

Energy and angular distributions of atmospheric muons at the Earth

Prashant Shukla*

*Nuclear Physics Division, Bhabha Atomic Research Centre,
Mumbai 400085, India
Homi Bhabha National Institute, Anushakti Nagar, Mumbai 400094, India
pshukla@barc.gov.in*

Sundaresh Sankrith

*Department of Physics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore 632014, Tamil Nadu, India
sankrithsundaresh@gmail.com*

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A fair knowledge of the atmospheric muon distributions at Earth is a prerequisite for the simulations of cosmic ray setups and rare event search detectors. A modified power law is proposed for atmospheric muon energy distribution which gives a good description of the cosmic muon data in low as well as high energy regime. Using this distribution, analytical forms for zenith angle (θ) distribution are obtained. Assuming a flat Earth, it leads to the $\cos^{n-1}\theta$ form where it is shown that the parameter n is nothing but the power of the energy distribution. Exact analytical function is obtained for inclined trajectory of muon. A new closed form for zenith angle distribution is obtained without assuming a flat Earth and which gives an improved description of the data at all angles even above 70° . These distributions are tested with the available atmospheric muon data of energy and angular distributions. The parameters of these distributions can be used to characterize the cosmic muon data as a function of energy, angle and altitude.

Keywords: Atmospheric muons; cosmic rays; zenith angle distribution.

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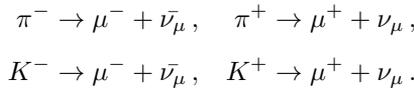
1. Introduction

The primary cosmic rays consisting of protons, alpha particles and heavier nuclei continuously bombard the Earth at the top of the Earth atmosphere.¹ Most of the

*Corresponding author.

cosmic rays originate in Galactic sources such as neutron stars, pulsars, supernovae and active galactic nuclei. The relative abundance of nuclei with charge number $Z > 1$ in cosmic rays is similar to that in the interstellar medium which indicates that cosmic rays are the normal interstellar matter in astrophysical process. The majority of cosmic rays from a few GeV to 100 TeV are accelerated in supernova blast. The magnetic field of the Sun tends to exclude lower energy particles ($E \approx 1$ GeV). During periods of low solar activity, more cosmic rays manage to reach the Earth. Earth's magnetic field also tends to exclude lower energy particles. The particles have greater difficulty penetrating the Earth's magnetic field near equator than the poles. Thus, the intensity of cosmic rays depends both on the location and the time.

Upon entering the atmosphere, the primary cosmic radiations interact with the air molecules (mainly oxygen and nitrogen nuclei) mostly at 10–15 km above the sea level.² All particles suffer energy losses through hadronic and/or electromagnetic processes. The most abundant particles emerging from the energetic hadronic collisions are pions. Particles such as kaons, hyperons, charmed particles and nucleon–antinucleon pairs are also produced. However, most of these particles interact further with the air or decay to other lighter particles. The charged pions and kaons decay to muons and neutrinos as shown below



For a given particle propagating in the atmosphere, the probabilities of decay and interaction become a function of energy, altitude and zenith angle. The atmospheric length increases from vertical to inclined direction and this means more energy loss of the particles which results in a smaller integrated muon flux at the surface of the Earth. The pion decay probability is also larger in the inclined direction which will enhance the muon flux in the intermediate energy. As the energy increases, the fraction of muons from kaon decays also increases because the longer-lived pions become more likely to interact before decaying than the shorter-lived kaons. Since muons are produced with $\nu_\mu(\bar{\nu}_\mu)$, the measurement of muons near the maximum intensity curve for the parent pions can be used to calibrate the atmospheric neutrinos.

Numerical calculations are needed to account accurately for decay and energy loss processes along with the knowledge of primary cosmic spectrum and the energy dependencies of their interaction cross-sections. Simple parametrizations have always been useful to characterize the measured muon momentum spectra.^{3,4} There are transport models treating the propagation of muons which can be used to numerically obtain the muon spectra at different depths of atmosphere.^{5–7} Detailed theoretical calculations have been performed to obtain both atmospheric muons and neutrino spectra at sea level.^{8,9} In addition, Monte Carlo codes such as CORSIKA¹⁰ are available which use different models to calculate the interactions of cosmic particles in the Earth's atmosphere. The goal of the present paper is to obtain simple analytical expressions to provide quick checks on how the muon distributions at

Earth behave and which can be readily used as input in the simulations of detectors for cosmic rays as well as for rare event searches.

In this work, a modified power law is assumed for the cosmic (atmospheric) muon energy distribution at Earth. Using this function and geometrical considerations, we obtain analytical form for zenith angle distribution. For a flat Earth, it leads to $\cos^{n-1} \theta$ form where n is the same as the power of energy distribution. Further, a new analytical form for zenith angle distribution is obtained without assuming a flat Earth. The parameters of these distributions are obtained with the help of measured energy and angular distributions of atmospheric muons.

2. Energy and Angular Distribution of Atmospheric Muons

Muons are produced at about 10–15 km height in the atmosphere and lose about 2 GeV of energy before reaching the ground. Their energy and angular distribution at ground reflect a convolution of production spectrum, energy loss in the atmosphere and the decay. The energy spectrum of muons is almost flat below 1 GeV and then steepens to reflect the primary energy spectrum in the 10–100 GeV range. It steepens further above 100 GeV since the pions above this energy would interact in the atmosphere before decaying to muons. Above 1 TeV, the energy spectrum of the muons is one power steeper than the primary spectrum.¹¹

The energy distribution of primary cosmic rays follow power law E^{-n} . The pion and the muon distributions also follow the same power law which is modified in the low energy region. The vertical flux as a function of energy can be described by

$$I(E, \theta = 0) = I_0 N (E_0 + E)^{-n}, \quad (1)$$

where I_0 is the vertical ($\theta = 0$) muon flux integrated over energy, which gives the normalization $N = (n - 1)(E_0 + E_c)^{(n-1)}$, where E_c is the cutoff value of the data. Here, we have added a parameter E_0 which accounts for energy loss due to both the hadronic as well as electromagnetic interactions with air molecules. We can introduce one more parameter ϵ which modifies the power in the high energy part and that should account for the finite lifetime of pions and kaons

$$I(E) = I_0 N (E_0 + E)^{-n} \left(1 + \frac{E}{\epsilon} \right)^{-1}. \quad (2)$$

Both Eqs. (1) and (2) assume that the energy loss (E_0) is independent of particle energy, an assumption which is good for minimum ionizing particles. At low energies, the energy loss varies as $1/E$ thus a more appropriate distribution would come with an additional parameter E_1 as

$$I(E) = I_0 N (E_0 + E_1/E + E)^{-n} \left(1 + \frac{E}{\epsilon} \right)^{-1}. \quad (3)$$

Here, the normalization constant N can be obtained numerically. However, we will not use Eq. (3) for the analysis presented in this paper.

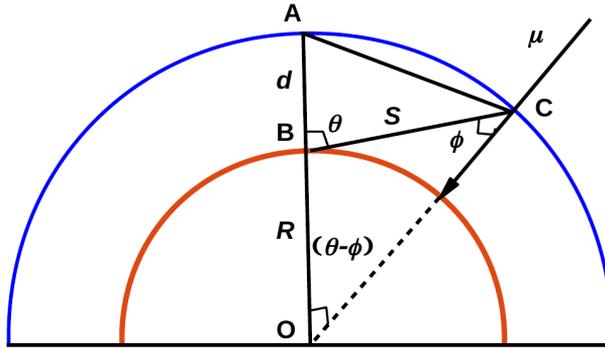


Fig. 1. Geometrical relation between the vertical pathlength d and the pathlength inclined at a zenith angle θ .

The muon flux measured on the Earth's surface has a weak dependence on the azimuthal angle but depends strongly on zenith angle which is defined as the angle made by the incident ray with the vertical direction at that point. One can obtain inclined distance S in terms of vertical distance d , the zenith angle θ and the Earth's radius R using a simple geometrical picture given in Fig. 1. Using sine law for $\triangle OBC$

$$\frac{\sin \phi}{R} = \frac{\sin(\theta - \phi)}{S} = \frac{\sin(180 - \theta)}{R + d}. \quad (4)$$

The relation between ϕ and θ is

$$\sin \phi = \frac{R}{R + d} \sin \theta \quad (5)$$

and the pathlength S in the inclined direction is

$$S = \frac{\sin(\theta - \phi)}{\sin(\theta)} (R + d). \quad (6)$$

Using Eqs. (5) and (6), the ratio of pathlengths of a muon from inclined direction to that of a muon from the vertical direction is obtained as

$$D(\theta) = \frac{S}{d} = \sqrt{\left(\frac{R^2}{d^2} \cos^2 \theta + 2\frac{R}{d} + 1\right)} - \frac{R}{d} \cos \theta. \quad (7)$$

Equation (7) provides a closed expression for column density for an inclined trajectory of muons in curved Earth's atmosphere and is a replacement of Chapman's function given in the first chapter of the book by Grieder.² Accuracy of the various approximations to the Chapman function is discussed by Swider and Gardner.¹² Figure 2 shows the comparison of the analytical expression for the ratio of inclined to vertical pathlengths of muon in Eq. (7) with the Chapman function.

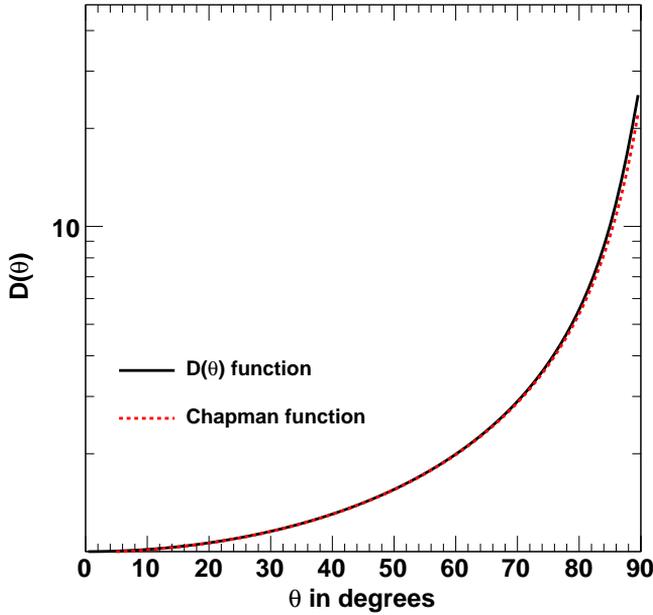


Fig. 2. Comparison of the analytical expression for the ratio of inclined to vertical pathlengths of muon in Eq. (7) with the Chapman function.²

The ratio of the integrated muon flux at θ with that at 0° can be obtained from Eq. (1) as

$$\begin{aligned} \frac{\Phi(\theta)}{\Phi(\theta = 0)} &= \frac{\int_0^\infty (E_0 + E_\theta + E)^{-n} dE}{\int_0^\infty (E_0 + E)^{-n} dE} \\ &= \left(\frac{E_0 + E_\theta}{E_0} \right)^{-(n-1)}, \end{aligned} \quad (8)$$

where $E_0 + E_\theta$ is the energy loss of the muon in the inclined direction. The integrated flux from Eq. (2) can also be considered the same as above since the higher energy term depending on ϵ will make a little difference to the integrated flux. The ratio of the energy loss from inclined to the vertical direction is given by the ratio of the pathlengths (same as the ratio of thicknesses) $D(\theta)$ in the respective directions and thus, the zenith angle distribution of energy integrated flux in terms of $I_0 = \Phi(\theta = 0)$ is obtained as

$$\Phi(\theta) = I_0 D(\theta)^{-(n-1)}. \quad (9)$$

The overall distribution function as a function of both the energy and the zenith angle then can be written as

$$I(E, \theta) = I_0 N (E_0 + E)^{-n} \left(1 + \frac{E}{\epsilon} \right)^{-1} D(\theta)^{-(n-1)}. \quad (10)$$

Here, the function $D(\theta)$ is given by Eq. (7). If the Earth is assumed to be flat then $D(\theta) = 1/\cos \theta$ which on putting in Eq. (9) leads to

$$\Phi(\theta) = I_0 \cos^{n-1} \theta. \tag{11}$$

With $n \simeq 3$, this gives the usual $\cos^2 \theta$ distribution which is widely used to describe zenith angle distribution. This expression gives good description of the data at lower zenith angle but not at higher angles because it assumes a flat Earth.

Gaisser had given the formula¹ for muon energy distribution assuming flat Earth and which is valid for high energy ($E_\mu > 100/\cos \theta$ GeV) which is written as

$$\frac{dN_\mu}{dE d\Omega} \approx 1400 E_\mu^{-2.7} / (\text{m}^2 \text{ s GeV sr}) \left(\frac{1}{1 + \frac{1.1 E \cos \theta}{\epsilon_\pi}} + \frac{0.054}{1 + \frac{1.1 E \cos \theta}{\epsilon_K}} \right), \tag{12}$$

where the two terms in the bracket give the contributions of pions and kaons in terms of two parameters $\epsilon_\pi \approx 115$ GeV and $\epsilon_K \approx 850$ GeV. In the distribution function given by Eq. (10), we use only one parameter ϵ which is obtained by fitting the experimental data.

3. Analysis of the Measured Data

We choose four datasets of muon energy distribution namely at sea level, at high altitude and at an inclined angle and fit them with the function given in Eq. (2) to obtain the parameters. We also analyze primary cosmic ray (protons and helium) distributions using the same Eq. (2). Figures 3 and 4 show the momentum distributions of atmospheric muons at 0° zenith angle at sea level measured at two different

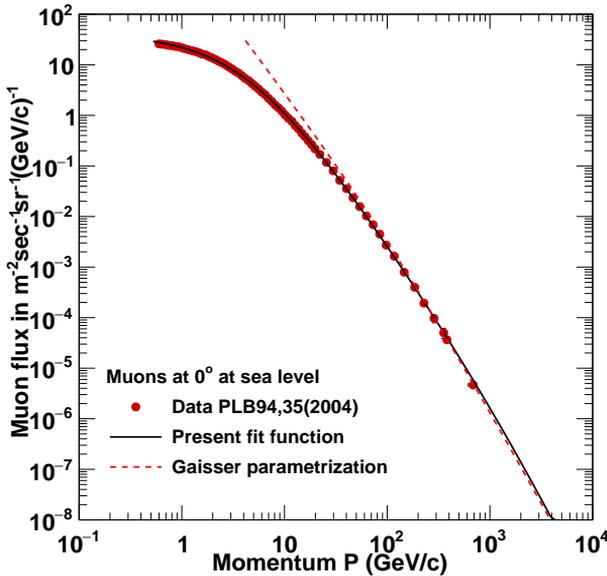


Fig. 3. Muon momentum distribution at 0° zenith angle at sea level.¹³ The lines show the fits with Eqs. (2) and (12).

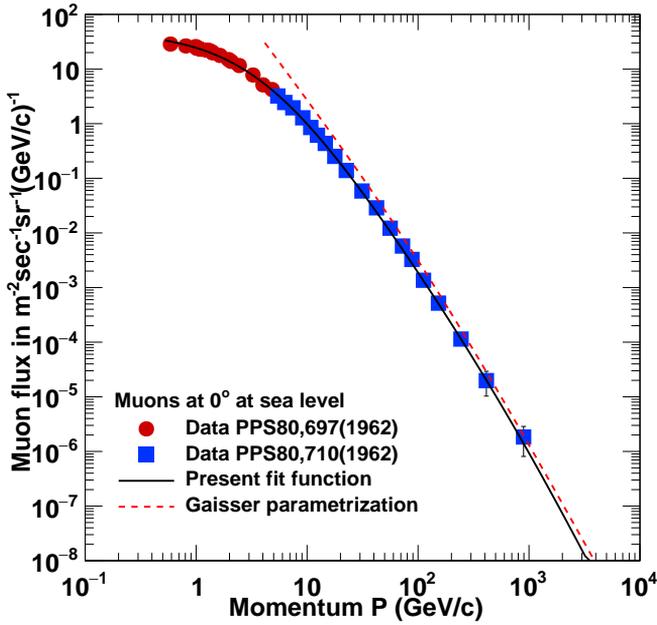


Fig. 4. Muon momentum distribution at 0° zenith angle at sea level.^{14,15} The lines show the fits with Eqs. (2) and (12).

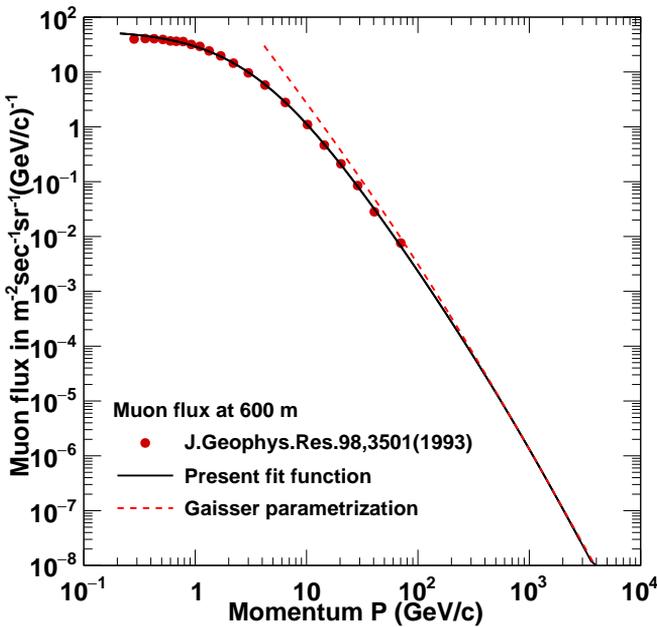


Fig. 5. Muon momentum distribution at 0° zenith angle at 600 m altitude.¹⁶ The lines show the fits with Eqs. (2) and (12).

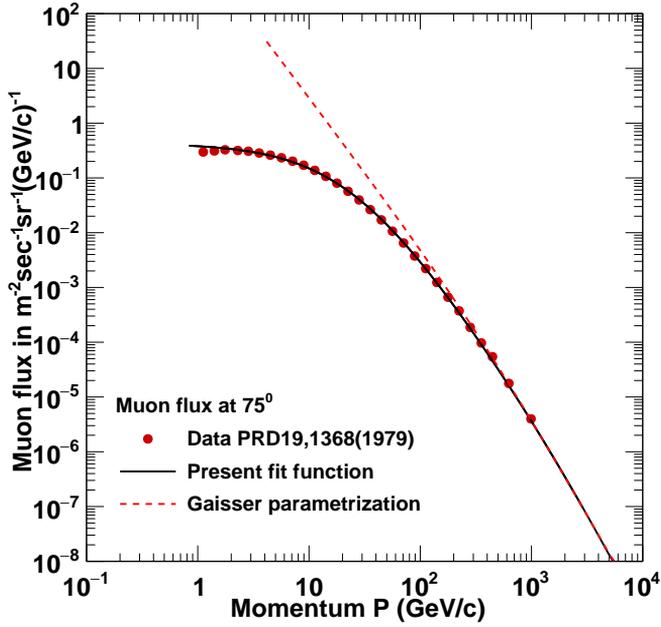


Fig. 6. Muon momentum distribution at sea level measured at zenith angle 75° ¹⁷ fitted with Eqs. (2) and (12).

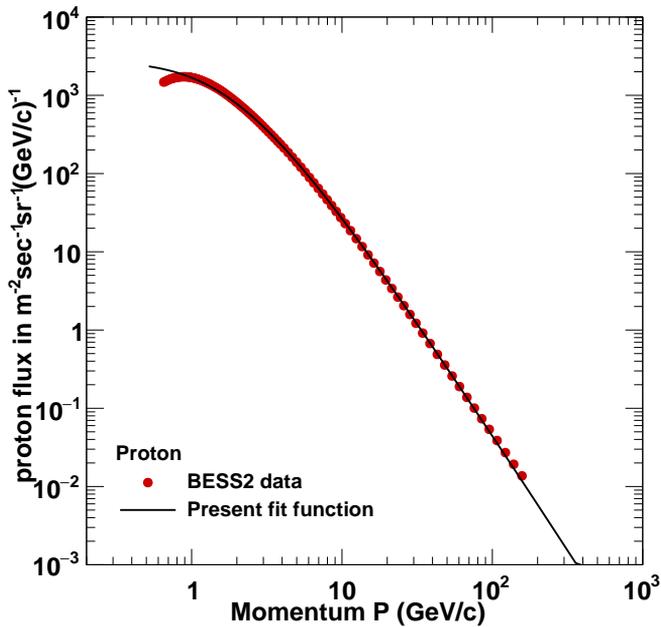


Fig. 7. Proton flux¹⁸ as a function of momentum at the top of the atmosphere fitted with Eq. (2).

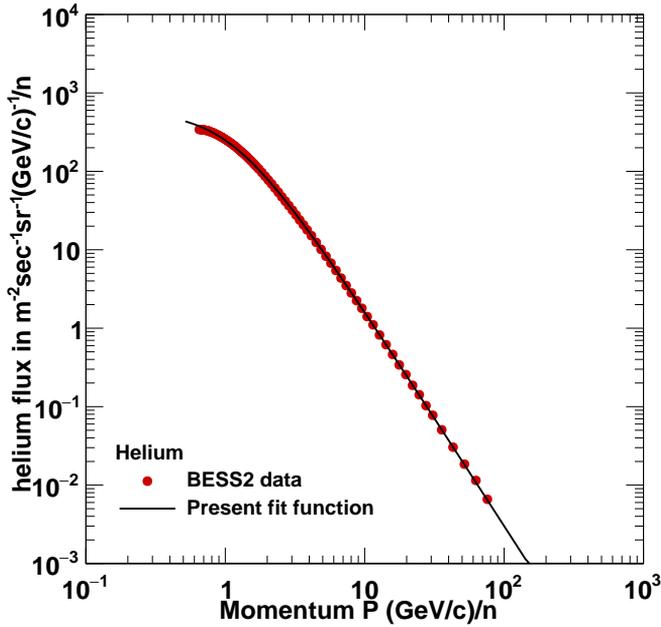


Fig. 8. Helium flux¹⁸ as a function of momentum at the top of the atmosphere fitted with Eq. (2).

locations Tsukuba¹³ and Durham,^{14,15} respectively. Figure 5 shows the same at 600 m altitude.¹⁶ The lines show the fits with Eqs. (2) and (12). The Gaisser function gives good description of the data at high momentum. The present function gives excellent description of both the low as well as the high momentum part of the muon distribution and thus the parameter I_0 gives a reliable estimate of the integrated flux at 0° zenith angle.

Figure 6 shows the muon momentum distribution at sea level but measured at zenith angle 75° ¹⁷ fitted with Eqs. (2) and (12). The function in Eq. (2) describes the data well though there is an expected mismatch at the lowest momentum.

Figure 7 shows proton flux and Fig. 8 shows the Helium flux¹⁸ as a function of momentum at the top of the atmosphere fitted with Eq. (2). The aim here is to get the shape and the power n for the primary cosmic spectra. There are recent measurements of primary cosmic particles from PAMELA detector¹⁹ in the high momentum range.

The values of the fit parameters corresponding to all the data analyzed are listed in Table 1. The value of the power n of the energy distribution is around 3 for muons at sea level, at 600 m altitude and at an inclined angle. For protons $n = 2.93$ and for helium it is 2.75 which means that the muon spectra become slightly steeper than the primary rays due to the interaction processes in the atmosphere. The integrated flux I_0 at $\theta = 0$ is $72.5 \pm 0.2 \text{ m}^{-2} \text{ s}^{-1} \text{ sr}^{-1}$ at sea level which increases to $98.8 \pm 0.5 \text{ m}^{-2} \text{ s}^{-1} \text{ sr}^{-1}$ at 600 m. The value of the parameter I_0 obtained at a zenith angle of 75° at sea level is $65.2 \pm 2 \text{ m}^{-2} \text{ s}^{-1} \text{ sr}^{-1}$. The value of parameter E_0 for

Table 1. Parameters of Eq. (2) obtained by fitting the measured muon distributions.

	I_0 (m^{-2} $\text{s}^{-1} \text{sr}^{-1}$)	n	E_0 (GeV)	$1/\epsilon$ (GeV) $^{-1}$	χ^2/ndf	Data reference
μ at 0° sea level ($E > 0.5$ GeV)	70.7 ± 0.2	3.01 ± 0.01	4.29 ± 0.04	1/854	128/63	Tsukuba, Japan (36.2° N, 140.1° W)
μ at 0° sea level ($E > 0.5$ GeV)	72.5 ± 0.2	3.06 ± 0.03	3.87 ± 0.07	1/854 (fixed)	60.8/23	Durham, UK (54.76° N, 1.57° W)
μ at 0° at 600 m ($E > 0.2$ GeV)	98.8 ± 0.5	3.00 ± 0.08	3.6 ± 0.1	1/854 (fixed)	60.4/18	Prince Albert, Canada (53.2° N, 105.75° W)
μ at 75° sea level ($E > 1.0$ GeV)	65.2 ± 1.5	3.00 ± 0.02	23.78 ± 0.30	1/2000 (fixed)	48.3/25	Hamburg, Germany (53.56° N, 10° E)
proton ($E > 0.5$ GeV)	5000 ± 52	2.93 ± 0.01	1.42 0.04	0.0	96.3/86	Antarctica
helium ($E > 0.5$ GeV)	863 ± 15	2.75 ± 0.02	0.28 ± 0.03	0.0	20.7/67	Antarctica

muons is 4.29 GeV at ground and becomes smaller at 600 m above the ground. The value of E_0 is very high for muons at 75° due to longer pathlength in the atmosphere. For proton and helium its value is small but finite, showing the interactions before they are detected. The parameter ϵ is 854 ± 105 for vertical flux. For the other datasets it is fixed so as to have an agreement with the Gaisser distribution since there is no data in the high energy region to constrain this parameter.

The value of the parameter R/d is fixed at 174.0 which is obtained by fitting the parameters of Eq. (9) with the zenith angle distributions measured by various experiments shown in Fig. 9. The data are taken from the collection of Ref. 20 with the original references.²¹⁻²⁴ The different datasets have different muon energy thresholds and we take the normalized data from the review.²⁰ Ideally, we should have a dataset from a single experiment covering large range of zenith angles.

Table 2 lists the parameters obtained from the measured zenith angle distribution. The fit with the function $\cos^{n-1} \theta$ has been restricted below 80° . With $n \sim 3$ this gives us the popular $\cos^2 \theta$ distribution. The present distribution Eq. (9) gives excellent description of the data at all angles. The parameter I_0 obtained from the present distribution and the $\cos^{n-1} \theta$ distribution match with each other. Another observation is the value of the power n obtained from the energy distribution is very close to the value obtained from fitting the zenith angle distribution. This is the most important result of this study.

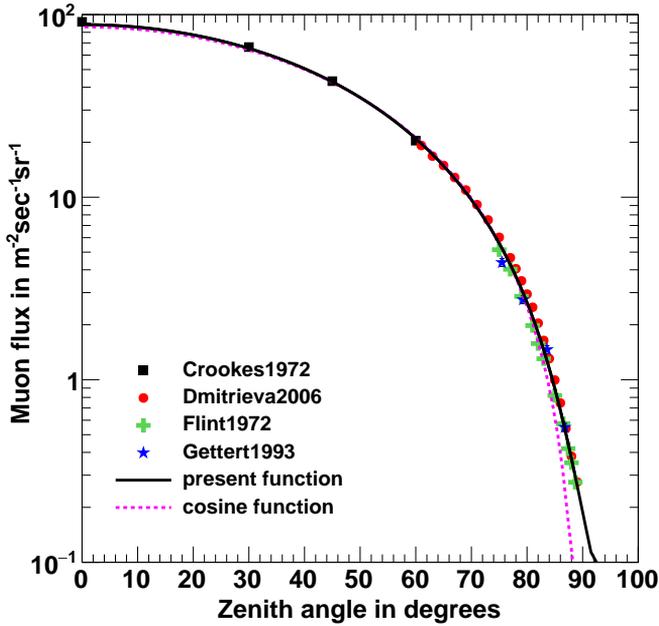


Fig. 9. Muon flux as a function of zenith angle²⁰ at sea level fitted with Eqs. (9) and (11).

Table 2. Parameters obtained from the measured zenith angle distribution.

Fit function	I_0	n	R/d	χ^2/ndf
$\Phi(\theta) = I_0 D(\theta)^{-(n-1)}$	88.0 ± 2.4	3.09 ± 0.03	174 ± 12	111/37
$\Phi(\theta) = I_0 \cos^{(n-1)} \theta$	85.6 ± 2.4	3.01 ± 0.03	—	52/17

4. Conclusions

In this work, analytical functions are proposed for muon energy and angle distributions. A modified power law gives a good description of the cosmic muon momentum distribution in low as well as high energy region. Using the modified power law form of energy distribution, analytical forms for zenith angle distribution are obtained. Assuming a flat Earth, it leads to the $\cos^{n-1} \theta$ form where it is shown that the parameter n is nothing but the power of the energy distribution. With $n \sim 3$ it leads to the famous $\cos^2 \theta$ distribution. Exact analytical function is obtained for inclined trajectory of muon. A new analytical form for zenith angle distribution is obtained without assuming a flat Earth which gives an excellent description of the data at all zenith angles. These functions explain the shape of the spectra and are useful to get the integrated flux. Their parameters are useful to characterize the data as a function of energy, angle and altitude.

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