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# Wirelength of hypercubes into certain trees

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ABSTRACT

A lot of research has been devoted to finding efficient embedding of trees into hypercubes. On the other hand, in this paper, we consider the problem of embedding hypercubes into $k$-rooted complete binary trees, $k$-rooted sibling trees, binomial trees and certain classes of caterpillars to minimize the wirelength.



## 1. Introduction

An important feature of an interconnection network is its ability to efficiently simulate programs written for other architecture. Such a simulation problem can be mathematically formulated as graph embedding. An embedding of a guest graph $G$ into a host graph $H$ is defined by an injective mapping $f: V(G) \to V(H)$ together with a mapping $P_f$ which assigns to each edge $(u, v)$ of $G$ a path $P_f((u, v))$ between $f(u)$ and $f(v)$ in $H$ [9,29,33]. Some of the parameters used to analyze the efficiency of an embedding are dilation, expansion, edge congestion and wirelength.

If $e = (u, v) \in E(G)$, then the length of $P_f(e)$ in $H$ is called the dilation of the edge $e$. The maximal dilation over all edges of $G$ is called the dilation of the embedding $f$. The expansion of an embedding $f$ is the ratio of the number of vertices of $H$ to the number of vertices of $G$. In this paper, we consider embeddings with expansion one.

The *edge congestion*of an embedding $f$ of $G$ into $H$ is the maximum number of edges of the graph $G$ that are embedded on any single edge of $H$. Let $EC_f(e)$ denote the number of edges $(u, v)$ of $G$ such that $e$ is in the path $P_f((u, v))$ between $f(u)$ and $f(v)$ in $H$. In other words,

$$EC_f(e) = \left|\left\{(u, v) \in E(G): e \in P_f((u, v))\right\}\right|.$$

If we think of $G$ as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum, over all embeddings $f: V(G) \to V(H)$, of the maximum number of wires that cross any edge of $H$ [5].

The wirelength [21,25] of an embedding $f$ of $G$ into $H$ is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} \left|P_f((u, v))\right| = \sum_{e\in E(H)} EC_f(e).$$

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The *wirelength* of $G$ into $H$ is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings $f$ of $G$ into $H$. The *wirelength problem* of a graph $G$ into $H$ is to find an embedding of $G$ into $H$ that induces the minimum wirelength $WL(G, H)$. Since our goal to construct embeddings of minimum wirelength, we will take $P_f$ to be a mapping that assigns to each edge $(u, v)$ of $G$ a shortest path between vertices $f(u)$ and $f(v)$ in $H$.

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [24,33]. Embedding problems have been considered for binary trees into paths [24], binary trees into hypercubes [3,7,13,14,17,20,22,26], binomial trees into hypercubes [31,32], generalized ladders into hypercubes [10], binary trees into grids [27], hypercubes into cycles [15,19], generalized wheels into arbitrary trees [28], and hypercubes into grids [25].

Even though there are numerous results and discussions on the embedding problem, most of them deal with only approximate results and the estimation of lower bounds [4,15]. The embeddings discussed in this paper produce exact wirelength.

## 2. Isoperimetric problem

The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [6], which is $NP$-complete [18].

**Version 1.** Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given $m$, if $\theta_G(m) = \min_{A\subseteq V, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $\theta_G(m) = |\theta_G(A)|$.

It is interesting to note that $\theta_G(\lfloor |V|/2 \rfloor)$ solves bisection width of $G$ [18].

**Version 2.** Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given $m$, if $I_G(m) = \max_{A\subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $I_G(m) = |I_G(A)|$.

We call such a set $A$ optimal. Clearly, if a subset of vertices is optimal with respect to Version 1, then its complement is also an optimal set. However, it is not true for Version 2 in general, although this is indeed the case if the graph is regular [6]. In the literature, Version 2 is defined as the maximum subgraph problem.

The hypercube is one of the most popular versatile and efficient topological structures of interconnection networks. The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machine based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel and Connection Machines have been implemented commercially [30].

**Definition 1** (*[33]*)**.** For $n \geq 1$, let $Q_n$ denote the graph of $n$-dimensional hypercube. The vertex set of $Q_n$ is formed by the collection of all $n$-dimensional binary representations. Two vertices $x, y \in V(Q_n)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit.

Equivalently if $|V(Q_n)| = 2^n$ then the vertices of $Q_n$ can also be identified with integers $0, 1, \ldots, 2^n - 1$ so that if a pair of vertices $i$ and $j$ are adjacent then $i - j = \pm 2^p$ for some $p \geq 0$.

**Definition 2** (*[23]*)**.** An incomplete hypercube on $i$ vertices of $Q_n$ is the subcube induced by $\{0, 1, \ldots, i - 1\}$ and is denoted by $L_i$, $1 \leq i \leq 2^n$.

**Theorem 1** (*[8,11,21]*)**.** *Let $Q_n$ be an n-dimensional hypercube. For $1 \leq i \leq 2^n$, $L_i$ is an optimal set.* □

**Lemma 1** (*[2,25]*)**.** *Let $Q_n$ be an n-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_l}$ such that $n \geq t_1 > t_2 > \cdots > t_l \geq 0$. Then $|E(Q_n[L_m])| = [t_1 \cdot 2^{t_1-1} + t_2 \cdot 2^{t_2-1} + \cdots + t_l \cdot 2^{t_l-1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \cdots + (l - 1)2^{t_l}]$.* □

In this paper we solve the wirelength problem of hypercubes into $k$-rooted complete binary trees, $k$-rooted sibling trees, binomial trees and certain classes of caterpillars. We begin with the following notation.

**Notation.** For any set $S$ of edges of $H$, $EC_f(S) = \sum_{e\in S} EC_f(e)$.

**Lemma 2** (*Congestion Lemma [25]*)**.** *Let $G$ be an r-regular graph and $f$ be an embedding of $G$ into $H$. Let $S$ be an edge cut of $H$ such that the removal of edges of $S$ leaves $H$ into 2 components $H_1$ and $H_2$ and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also $S$ satisfies the following conditions:*

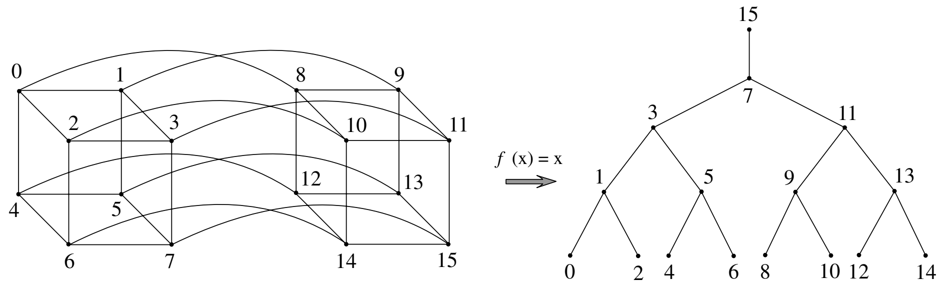


**Fig. 1.** Embedding of $Q_4$ with vertices labeled by lexicographic order into $T_4^1$ with vertices labeled by inorder traversal.

(i) *For every edge* $(a, b) \in G_i$, $i = 1, 2$, $P_f((a, b))$ has no edges in $S$.
(ii) *For every edge* $(a, b)$ in $G$ with $a \in G_1$ and $b \in G_2$, $P_f((a, b))$ has exactly one edge in $S$.
(iii) $G_1$ *is a maximum subgraph on* $k$ vertices where $k = |V(G_1)|$.
*Then* $EC_f(S)$ *is minimum and* $EC_f(S) = r\,|V(G_1)| - 2\,|E(G_1)|$. □

**Lemma 3** (*Partition Lemma [25]*). *Let* $f : G \to H$ *be an embedding. Let* $\{S_1, S_2, \ldots, S_p\}$ *be a partition of* $E(H)$ *such that each* $S_i$ *is an edge cut of* $H$. *Then*

$$WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i). \quad \square$$

**Lemma 4** (*k-Partition Lemma [1]*). *Let* $f : G \to H$ *be an embedding. Let* $E^k(H)$ *denote a collection of edges of* $H$ *with each edge in* $H$ *repeated exactly* $k$ *times. Let* $\{X_1, X_2, \ldots, X_m\}$ *be a partition of* $E^k(H)$ *such that each* $X_i$ *is an edge cut of* $H$. *Then*

$$WL_f(G, H) = \frac{1}{k}\sum_{i=1}^{m} EC_f(X_i). \quad \square$$

## 3. Wirelength of hypercubes into *k*-rooted complete binary trees

A tree is a connected graph that contains no cycles. Trees are the most fundamental graph-theoretic models used in many fields: information theory, automatics classification, data structure and analysis, artificial intelligence, design of algorithms, operation research, combinatorial optimization, theory of electrical networks, and design of network [33].

Let $T$ be a rooted tree. Suppose that vertex $u$ of $T$ adjacent to $v$, lies in the level below $v$, we say that $u$ is a child of $v$ and $v$ is the parent of $u$. Suppose that there is a path from $v$ to $w$ in $T$ such that $w$ lies below $v$, we say that $w$ is a descendant of $v$ and $v$ is an ancestor of $w$. A vertex with no children is called a leaf. All other vertices are called internal vertices. The most common type of tree is the binary tree. A binary tree is a rooted tree in which each vertex has at most two children and each child is designated as its left child or right child. A binary tree is said to be a complete binary tree if each internal vertex has exactly two children. Binary trees are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved. Also, many operations such as searching and storing can be easily performed on tree data structures. Furthermore, binary trees appear in communication pattern of divide-and-conquer type algorithms, functional and logic programming, and graph algorithms [33].

For any non-negative integer $n$, the complete binary tree of height $n$, denoted by $T_n$, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree $T_n$ has $n$ levels and level $i$, $1 \leq i \leq n$, contains $2^{i-1}$ vertices. Thus $T_n$ has exactly $2^n - 1$ vertices. The 1-rooted complete binary tree $T_n^1$ is obtained from a complete binary tree $T_n$ by attaching to its root a pendent edge. The new vertex is called the root of $T_n^1$ and is considered to be at level 0. The $k$-rooted complete binary tree $T_n^k$ is obtained by taking $k$ vertex disjoint 1-rooted complete binary trees $T_n^1$ on $2^n$ vertices with roots say $r_1, r_2, \ldots, r_k$ and adding the edges $(r_i, r_{i+1})$, $1 \leq i \leq k - 1$. The 2-rooted complete binary tree has been considered in [12].

**Embedding Algorithm A**
*Input*: The $n$-dimensional hypercube $Q_n$ and the 1-rooted complete binary tree $T_n^1$ on $2^n$ vertices.
*Algorithm*: Label the vertices of $Q_n$ and $T_n^1$ by lexicographic order [4] and inorder traversal [16,28] from 0 to $2^n - 1$ respectively. See Fig. 1.
*Output*: An embedding $f$ of $Q_n$ into $T_n^1$ given by $f(x) = x$ with minimum wirelength.

**Lemma 5.** *For* $j = 1, 2, \ldots, n$ *and* $i = 1, 2, \ldots, 2^{n-j}$, $Tcut_i^{2^j-1} = \{2^j(i-1), 2^j(i-1)+1, 2^j(i-1)+2, \ldots, 2^j(i-1)+(2^j-2)\}$ *is an optimal set in* $Q_n$.

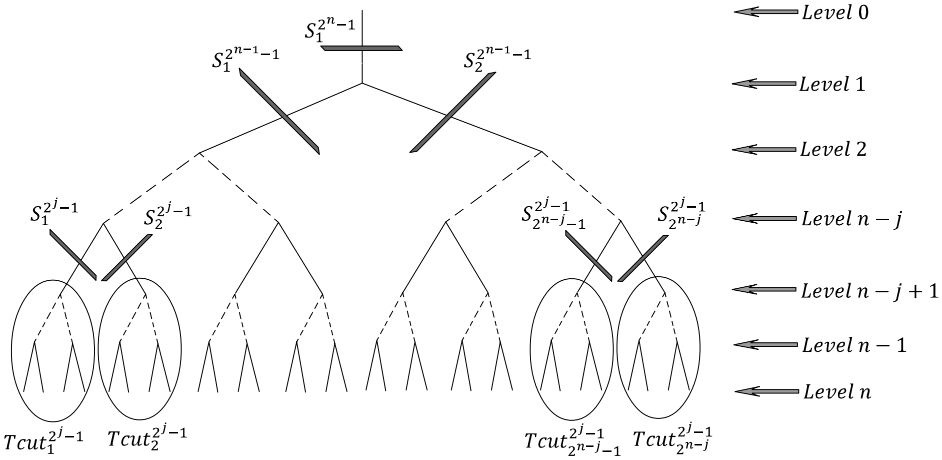


**Fig. 2.** Cut edges of 1-rooted binary tree.

**Proof.** Define $\varphi: Tcut_i^{2^j-1} \to L_{2^j-1}$ by $\varphi(2^j(i-1)+k) = k$. If the binary representation of $2^j(i-1)+k$ is $\alpha_1\alpha_2\cdots\alpha_n$ then the binary representation of $k$ is $\underbrace{00\cdots00}_{n-j \text{ times}}\alpha_{n-j+1}\alpha_{n-j+2}\cdots\alpha_n$. Thus the binary representation of two numbers $x$ and $y$ differ in exactly one bit $\Leftrightarrow$ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore $(x, y)$ is an edge in $Tcut_i^{2^j-1} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in $L_{2^j-1}$. Hence $Tcut_i^{2^j-1}$ and $L_{2^j-1}$ are isomorphic. By Theorem 1, $Tcut_i^{2^j-1}$ is an optimal set in $Q_n$. □

**Remark.** By Lemma 1, we have for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, $\left|E(Q_n[Tcut_i^{2^j-1}])\right| = j(2^{j-1}-1)$.

**Lemma 6.** *The Embedding Algorithm A of hypercube $Q_n$ into 1-rooted complete binary tree $T_n^1$ induces a minimum wirelength $WL(Q_n, T_n^1)$.*

**Proof.** For $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, let $S_i^{2^j-1}$ be the cut edge of the 1-rooted complete binary tree $T_n^1$, which has one vertex in level $n-j$ and the other vertex in level $n-j+1$, such that $S_i^{2^j-1}$ disconnects $T_n^1$ into two components $X_i^{2^j-1}$ and $\overline{X}_i^{2^j-1}$ where $V(X_i^{2^j-1})$ is $Tcut_i^{2^j-1}$. See Fig. 2. Let $G_i^{2^j-1}$ and $\overline{G}_i^{2^j-1}$ be the inverse images of $X_i^{2^j-1}$ and $\overline{X}_i^{2^j-1}$ under $f$ respectively. By Lemma 5, $G_i^{2^j-1}$ is an optimal set in $Q_n$. Thus the cut edge $S_i^{2^j-1}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i^{2^j-1})$ is minimum for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$. The Partition Lemma implies that $WL_f(Q_n, T_n^1)$ is minimum. □

**Theorem 2.** *The exact wirelength of $Q_n$ into $T_n^1$ is given by*

$$WL(Q_n, T_n^1) = 2^{n-1}(n^2 - 3n + 8) - n - 4.$$

**Proof.** By Congestion Lemma, for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, $EC_f(S_i^{2^j-1}) = n(2^j-1) - 2j(2^{j-1}-1)$. Therefore $WL(Q_n, T_n^1) = \sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^j-1}) = \sum_{j=1}^{n} 2^{n-j}[n(2^j-1) - 2j(2^{j-1}-1)] = 2^{n-1}(n^2-3n+8) - n - 4$. □

**Embedding Algorithm B**

*Input*: The $n$-dimensional hypercube $Q_n$ and the $k$-rooted complete binary tree $T_{n_1}^k$, $k = 2^{n-n_1}$.

*Algorithm*: Label the vertices of $Q_n$ by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of $T_{n_1}^k$, $k = 2^{n-n_1}$, as follows: Let $T_{n_1}^{1,1}, T_{n_1}^{1,2}, \ldots, T_{n_1}^{1,k}$ be the $k$ vertex disjoint 1-rooted complete binary trees of $T_n^k$. Label the vertices of $T_{n_1}^{1,i}$, $1 \le i \le k$, by inorder traversal [16,28] from $(i-1)2^{n_1}$ to $i2^{n_1} - 1$. See Fig. 3.

*Output*: An embedding $f$ of $Q_n$ into $T_{n_1}^k$ given by $f(x) = x$ with minimum wirelength.

**Lemma 7.** *For $i = 1, 2, \ldots, 2^{n-n_1}$, $Tcut_i^{2^{n_1}} = \{2^{n_1}(i-1), 2^{n_1}(i-1)+1, 2^{n_1}(i-1)+2, \ldots, 2^{n_1}(i-1)+(2^{n_1}-1)\}$ is an optimal set in $Q_n$.*

**Proof.** Define $\varphi: Tcut_i^{2^{n_1}} \to L_{2^{n_1}}$ by $\varphi(2^{n_1}(i-1)+k) = k$. If the binary representation of $2^{n_1}(i-1)+k$ is $\alpha_1\alpha_2\cdots\alpha_n$ then the binary representation of $k$ is $\underbrace{00\cdots00}_{n-n_1 \text{ times}}\alpha_{n-n_1+1}\alpha_{n-n_1+2}\cdots\alpha_n$. Thus the binary representation of two numbers $x$ and $y$

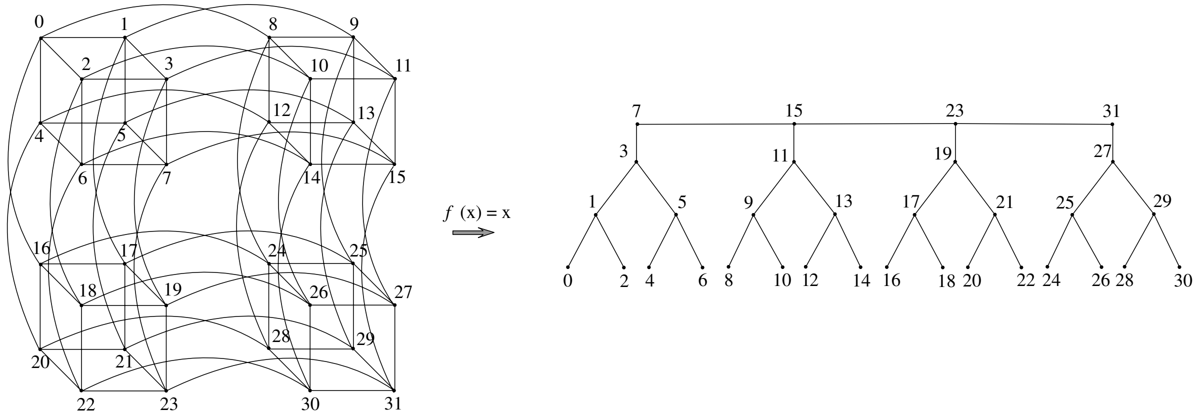


**Fig. 3.** Embedding of hypercube $Q_5$ into a 4-rooted complete binary tree $T_3^4$.

differ in exactly one bit ⇔ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore $(x, y)$ is an edge in $Tcut_i^{2^{n_1}}$ ⇔ $(\varphi(x), \varphi(y))$ is an edge in $L_{2^{n_1}}$. Hence $Tcut_i^{2^{n_1}}$ and $L_{2^{n_1}}$ are isomorphic. By Theorem 1, $Tcut_i^{2^{n_1}}$ is an optimal set in $Q_n$. □

**Lemma 8.** *The Embedding Algorithm B of hypercube $Q_n$ into k-rooted complete binary tree $T_{n_1}^k$, $k = 2^{n-n_1}$ induces a minimum wirelength $WL(Q_n, T_{n_1}^k)$.*

**Proof.** By Lemmas 6 and 7, it is enough to prove that the cut edge $(r_i, r_{i+1})$, $1 \le i \le k-1$, where $r_i$ is the root of $T_{n_1}^{1,i}$, $1 \le i \le k$, has minimum edge congestion. The cut edge $(r_i, r_{i+1})$, $1 \le i \le k-1$ of $T_{n_1}^k$, disconnects $T_{n_1}^k$ into two components $X_i$ and $\overline{X}_i$ where $V(X_i) = \{0, 1, \ldots, i2^{n_1} - 1\}$. Let $G_i$ and $\overline{G}_i$ be the inverse images of $X_i$ and $\overline{X}_i$ under $f$ respectively. By Theorem 1, $G_i$ is an optimal set in $Q_n$. Thus the cut edge $(r_i, r_{i+1})$, $1 \le i \le k-1$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f((r_i, r_{i+1}))$ is minimum for $i = 1, 2, \ldots, k-1$. The Partition Lemma implies that $WL_f(Q_n, T_{n_1}^k)$ is minimum. □

**Theorem 3.** *The exact wirelength of $Q_n$ into $T_{n_1}^k$, $k = 2^{n-n_1}$ is given by*

$$WL(Q_n, T_{n_1}^k) = 2^{n-1}[2nn_1 - 3n - n_1(n_1+1) + 8] + 2^{n-n_1}[n(2^{n-1}+1) - 2n_1 - 4] - 2\sum_{i=1}^{k-1} |E(Q_n[L_{i2^{n_1}}])| .$$

**Proof.** By Theorem 2 and Congestion Lemma, $WL(Q_n, T_{n_1}^k) = k\sum_{j=1}^{n_1}\sum_{i=1}^{2^{n_1-j}} EC_f(S_i^{2^j-1}) + \sum_{i=1}^{k-1} EC_f((r_i, r_{i+1})) = k\sum_{j=1}^{n_1} 2^{n_1-j}[n(2^j-1) - 2j(2^{j-1}-1)] + \sum_{i=1}^{k-1}[ni2^{n_1} - 2\,|E(Q_n[L_{i2^{n_1}}])|] = k\{2^{n_1-1}[2nn_1 - 2n - n_1(n_1+1) + 8] + n - 2n_1 - 4\} + n2^{n-n_1}2^{n-1} - n2^{n-1} - 2\sum_{i=1}^{k-1} |E(Q_n[L_{i2^{n_1}}])|$. □

## 4. Wirelength of hypercubes into *k*-rooted sibling trees

The 1-rooted sibling tree $ST_n^1$ is obtained from the 1-rooted complete binary tree $T_n^1$ by adding edges (sibling edges) between left and right children of the same parent node. See Fig. 4(a). The $k$-rooted sibling tree $ST_n^k$ is obtained by taking $k$ vertex disjoint 1-rooted sibling tree $ST_n^1$ on $2^n$ vertices with roots say $r_1, r_2, \ldots, r_k$ and adding the edges $(r_i, r_{i+1})$, $1 \le i \le k-1$.

Since $V(ST_n^1) = V(T_n^1)$, we show that the Embedding Algorithm A of hypercube $Q_n$ into 1-rooted sibling tree $ST_n^1$ induces a minimum wirelength.

**Lemma 9.** *For $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$,*

$$STcut_i^{2(2^j-1)} = \{2^j(2i-2), 2^j(2i-2)+1, 2^j(2i-2)+2, \ldots 2^j(2i-2)+2^j-2,$$
$$2^j(2i-1), 2^j(2i-1)+1, 2^j(2i-1)+2, \ldots 2^j(2i-1)+2^j-2\}$$

*is an optimal set in $Q_n$.*

**Proof.** By Lemma 5, the sets $\{2^j(2i-2), 2^j(2i-2)+1, 2^j(2i-2)+2, \ldots, 2^j(2i-2)+2^j-2\}$ and $\{2^j(2i-1), 2^j(2i-1)+1, 2^j(2i-1)+2, \ldots, 2^j(2i-1)+2^j-2\}$ are isomorphic to $L_{2^j-1}$. Also the binary representation of $2^j(2i-2)$ and $2^j(2i-1)$

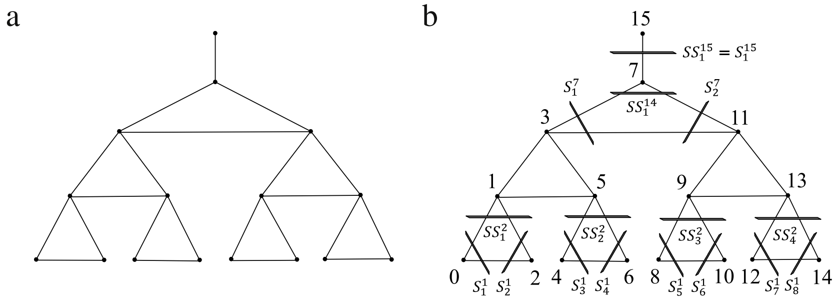


**Fig. 4.** (a) The 1-rooted sibling tree $ST_4^1$ (b) edge cut of $ST_4^1$.

differ exactly in one bit. Therefore $\left|E(Q_n[STcut_i^{2(2^j-1)}])\right| = 2\left|E(Q_n[L_{2^j-1}])\right| + 2^j - 1 = 2j(2^{j-1}-1)+2^j-1 = (j+1)2^j-2j-1$. But by Lemma 1, $\left|E(Q_n[L_{2(2^j-1)}])\right| = (j+1)2^j - 2j - 1$ and hence by Theorem 1, $STcut_i^{2(2^j-1)}$ is an optimal set in $Q_n$. □

**Lemma 10.** *The Embedding Algorithm A of hypercube $Q_n$ into 1-rooted sibling tree $ST_n^1$ induces a minimum wirelength $WL(Q_n, ST_n^1)$.*

**Proof.** For $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, let $S_i^{2^j-1}$ be an edge cut of the 1-rooted sibling tree $ST_n^1$ consisting of edges induced by the $\lceil i/2 \rceil$th parent vertex from left to right in level $n-j$ with its left child if $i$ is odd and its right child if $i$ is even together with the corresponding sibling edge which is the same edge in either case, such that $S_i^{2^j-1}$ disconnects $ST_n^1$ into two components $X_i^{2^j-1}$ and $\overline{X}_i^{2^j-1}$ where $V(X_i^{2^j-1})$ is $Tcut_i^{2^j-1}$. See Fig. 4(b). Let $G_i^{2^j-1}$ and $\overline{G}_i^{2^j-1}$ be the inverse images of $X_i^{2^j-1}$ and $\overline{X}_i^{2^j-1}$ under $f$ respectively. By Lemma 5, $G_i^{2^j-1}$ is an optimal set in $Q_n$. Thus the edge cut $S_i^{2^j-1}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i^{2^j-1})$ is minimum for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$.

For $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$, let $SS_i^{2(2^j-1)}$ be an edge cut of the 1-rooted sibling tree $ST_n^1$ consisting of the edges induced by the $i$th parent vertex from left to right in level $n-j$ and its two children, such that $SS_i^{2(2^j-1)}$ disconnects $ST_n^1$ into two components $X_i^{2(2^j-1)}$ and $\overline{X}_i^{2(2^j-1)}$ where $V(X_i^{2(2^j-1)})$ is $STcut_i^{2(2^j-1)}$. See Fig. 4(b). Let $G_i^{2(2^j-1)}$ and $\overline{G}_i^{2(2^j-1)}$ be the inverse images of $X_i^{2(2^j-1)}$ and $\overline{X}_i^{2(2^j-1)}$ under $f$ respectively. By Lemma 9, $G_i^{2(2^j-1)}$ is an optimal set in $Q_n$. Thus the edge cut $SS_i^{2(2^j-1)}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(SS_i^{2(2^j-1)})$ is minimum for $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$. Let $SS_1^{2^n-1} = S_1^{2^n-1}$ and it is easy to see that the conditions of the Congestion Lemma are satisfied. We note that the set $\{S_i^{2^j-1} : 1 \le j \le n, 1 \le i \le 2^{n-j}\} \cup \{SS_i^{2(2^j-1)} : 1 \le j \le n-1, 1 \le i \le 2^{n-j-1}\} \cup \{SS_1^{2^n-1}\}$ forms a partition of $E^2(ST_n^1)$. The 2-Partition Lemma implies that $WL_f(Q_n, ST_n^1)$ is minimum. □

**Theorem 4.** *The exact wirelength of $Q_n$ into $ST_n^1$ is given by*

$$WL(Q_n, ST_n^1) = 2^{n-1}(n^2 - 4n + 10) - n - 5.$$

**Proof.** By Lemma 2, for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, $\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^j-1}) = 2^{n-1}(n^2 - 3n + 8) - n - 4$. Again for $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$, $\sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} EC_f(SS_i^{2(2^j-1)}) = \sum_{j=1}^{n-1} 2^{n-j-1}[2n(2^j - 1) - 2(j+1)2^j + 4j + 2] = 2^{n-1}(n^2 - 5n + 12) - 2n - 6$. Also $EC_f(SS_1^{2^n-1}) = n$. Hence $WL(Q_n, ST_n^1) = \frac{1}{2}\{\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^j-1}) + \sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} EC_f(SS_i^{2(2^j-1)}) + EC_f(SS_1^{2^n-1})\} = \frac{1}{2}\{2^{n-1}(n^2 - 3n + 8) - n - 4 + 2^{n-1}(n^2 - 5n + 12) - 2n - 6 + n\} = 2^{n-1}(n^2 - 4n + 10) - n - 5$. □

As $V(ST_{n_1}^k) = V(T_{n_1}^k)$, $k = 2^{n-n_1}$, using the proof techniques of Lemmas 8 and 10, we have the following result.

**Theorem 5.** *The Embedding Algorithm B of hypercube $Q_n$ into k-rooted sibling tree $ST_{n_1}^k$, $k = 2^{n-n_1}$, induces a minimum wirelength $WL(Q_n, ST_{n_1}^k)$.* □

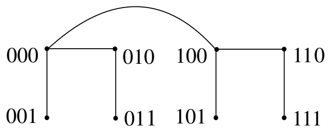


**Fig. 5.** Binary labeling of $B_3$.

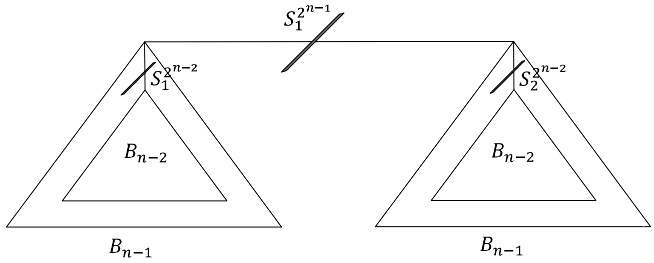


**Fig. 6.** Cut edges of binomial tree.

## 5. Wirelength of hypercubes into binomial trees

A binomial tree $B_0$ of height 0 is a single vertex. For all $n > 0$, a binomial tree $B_n$ of height $n$ is a tree formed by joining the roots of two binomial trees of height $n-1$ with a new edge and designating one of these roots to be the root of the new tree. A binomial tree of height $n$ has $2^n$ vertices [30,32].

*Binary Labeling of* $B_n$: Define $B_0 = K_1$ and $B_1 = Q_1$. For $n \geq 2$, $B_n$ is obtained recursively by taking two copies of the binomial tree $B_{n-1}$ with $V(B_{n-1}) = \{\alpha_2\alpha_3\cdots\alpha_n : \alpha_i \in \{0,1\}, 2 \leq i \leq n\}$, denoted by $B_{n-1}^0, B_{n-1}^1$ and adding an edge between these copies as follows: Let $V(B_{n-1}^0) = \{0\alpha_2\alpha_3\cdots\alpha_n : \alpha_i \in \{0,1\}, 2 \leq i \leq n\}$ and $V(B_{n-1}^1) = \{1\alpha_2\alpha_3\cdots\alpha_n : \alpha_i \in \{0,1\}, 2 \leq i \leq n\}$. A vertex $0\alpha_2\alpha_3\cdots\alpha_n \in V(B_{n-1}^0)$ is adjacent to $1\alpha_2\alpha_3\cdots\alpha_n \in V(B_{n-1}^1)$ if $\alpha_i = 0, 2 \leq i \leq n$. See Fig. 5.

**Embedding Algorithm C**

*Input*: The $n$-dimensional hypercube $Q_n$ and the binomial tree $B_n$.

*Algorithm*: Label the vertices of $Q_n$ by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of $B_n$ by binary labeling identified with integers $0, 1, \ldots, 2^n - 1$.

*Output*: An embedding $f$ of $Q_n$ into $B_n$ given by $f(x) = x$ with minimum wirelength.

**Lemma 11.** *For $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, $Bcut_i^{2^{j-1}} = \{2^{j-1}(2i-1), 2^{j-1}(2i-1)+1, 2^{j-1}(2i-1)+2, \ldots, 2^{j-1}(2i-1)+(2^{j-1}-1)\}$ is an optimal set in $Q_n$.*

**Proof.** Define $\varphi: Bcut_i^{2^{j-1}} \to L_{2^{j-1}}$ by $\varphi(2^{j-1}(2i-1)+k) = k$. If the binary representation of $2^{j-1}(2i-1)+k$ is $\alpha_1\alpha_2\cdots\alpha_n$ then the binary representation of $k$ is $\underbrace{00\cdots00}_{n-j+1 \text{ times}}\alpha_{n-j+2}\alpha_{n-j+3}\cdots\alpha_n$. Thus the binary representation of two numbers $x$ and $y$ differ in exactly one bit $\Leftrightarrow$ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore $(x, y)$ is an edge in $Bcut_i^{2^{j-1}} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in $L_{2^{j-1}}$. Hence $Bcut_i^{2^{j-1}}$ and $L_{2^{j-1}}$ are isomorphic. By Theorem 1, $Bcut_i^{2^{j-1}}$ is an optimal set in $Q_n$. □

**Lemma 12.** *The Embedding Algorithm C of hypercube $Q_n$ into binomial tree $B_n$ induces a minimum wirelength $WL(Q_n, B_n)$.*

**Proof.** For $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, let $S_i^{2^{j-1}} = \{(2^{j-1}(2i-1), 2^{j-1}(2i-2))\}$ be a cut edge of $B_n$ such that $S_i^{2^{j-1}}$ disconnects $B_n$ into two components $X_i^{2^{j-1}}$ and $\overline{X}_i^{2^{j-1}}$ where $V(X_i^{2^{j-1}})$ is $Bcut_i^{2^{j-1}}$. See Fig. 6. Let $G_i^{2^{j-1}}$ and $\overline{G}_i^{2^{j-1}}$ be the inverse images of $X_i^{2^{j-1}}$ and $\overline{X}_i^{2^{j-1}}$ under $f$ respectively. By Lemma 11, $G_i^{2^{j-1}}$ is an optimal set in $Q_n$. Thus the cut edge $S_i^{2^{j-1}}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i^{2^{j-1}})$ is minimum for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$. The Partition Lemma implies that $WL_f(Q_n, B_n)$ is minimum. □

**Theorem 6.** *The exact wirelength of $Q_n$ into $B_n$ is given by*

$$WL(Q_n, B_n) = n(n+1)2^{n-2}.$$

**Proof.** By Congestion Lemma, for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, $EC_f(S_i^{2^{j-1}}) = n2^{j-1} - 2(j-1)2^{j-2}$. Therefore $WL(Q_n, B_n) = \sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^{j-1}}) = \sum_{j=1}^{n} 2^{n-j}[n2^{j-1} - 2(j-1)2^{j-2}] = n(n+1)2^{n-2}$. □

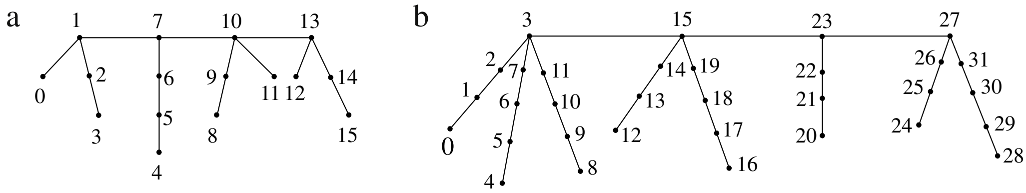


**Fig. 7.** (a) The caterpillar $C$ with each leg on 4 vertices (b) $CAT(3, 2, 1, 2; 4)$.

## 6. Wirelength of hypercubes into caterpillars

A caterpillar $C$ is a tree of maximal degree 4 where there exists a path $B$ (called the backbone of $C$) so that, after deleting all edges of $B$, $C$ consists of a set of paths. These paths are called the legs of $C$. Bezrukov et al. [9] considered caterpillar of maximal degree 3.

**Embedding Algorithm D**

*Input*: The $n$-dimensional hypercube $Q_n$ and a caterpillar $C$, each leg of which has $2^{n_1}$ vertices and the backbone has $2^{n_2}$ vertices such that $n_1 + n_2 = n$.
*Algorithm*: Label the vertices of $Q_n$ by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of $C$ as follows: Let $v_1, v_2, \ldots, v_{2^{n_2}}$ be the vertices in the backbone path of $C$. Label the vertices of each leg contains the vertex $v_i$, $1 \le i \le 2^{n_2}$, as $2^{n_1}(i-1), 2^{n_1}(i-1)+1, 2^{n_1}(i-1)+2, \ldots, i2^{n_1}-1$. See Fig. 7(a).
*Output*: An embedding $f$ of $Q_n$ into $C$ given by $f(x) = x$ with minimum wirelength.

**Lemma 13.** *For $j = 1, 2, \ldots, 2^{n_1}$ and $i = 1, 2, \ldots, 2^{n_2}$,*

(i) $Ccut_i^j = \{2^{n_1}(i-1), 2^{n_1}(i-1)+1, 2^{n_1}(i-1)+2, \ldots, 2^{n_1}(i-1)+(j-1)\}$,
(ii) $\overline{Ccut_i^j} = \{i2^{n_1}-1, i2^{n_1}-2, i2^{n_1}-3, \ldots, i2^{n_1}-j\}$

are optimal sets in $Q_n$, $n_1 + n_2 = n$.

**Proof.** (i) Define $\varphi: Ccut_i^j \to L_j$ by $\varphi(2^{n_1}(i-1)+k) = k$. If the binary representation of $2^{n_1}(i-1)+k$ is $\alpha_1\alpha_2\cdots\alpha_n$ then the binary representation of $k$ is $\underbrace{00\cdots00}_{n-n_1 \text{ times}}\alpha_{n-n_1+1}\alpha_{n-n_1+2}\cdots\alpha_n$.

(ii) Define $\varphi: \overline{Ccut_i^j} \to L_j$ by $\varphi(i2^{n_1}-k) = k-1$. If the binary representation of $i2^{n_1}-j$ is $\alpha_1\alpha_2\cdots\alpha_n$ then the binary representation of $k-1$ is $\underbrace{00\cdots00}_{n-n_1 \text{ times}}\overline{\alpha}_{n-n_1+1}\overline{\alpha}_{n-n_1+2}\cdots\overline{\alpha}_n$.

Thus the binary representation of two numbers $x$ and $y$ differ in exactly one bit $\Leftrightarrow$ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore $(x, y)$ is an edge in $Ccut_i^j$ (resp. $\overline{Ccut_i^j}$) $\Leftrightarrow$ $(\varphi(x), \varphi(y))$ is an edge in $L_j$. Hence $Ccut_i^j$ (resp. $\overline{Ccut_i^j}$) and $L_j$ are isomorphic. By Theorem 1, $Ccut_i^j$ (resp. $\overline{Ccut_i^j}$) is an optimal set in $Q_n$. □

**Lemma 14.** *The Embedding Algorithm D of hypercube $Q_n$ into caterpillar $C$ induces a minimum wirelength $WL(Q_n, C)$.*

**Proof.** Using the proof techniques employed in Lemma 8 and by Lemma 13, the result follows. □

Now, we consider another class of caterpillars. A caterpillar $CAT(k_1, k_2, \ldots, k_m; 2^{n_1})$, $k_i \ge 1, i = 1, 2, \ldots, m$, on $2^{n_1}(k_1 + k_2 + \cdots + k_m) = 2^n$ vertices is obtained from a path $v_1, v_2, \ldots, v_m$ (called the backbone) by joining $k_i$ number of vertex disjoint paths to vertex $v_i$, $1 \le i \le m$ such that it has exactly one path of length $2^{n_1} - 1$ and the remaining paths are of length $2^{n_1}$ (counted together with the vertex in the backbone).

**Embedding Algorithm E**

*Input*: The $n$-dimensional hypercube $Q_n$ and a caterpillar $CAT(k_1, k_2, \ldots, k_m; 2^{n_1})$, $k_i \ge 1, i = 1, 2, \ldots, m$, $2^{n_1}(k_1 + k_2 + \cdots + k_m) = 2^n$.
*Algorithm*: Label the vertices of $Q_n$ by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of $CAT(k_1, k_2, \ldots, k_m; 2^{n_1})$ as follows: Deletion of backbone edges of caterpillar leaves $m$ vertex disjoint subtrees. Label the vertices of subtree which contains the backbone vertex $v_i$, $1 \le i \le m$, treating it as the root of the tree by inorder traversal [16,28] from $(k_1 + k_2 + \cdots + k_{i-1})2^{n_1}$ to $(k_1 + k_2 + \cdots + k_i)2^{n_1} - 1$, where $k_0 = 0$. See Fig. 7(b).
*Output*: An embedding $f$ of $Q_n$ into $CAT(k_1, k_2, \ldots, k_m; 2^{n_1})$ given by $f(x) = x$ with minimum wirelength.

As a consequence of Lemma 14, we have the following result.

**Theorem 7.** *The Embedding Algorithm E of hypercube $Q_n$ into caterpillar $CAT(k_1, \ldots, k_m; 2^{n_1})$, induces a minimum wirelength $WL(Q_n, CAT(k_1, \ldots, k_m; 2^{n_1}))$, $k_i \ge 1, i = 1, 2, \ldots, m$, $2^{n_1}(k_1 + \cdots + k_m) = 2^n$.* □

## 7. Conclusion

In this paper we have solved the wirelength problem of hypercubes into $k$-rooted complete binary trees, $k$-rooted sibling trees, binomial trees and certain classes of caterpillars. It would be interesting to identify various classes of trees such that exact wirelength of hypercubes into such host trees can be found.

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