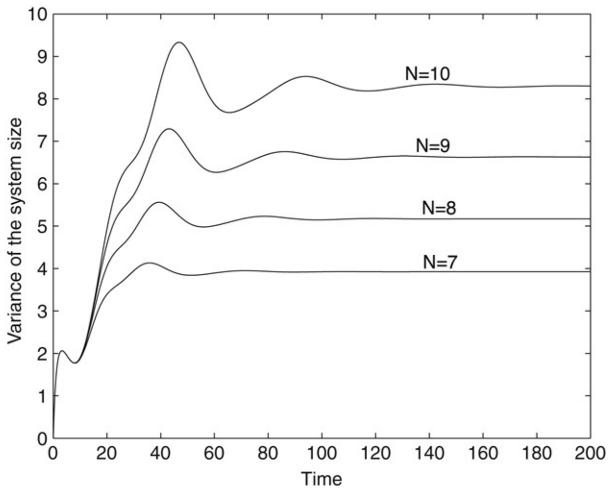

Fig. 6. The variance of the system size for different values of $N = 7, 8, 9, 10$.

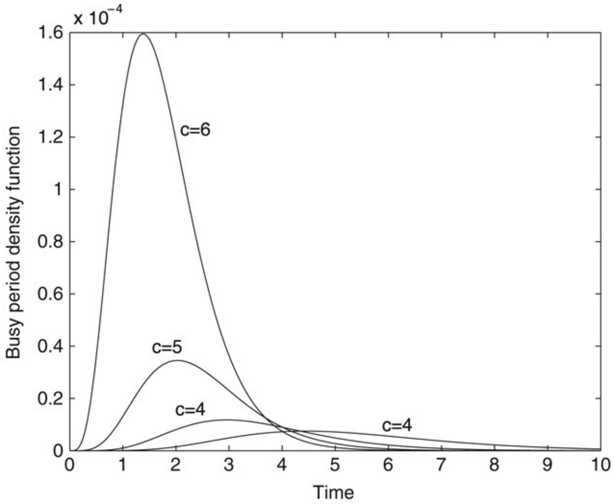

Fig. 7. The Busy period density function for different number of servers.

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