



A lower bound and several exact results on the d -lucky number



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ABSTRACT

If $\ell : V(G) \rightarrow \mathbb{N}$ is a vertex labeling of a graph $G = (V(G), E(G))$, then the d -lucky sum of a vertex $u \in V(G)$ is $d_\ell(u) = d_G(u) + \sum_{v \in N(u)} \ell(v)$. The labeling ℓ is a d -lucky labeling if $d_\ell(u) \neq d_\ell(v)$ for every $uv \in E(G)$. The d -lucky number $\eta_{dl}(G)$ of G is the least positive integer k such that G has a d -lucky labeling $V(G) \rightarrow [k]$. A general lower bound on the d -lucky number of a graph in terms of its clique number and related degree invariants is proved. The bound is sharp as demonstrated with an infinite family of corona graphs. The d -lucky number is also determined for the so-called $G_{m,n}$ -web graphs and graphs obtained by attaching the same number of pendant vertices to the vertices of a generalized cocktail-party graph.

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1. Introduction

In the celebrated paper [14], Karoński et al. asked whether the edges of any graph with no component K_2 can be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights, in other words, such that the resultant vertex weighting is a proper coloring. Although the paper does not use the word “conjecture” for the question, it later (quite naturally) became known as the 1–2–3 Conjecture. The progress on the conjecture until 2012 has been surveyed in [21], while for recent progress see [11,13] and references therein.

The paper [14] can also be seen as the seed for the investigation of other types of graph labelings in which integers are assigned to some elements of the graph (vertices, edges, or both of them), such that the labeling yields a proper vertex coloring. For us, the most important such labelings are due to Czerwiński et al., who in [7] introduced the concept of the lucky labeling and proposed the conjecture $\eta(G) \leq \chi(G)$, where $\eta(G)$ is the lucky number of G (and, of course, $\chi(G)$ is the chromatic number of G). For more information on the lucky labelings see [1,2]. Similar to the lucky number, Chartrand et al. [5] introduced sigma colorings, where the value at a vertex is obtained as the sum of the weights in its neighborhood. Club scheduling problems and hospital planning are real life applications of sigma colorings, cf. [15]. For additional related labelings we refer to [8]. We mention in passing that graph labelings have a variety of applications such as incorporating redundancy in disks, designing drilling machines, creating layouts for circuit boards, and configuring resistor networks, see [23].

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Finally, different graph labelings were and are still extensively investigated, we refer to the recent developments on antimagic labelings [4,16,17], $(d, 1)$ -labelings [10,19], and cordial labelings [20,22], to mention just some of them.

In this paper we are interested in d -lucky labelings that were introduced by Miller et al. [18] as a variant of the lucky labelings as follows. Let $N(u) = \{v \in V(G) : uv \in E(G)\}$ be the open neighborhood of a vertex u in a graph G . If $\ell : V(G) \rightarrow \mathbb{N}$ is a vertex labeling, then the d -lucky sum of a vertex $u \in V(G)$ with respect to ℓ is

$$d_\ell(u) = d_G(u) + \sum_{v \in N(u)} \ell(v),$$

where $d_G(u)$ is the degree of u . The labeling ℓ is a d -lucky labeling if $d_\ell(u) \neq d_\ell(v)$ holds for every pair of adjacent vertices u and v . The d -lucky number $\eta_{dl}(G)$ of G is the least positive integer k such that G admits a d -lucky labeling $\ell : V(G) \rightarrow [k] = \{1, \dots, k\}$. Lucky labelings are obtained from d -lucky labelings by omitting the additive term $d_G(u)$. A closely related concept of the adjacent vertex distinguishing colorings is defined analogously, except that one adds up the labels in the closed neighborhood of a vertex, see [3,9].

In the next section we prove a general lower bound on the d -lucky number of a graph in terms of its clique number and related degree invariants. The bound is sharp as demonstrated with an infinite family of corona graphs. The latter result is in turn used in Section 3 to determine the d -lucky number of the so called $G_{m,n}$ -web graphs. We conclude the paper with the d -lucky number of graphs obtained by attaching the same number of pendant vertices to the vertices of a generalized cocktail-party graph.

2. A lower bound on the d -lucky number

In this section we give a lower bound on $\eta_{dl}(G)$ of a graph G in terms of its clique number $\omega(G)$ (that is, the size of a largest complete subgraph) and demonstrate that the bound is sharp.

To state the main result we need the following notation. If Q is a clique of G , then let $\delta_G(Q)$ and $\Delta_G(Q)$ be the minimum and the maximum degree in G among the vertices from Q , respectively. Let further $\mathcal{Q}(G)$ be the set of largest cliques of G . Then we have:

Theorem 2.1. *If G is a connected graph, then*

$$\eta_{dl}(G) \geq \max_{Q \in \mathcal{Q}(G)} \left\lceil \frac{2\delta_G(Q) - \Delta_G(Q) + 1}{\Delta_G(Q) - \omega(G) + 2} \right\rceil.$$

Proof. Let $\omega(G) = s$ and let $Q \in \mathcal{Q}(G)$, so that $|n(Q)| = s$. Let $\eta_{dl}(G) = k$ and let $\ell : V(G) \rightarrow [k]$ be a d -lucky labeling of G . Set

$$x = \sum_{u \in V(Q)} \ell(u).$$

If $u \in V(Q)$ and $\ell(u) = i \in [k]$, then

$$(x - i) + (d_G(u) - (s - 1)) + d_G(u) \leq d_\ell(u) \leq (x - i) + k(d_G(u) - (s - 1)) + d_G(u).$$

Since $\ell(u) \in [k]$, the largest possible value of $d_\ell(u)$ is $(x - 1) + k(\Delta_G(Q) - s + 1) + \Delta_G(Q)$, and the smallest possible value of $d_\ell(u)$ is $(x - k) + (\delta_G(Q) - s + 1) + \delta_G(Q)$. Therefore, vertices from Q receive at most

$$N = [(x - 1) + k(\Delta_G(Q) - s + 1) + \Delta_G(Q)] - [(x - k) + (\delta_G(Q) - s + 1) + \delta_G(Q)] + 1$$

distinct d_ℓ -sum values. Since the vertices of Q receive pairwise different labels, $N \geq s$ must hold. From this inequality a straightforward computation yields

$$k(\Delta_G(Q) - s + 2) \geq 2\delta_G(Q) - \Delta_G(Q) + 1.$$

The assertion now follows because this inequality holds true for any clique from \mathcal{Q} and since the d -lucky number is an integer. \square

Theorem 2.1 significantly simplifies for certain classes of graphs, an instance is presented in the next result.

Corollary 2.2. *If G is a connected graph and every vertex v in any largest clique of G has $\deg_G(v) = r$, then*

$$\eta_{dl}(G) \geq \left\lceil \frac{r + 1}{r - \omega(G) + 2} \right\rceil.$$

To see that Corollary 2.2 (and hence also Theorem 2.1) is sharp, consider complete graphs K_n which are $(n - 1)$ -regular with $\omega(K_n) = n$. Thus Corollary 2.2 implies $\eta_{dl}(K_n) \geq n$. To find another (non-trivial) family of such graphs recall that the corona $G \circ H$ of graphs G and H is defined as follows. Let $V(G) = [n]$ and let H be a graph. Then $G \circ H$ is obtained from the disjoint union of G and n disjoint copies H_1, \dots, H_n of the graph H , where the vertex $i \in V(G)$ is connected with an edge to every vertex of H_i . Let further \bar{G} denote the complement of G . Then we have:

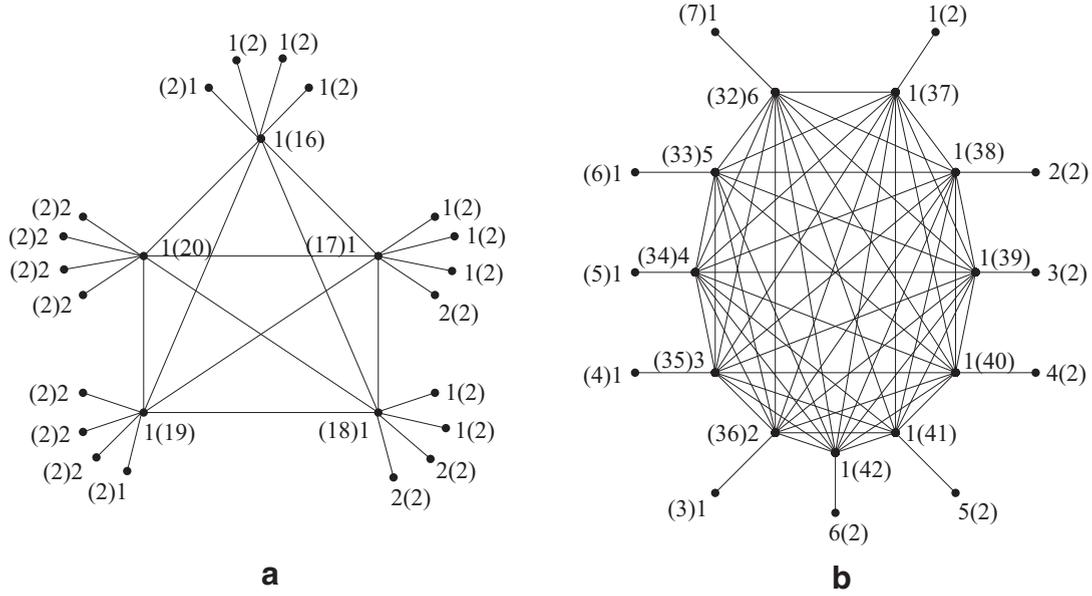


Fig. 1. (a) d -lucky labeling of $G_{5,4}$ (b) d -lucky labeling of $G_{11,3}$.

Theorem 2.3. If $n \geq 2$ and $r \geq 1$, then

$$\eta_{dl}(K_n \circ \overline{K_r}) = \left\lceil \frac{n+r}{r+1} \right\rceil.$$

Proof. To shorten the notation, set $G_{n,r} = K_n \circ \overline{K_r}$. Clearly, $\omega(G_{n,r}) = n$ and every vertex v in the largest clique K_n of $G_{n,r}$ has $d_{G_{n,r}}(v) = n - 1 + r$. Hence Corollary 2.2 gives $\eta_{dl}(K_n \circ \overline{K_r}) \geq \left\lceil \frac{n+r}{r+1} \right\rceil$.

To prove the reverse inequality, let $X = \{v_1, \dots, v_n\}$ be the vertex set of the subgraph K_n of $G_{n,r}$ and let $X_i, i \in [n]$, be the set of pendant vertices of $G_{n,r}$ adjacent to v_i .

Suppose first that $n \leq r + 1$. In this case label all the vertices of X with 1. Further, if $i \in [n]$, then label $(i - 1)$ vertices from X_i with 2 and the other $r - i + 1$ vertices of X_i with label 1, See Fig. 1(a). As $n \leq r + 1$, the sums of the labels in X_i and X_j are different for all $i \neq j$. Hence, this labeling is a d -lucky labeling with two labels and so $\eta_{dl}(K_n \circ \overline{K_r}) \leq 2$. We observe that as $n \geq 2$,

$$\frac{n+r}{r+1} \geq \frac{2+r}{r+1} > 1$$

and as $n \leq r + 1$,

$$\frac{n+r}{r+1} \leq \frac{2r+1}{r+1} = 1 + \frac{r}{r+1} < 2.$$

Therefore $\left\lceil \frac{n+r}{r+1} \right\rceil = 2$. Thus $\eta_{dl}(K_n \circ \overline{K_r}) \leq \left\lceil \frac{n+r}{r+1} \right\rceil$.

Assume now that $n > r + 1$. In this case label the vertices $v_1, v_2, \dots, v_{kr-r+1}$ of X with 1 and the remaining $n - (kr - r + 1)$ vertices in X with labels $2, 3, \dots, n - (kr - r + 1) + 1$, where $k = \left\lceil \frac{n+r}{r+1} \right\rceil$. Label vertices in $X_i, 1 \leq i \leq kr - r + 1$ with labels such that the sums x_i 's, $1 \leq i \leq kr - r + 1$ are all distinct lying between r and kr . Further, label all other pendant vertices as 1. See Fig. 1(b). The contribution to the d_ℓ -sums of the $kr - r + 1$ vertices of X which are labeled 1 are, respectively,

$$x - 1 + r, x - 1 + (r + 1), \dots, x - 1 + kr$$

and the contribution to the d_ℓ -sums of the remaining $n - (kr - r + 1)$ vertices of X are, respectively,

$$x - 2 + r, x - 3 + r, \dots, x - (n - kr + r) + r.$$

Since the degree of every vertex in X is $t = n - 1 + r$, the d_ℓ -sums of the vertices of X are $t + x + r - (n - kr + r), t + x + r - (n - kr + r) + 1, \dots, t + x + r - 3, t + x + r - 2, t + x + r - 1, t + x + r, t + x + r + 1, \dots, t + x + r + (kr - r - 1)$, which are consecutive integers between $t + x + kr - n$ and $t + x + kr - 1$. Thus all vertices of X receive distinct consecutive d_ℓ -sums. Since the pendant vertices form an independent set, the labels of these vertices do not contribute to the d -lucky number. As $n > r + 1$, the sums of the labels in X_i and X_j are different for all $i \neq j$. Therefore,

$$\begin{aligned} \eta_{dl}(K_n \circ \overline{K_r}) &\leq n - (kr - r + 1) + 1 = (n + r) - \left\lceil \frac{n+r}{r+1} \right\rceil r \\ &\leq (n + r) - \left(\frac{n+r}{r+1} \right) r \leq \frac{n+r}{r+1} \leq \left\lceil \frac{n+r}{r+1} \right\rceil. \end{aligned}$$

We conclude that $\eta_{dl}(K_n \circ \overline{K_r}) = \left\lceil \frac{n+r}{r+1} \right\rceil$. \square

Theorem 2.3 thus gives an infinite, non-trivial family of graphs for which the equality is achieved in Theorem 2.1.

3. More exact d -lucky numbers

In this section we determine the d -lucky number of two infinite families of graphs. With the aid of Theorem 2.3 we obtain the d -lucky number of the $G_{m,n}$ -web graphs defined below. At the end of the section we then present the d -lucky number of graphs obtained by attaching the same number of pendant vertices to the vertices of a generalized cocktail-party graph.

For $m, n \geq 3$, set $C_{m,n} = P_m \square C_n$, where \square denotes the standard Cartesian product of graphs [12]. The graphs $C_{m,n}$ are sometimes called *cylinders*. Let us call the edges of $C_{m,n}$ that project on P_m *radial edges* of $C_{m,n}$, and the other edges (that is, those that project on C_n) *cycle edges*. Further, the two C_n -layers whose vertices are of degree 3 will be called the *top layer* and the *bottom layer*, respectively, while the other C_n -layers will be referred to as *layer 1, ..., layer (m - 2)*, see Fig. 2(a). Now, the $G_{m,n}$ -web graph is obtained from the disjoint union of $C_{m,n}$ and K_n by adding a matching between the top layer of $C_{m,n}$ and the vertices of K_n , subdividing each of these matching edges, and subdividing all the edges of the C_m -layers of $C_{m,n}$. See Fig. 2(b) and (c) for $G_{3,6}$ and $G_{4,6}$, respectively.

The edges of $G_{m,n}$ obtained by subdividing the matching edges between $C_{m,n}$ and K_n will be called *pendant-subdivided edges*, and the edges obtained by subdividing the C_m -layers *cycle-subdivided edges*, see Fig. 2(c) again. The result of this section now reads as follows.

Theorem 3.1. *If $m \geq 3$ and $n \geq 5$, then $\eta_{dl}(G_{m,n}) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Proof. To simplify the notation, set $G = G_{m,n}$ for the rest of the proof. Note first that K_n is the unique largest clique of G , its vertices being of degree n in G , hence by Corollary 2.2, $\eta_{dl}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil$. It thus remains to prove that G admits a d -lucky labeling using $\left\lceil \frac{n+1}{2} \right\rceil$ labels.

Let $V(K_n) = \{v_1, \dots, v_n\}$, let u_1, \dots, u_n be the respective adjacent vertices of G obtained by subdividing the matching edges between K_n and $C_{m,n}$, and let w_1, \dots, w_n be the corresponding vertices of the top layer of C_n . Let H be the subgraph of G induced by the vertices $u_1, \dots, u_n, v_1, \dots, v_n$.

Set $k = \left\lceil \frac{n+1}{2} \right\rceil$ and construct a labeling $\ell: V(G) \rightarrow [k]$ as follows. Let ℓ restricted to H be the labeling of $K_n \circ K_1 = H$ with $\eta_{dl}(H) = k$ labels from the proof of Theorem 2.3. This labeling yields consecutive numbers $\frac{n^2+10n}{8} + j, j \in [n]$, as d_ℓ -sums of vertices of K_n , when n is even, and consecutive numbers $\frac{n^2+12n-5}{8} + j, j \in [n]$, as d_ℓ -sums of vertices of K_n when n is odd. Next, for every $i \in [n]$ set $\ell(w_i) = 1$ when m is odd, and $\ell(w_i) = 2$ when m is even. Next, label the remaining unlabeled vertices with labels 1 and 2 such that each vertex labeled 1 has all neighbors labeled 2 and vice versa, every edge adjacent to vertex labeled 1 as 2 and vice versa. (Note that this is possible as $G - K_n$ is bipartite.) Finally, if m is even and $n \geq 15$, redefine $\ell(w_{k+7}) = 3$, and if m is odd and $n \geq 8$, redefine $\ell(w_5) = 3$ and $\ell(w_{k+3}) = 3$.

For every edge $e = uv$ of G , we have to prove that $d_\ell(u) \neq d_\ell(v)$ and consider typical edges.

Case 1: $e = u_i v_i, i \in [n]$.

Since the degree of each vertex of K_n in G is n and it is possible to label each vertex of K_n as 1, the d_ℓ -sum of a vertex $v_i \in V(K_n), i \in [n]$, is at least $(n - 1) + 1 + n = 2n$ if m is even and at least $(n - 1) + 2 + n = 2n + 1$ if m is odd. In other words, for all n , the minimum d_ℓ -sum of a vertex in K_n is $2n$. On the other hand, for all n , the maximum d_ℓ -sum of a vertex $u_i, i \in [n]$, is $k + 5 = \left\lceil \frac{n+1}{2} \right\rceil + 5$. By a straightforward induction we can see that $\left\lceil \frac{n+1}{2} \right\rceil + 5 < 2n$ holds for $n \geq 5$. Hence the d_ℓ -sums of the adjacent vertices v_i and u_i are distinct.

Case 2: $e = u_i w_i, i \in [n]$.

By our labeling, k vertices, say v_1, \dots, v_k are labeled 1 and the corresponding adjacent vertices u_1, \dots, u_k are labeled $1, \dots, k$. Further, $\ell(v_{k+i}) = 2, 3, \dots, n - k + 1, i \in [n - k]$ and $\ell(u_{k+i}) = 1, i \in [n - k]$.

Suppose first that m is even. For $5 \leq n \leq 14, \ell(w_i) = 1, i \in [n]$, and for $n \geq 15$,

$$\ell(w_i) = \begin{cases} 1; & i \in [n] \text{ and } i \neq k + 7, \\ 3, & i = k + 7. \end{cases}$$

Hence when $5 \leq n \leq 14$ we have

$$d_\ell(u_i) = \begin{cases} 4; & i \in [k], \\ i - k + 4; & k + 1 \leq i \leq n, \end{cases}$$

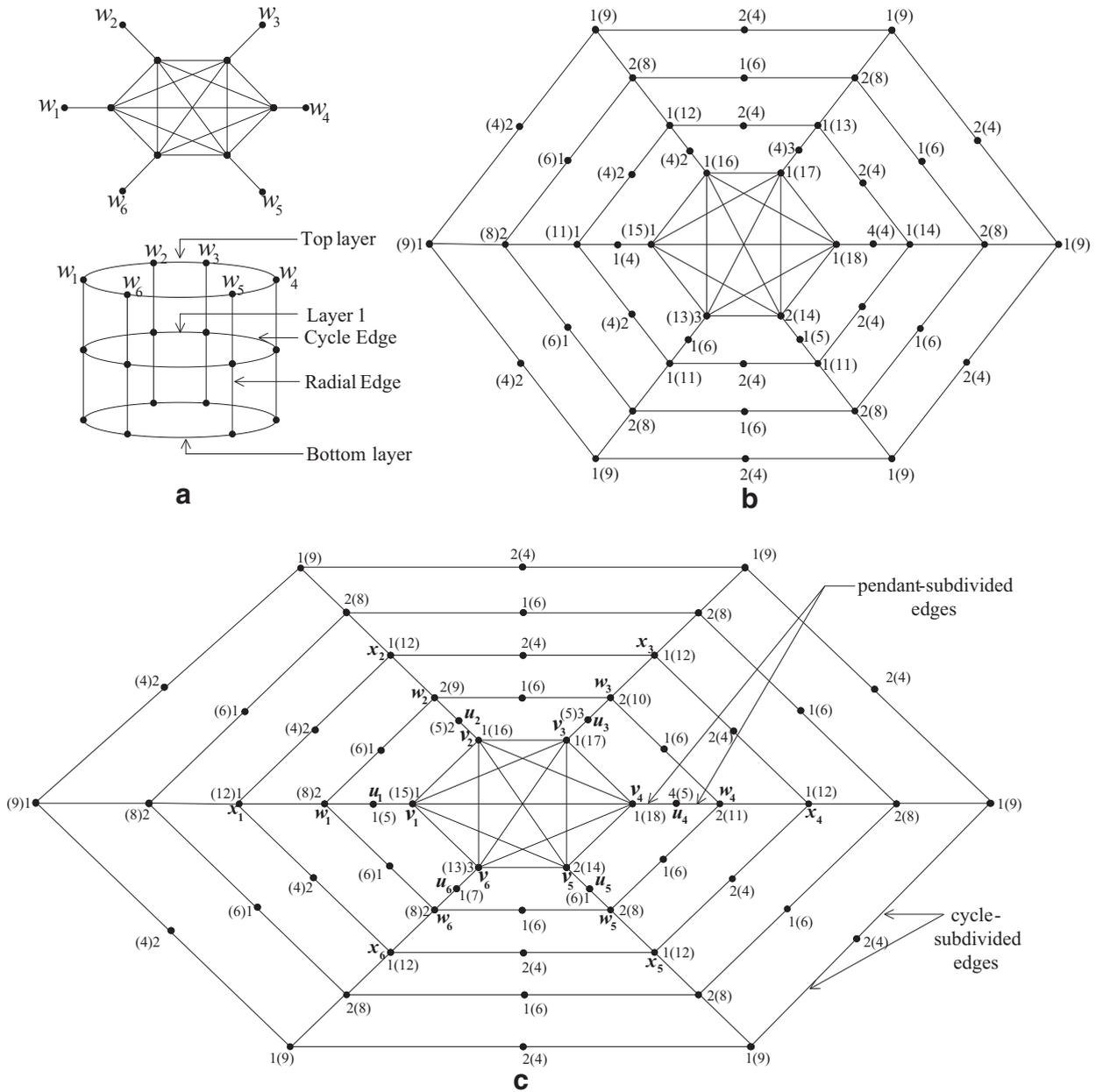


Fig. 2. (a) $P_3 \square C_6$ (b) d -lucky labeling of the $G_{5,6}$ -web graph (c) d -lucky labeling of the $G_{4,6}$ -web graph.

and

$$d_\ell(w_i) = \begin{cases} i + 10; & i \in [k], \\ 11; & k + 1 \leq i \leq n. \end{cases}$$

If $n \geq 15$, then we have

$$d_\ell(u_i) = \begin{cases} 4; & i \in [k], \\ i - k + 4; & k + 1 \leq i \leq n \text{ and } i \neq k + 7, \\ 13; & i = k + 7, \end{cases}$$

and

$$d_\ell(w_i) = \begin{cases} i + 10; & i \in [k], \\ 11, & k + 1 \leq i \leq n. \end{cases}$$

Suppose now that $d_\ell(u_i) = d_\ell(w_i)$. Since $n \geq 5$ this implies that $i - k + 4 = 11$, that is, $i = k + 7$, where $k + 1 \leq i \leq n$. Now $k + 7 \leq n$, that is, $\lceil \frac{n+1}{2} \rceil + 7 \leq n$, yields $n \geq 15$. But then $d_\ell(u_i) = 13 \neq 11 = d_\ell(w_i)$ for $i = k + 7$. This completes the argument when m is even.

Suppose next that m is odd. For $5 \leq n \leq 14$, $\ell(w_i) = 2$, $i \in [n]$, and for $n \geq 15$,

$$\ell(w_i) = \begin{cases} 2; & i \in [n] \text{ and } i \neq 5, k + 3, \\ 3; & i = 5 \text{ or } i = k + 3. \end{cases}$$

Therefore, if $5 \leq n \leq 7$ we have

$$d_\ell(u_i) = \begin{cases} 5; & i \in [k], \\ i - k + 5; & k + 1 \leq i \leq n, \end{cases}$$

and

$$d_\ell(w_i) = \begin{cases} i + 7; & i \in [k], \\ 8, & k + 1 \leq i \leq n. \end{cases}$$

And if $n \geq 8$, then we have

$$d_\ell(u_i) = \begin{cases} 5; & i \in [4] \text{ or } 6 \leq i \leq k, \\ 6; & i = 5, \\ i - k + 5; & k + 1 \leq i \leq n \text{ and } i \neq k + 3, \\ 9; & i = k + 3, \end{cases}$$

and

$$d_\ell(w_i) = \begin{cases} i + 7; & i \in [k], \\ 8; & k + 1 \leq i \leq n. \end{cases}$$

As in the case when m was even, we can prove that for if $n \geq 5$, then $d_\ell(u_i) \neq d_\ell(w_i)$ for $i \in [n]$.

Case 3: e is a radial edge.

We begin with the radial edges $w_i x_i$, $i \in [n]$, where x_i are vertices of layer 1 of the subgraph $C_{m,n}$ of G .

Suppose first that m is even. Then $d_\ell(x_i) = 8$ and $d_\ell(w_i) \geq 11$ for $i \in [n]$. It follows that the end vertices of the radial edges $w_i x_i$ receive distinct d_ℓ -sums.

Assume next that m is odd. Then

$$d_\ell(x_i) = \begin{cases} 12; & i \in [4] \text{ or } 6 \leq i \leq n \text{ or } i \leq k + 3, \\ 13, & i = 5 \text{ or } i = k + 3. \end{cases}$$

If $8 \leq d_\ell(w_i) \leq k + 7$, then

$$d_\ell(w_i) = \begin{cases} 7 + i; & i \in [k], \\ 8, & k + 1 \leq i \leq n. \end{cases}$$

If $i = 5$, then $d_\ell(w_i) = 12 \neq 13 = d_\ell(x_i)$, and if $i \neq 5$, then $d_\ell(w_i) \neq d_\ell(x_i)$. Thus the end vertices of the edges $e = w_i x_i$, $i \in [n]$ receive distinct d_ℓ -sums.

The end vertices of radial edges which are in layers $1, \dots, m - 2$ receive d_ℓ -sums 8 and 12, respectively. The radial edges with one end-vertex in layer $m - 2$ and the other end-vertex in the bottom layer, receive d_ℓ -sums 8 and 9, respectively.

Case 4: e is a cycle-subdivided edge.

If v is a vertex subdividing a cycle edge, then, for all m, n we have $\max d_\ell(v) = 6$, whereas $\min_{i \in [n]} d_\ell(w_i) = 8$. Hence the d_ℓ -sums of end vertices of cycle-subdivided edges are also distinct. \square

A generalized cocktail-party graph $H_{n,t}$, $n, t \geq 1$, is the complete t -partite graph with each partite set of order n , cf. [6]. If $n, t, r \geq 1$, set $H_{n,t,r} = H_{n,t} \circ \overline{K}_r$. That is, $H_{n,t,r}$ is obtained from $H_{n,t}$ by attaching r pendant vertices at each of its vertices.

Theorem 3.2. *If $n, r \geq 1$ and $t \geq 2$, then $\eta_{dl}(H_{n,t,r}) = \lceil \frac{t+n+r-1}{n+r} \rceil$.*

Proof. To simplify the notation, set $G = H_{n,t,r}$ for the rest of the proof.

Let $\ell: V(G) \rightarrow [k]$ be an optimal d -lucky labeling of G and set

$$x = \sum_{u \in V(H_{n,t})} \ell(u).$$

The minimum d_ℓ -sum and the maximum d_ℓ -sum of a vertex of G from $H_{n,t}$ are $r + r + (nt - n) + (x - nk)$ and $rk + r + (nt - n) + (x - n)$ respectively. Therefore the number of distinct d_ℓ -sums of the vertices from $H_{n,t}$ is at most

$$[(rk + r) + (nt - n) + (x - n)] - [r + r + (nt - n) + (x - nk)] + 1,$$

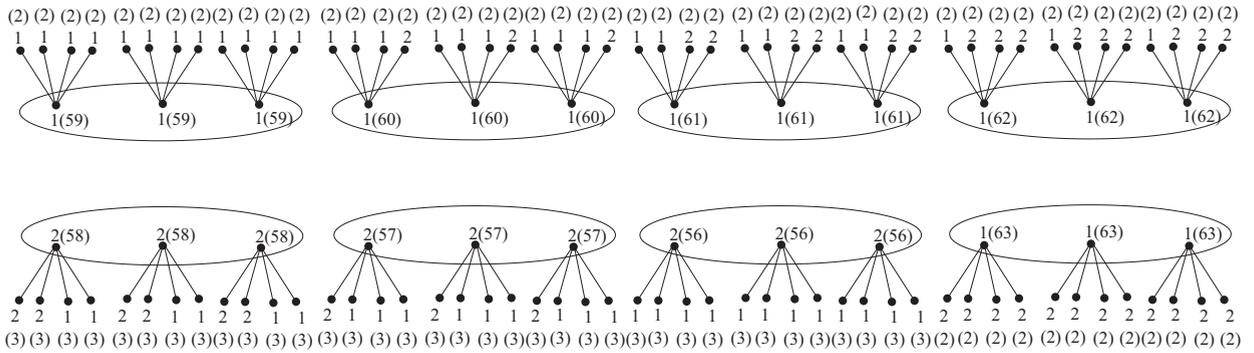


Fig. 3. d -lucky labeling of $H_{3,8,4}$ -generalized cocktail-party graph.

which simplifies to $k(n+r) - (n+r) + 1$. Since the d_ℓ -sums of vertices from $H_{n,t}$ that belong to different partite sets are different, this implies that $t \leq k(n+r) - (n+r) + 1$. Thus $\eta_{dl}(G) = k \geq \lceil \frac{t+n+r-1}{n+r} \rceil$.

For the other inequality, set $k = \lceil \frac{t+n+r-1}{n+r} \rceil$, let $p = \lfloor \frac{t-(kr-r+1)}{n} \rfloor$, and let $q = (t - (kr - r + 1)) \bmod n$. Let V_1, \dots, V_t be the t -partite sets of $H_{n,t}$. Let $\mathcal{W} = \{V_1, \dots, V_{kr-r+1}\}$, let $\mathcal{U}_j = \{V_{kr-r+1+(j-1)n+1}, \dots, V_{kr-r+1+(j-1)n+n}\}$, $j \in [p]$, and let $\mathcal{U}_{p+1} = \{V_{kr-r+1+pn+1}, \dots, V_t\}$. Note that $|\mathcal{W}| = kr - r + 1$, $|\mathcal{U}_j| = n$, and $|\mathcal{U}_{p+1}| = q$. In particular, if n divides $t - (kr - r + 1)$, then $\mathcal{U}_{p+1} = \emptyset$.

We now define $\ell: V(G) \rightarrow [k]$ as follows.

- The vertices in all the parts of \mathcal{W} are labeled 1.
- For $j \in [p+1]$ label all the vertices of the partite sets in \mathcal{U}_j with $j+1$.
- Label every set of r pendant vertices adjacent to each vertex of the partite set V_i from \mathcal{W} with equal label sequences $(\ell_1^i, \dots, \ell_r^i)$, $i \in [kr - r + 1]$, such that $(\ell_1^1, \dots, \ell_r^1) = (1, \dots, 1)$, and such that the Hamming distance $(\ell_1^i, \dots, \ell_r^i)$ and $(\ell_1^{i+1}, \dots, \ell_r^{i+1})$ is 1 for $i \in [kr - r]$.
- To label the rest of the pendant vertices in G , repeat the same procedure for each of the n number of partite sets in \mathcal{U}_j , $j \in [p]$, as well as for the partite sets in \mathcal{U}_{p+1} . See Fig. 3.

The d_ℓ -sums of vertices in a partite set are all equal. Let s_i denote the d_ℓ -sum of a representative vertex in V_i , $i \in [t]$. Let x be the sum of all labels of vertices in $H_{n,t}$. Then $x - n + r, x - n + (r + 1), \dots, x - n + kr$ are the d_ℓ -sums of representative vertices in $V_1, V_2, \dots, V_{kr-r+1}$ respectively. Similarly, the d_ℓ -sums of representative vertices in the n partite sets in \mathcal{U}_j are $x - (j + 1)n + r, x - (j + 1)n + (r + 1), \dots, x - (j + 1)n + (n + r - 1) = x - jn + (r - 1)$, $j \in [p]$. The same is true for \mathcal{U}_{p+1} . This implies that s_t, s_{t-1}, \dots, s_1 are distinct consecutive integers. The d_ℓ -sums of pendant vertices and vertices from the same partite set do not affect the d -lucky number. Further, the value of k is optimum when t is such that all vertices in V_t are labeled k and n divides $(t - (kr - r + 1))$. Then $t = kr - r + 1 + \underbrace{n + \dots + n}_{(k-1) \text{ times}}$. We conclude that $\eta_{dl}(G) \leq k = \lceil \frac{t+n+r-1}{n+r} \rceil$. \square

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