Czechoslovak Mathematical Journal, 64 (139) (2014), 251–267

RAMANUJAN-FOURIER SERIES AND THE CONJECTURE D OF HARDY AND LITTLEWOOD

H. Gopalakrishna Gadiyar, Ramanathan Padma, Vellore

(Received April 5, 2013)

Abstract. We give a heuristic proof of a conjecture of Hardy and Littlewood concerning the density of prime pairs to which twin primes and Sophie Germain primes are special cases. The method uses the Ramanujan-Fourier series for a modified von Mangoldt function and the Wiener-Khintchine theorem for arithmetical functions. The failing of the heuristic proof is due to the lack of justification of interchange of certain limits. Experimental evidence using computer calculations is provided for the plausibility of the result. We have also shown that our argument can be extended to the m-tuple conjecture of Hardy and Littlewood.

Keywords: Ramanujan-Fourier series; von Mangoldt function; twin primes; Sophie Germain prime; Wiener-Khintchine theorem

MSC 2010: 11N05, 11K65, 62M10

1. Introduction

The twin prime problem asks the question: Are there infinitely many prime pairs of the form $(p, p+2)$? Another outstanding problem in number theory is regarding the Sophie Germain primes. A positive integer p is called a Sophie Germain prime if both p and $2p+1$ are primes, $(2,5)$, $(3,7)$, $(5,11)$, $(11,23)$ for example. Again one asks the question: Are there infinitely many Sophie Germain primes? In this paper we offer a heuristic proof based on the Ramanujan-Fourier series and numerical evidence to affirm the truth of the conjectures in a more general setting. In [12], Hardy and Littlewood made several conjectures regarding the expression of a number as a sum of primes. We state their Conjecture D (page 45 of [12]) below.

Conjecture D. Let a, b and l be positive integers, where $(a,b)=1$. Let $\pi_{(a,b,l)}(N)$ denote the number of prime pairs (p,p') satisfying the condition $ap'-bp=$

l such that $p' < N$. Then

$$\pi_{(a,b,l)}(N) = o\Big(\frac{N}{\log^2 N}\Big) \tag{1.1}$$

unless $(l,a) = 1$, $(l,b) = 1$, and just one of a, b, l is even. But if these conditions are satisfied then

$$\pi_{(a,b,l)}(N) \sim \frac{2C}{a}\,\frac{N}{\log^2 N} \prod_{\substack{p \mid abl \\ p>2}} \frac{p-1}{p-2}, \tag{1.2}$$

where

$$C = \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big), \tag{1.3}$$

and p denotes a prime.

The twin prime problem corresponds to the case $a = b = 1$ and $l = 2$. The Sophie Germain prime problem corresponds to the case $a = 1$, $b = 2$ and $l = 1$. The recent work of Yitang Zhang [21] has led to the revival of interest in the twin prime problem. Sophie Germain primes are currently of great interest after the famous AKS algorithm for primality testing [1]. If the conjecture about the density of Sophie Germain primes is true, then the complexity of the AKS algorithm can be brought down to $O(\log^6 n)$. Sophie Germain primes are the most sought after primes for the RSA algorithm as they are robust against Pollard's $p - 1$ method of factoring [18].

Note that (1.1) can be easily proved. If $(l,a) > 1$ (similarly if $(l,b) > 1$), then $(l,a) \mid bp$ and since $(a,b) = 1$, $(l,a) \mid p$. If (l,a) is composite, then there are no solutions and if (l,a) is prime, then there will be at most one solution. Hence (1.1) holds trivially.

The present paper consists of the following sections. In Section 2, we give a brief historical overview, in Section 3 we state the main result and give the heuristic proof, in Section 4, we give numerical evidence of our main result for various choices of a and b and in Section 5 we show that the same argument can be extended to prove the m-tuple conjecture of Hardy and Littlewood.

2. Historical overview

This section consists of three independent parts. Section 2.1 traces the conjectures made in the additive number theory and the methods to prove them. Section 2.2 gives an introduction to the Ramanujan-Fourier series. In Section 2.3 we state the Wiener-Khintchine theorem.

2.1. Main conjectures in additive number theory. It is well known that the two main methods in the additive number theory are the circle method and the sieve method. We will give a brief historical overview of the subject closely following the references [2], [4], [5], [9], and [12]. The conjecture of the type given above (Conjecture D) was first made by J. J. Sylvester regarding the Goldbach problem. It is not known how he arrived at the conjecture. If $\nu(n)$ denotes the number of ways an even integer n can be expressed as a sum of two primes, then Sylvester conjectured that

$$\nu(n) \sim 2\pi(n) \prod_{\substack{3\leqslant p\leqslant n \\ p\nmid n}} \frac{p-2}{p-1}, \tag{2.1}$$

where $\pi(n)$ denotes the number of primes up to n. Using the prime number theorem and Merten's theorem

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) \sim \frac{\mathrm{e}^{-\gamma}}{\log x}, \tag{2.2}$$

where γ is Euler's constant, it can be shown that

$$\nu(n) \sim 4C\mathrm{e}^{-\gamma}\frac{n}{\log^2 n}\prod_{\substack{p\mid n \\ p>2}}\frac{p-1}{p-2}. \tag{2.3}$$

Stäckel made a similar conjecture (page 423, Chapter XVIII of [5]):

$$\nu(n) \sim \frac{n}{\log^2 n}\prod_{p\mid n}\frac{p}{p-1} \tag{2.4}$$

which was proved to be incorrect by Landau. It was Merlin and Brun who used the sieve method to attack the Goldbach problem and the twin prime problem. It was shown by Hardy [9] that Brun's argument will lead to

$$\nu(n) \sim 8C\mathrm{e}^{-2\gamma}\frac{n}{\log^2 n}\prod_{\substack{p\mid n \\ p>2}}\frac{p-1}{p-2}. \tag{2.5}$$

However, Hardy showed that both the formulae (2.3) and (2.5) contain an erroneous factor involving $\mathrm{e}^{-\gamma}$ and the correct formula should be

$$\nu(n) \sim 2C\frac{n}{\log^2 n}\prod_{\substack{p\mid n\\ p>2}}\frac{p-1}{p-2}. \tag{2.6}$$

Using his sieve method, Brun [2] proved that every large integer n can be expressed as m_1+m_2 where both m_1 and m_2 contain at most 9 prime factors, and also that the number of such decompositions is of the order $n/\log^2 n$ at least. Also Brun proved that

$$\pi_{(1,1,2)}(N) \leqslant 100\,\frac{N}{\log^2 N} \tag{2.7}$$

for the number of twin primes up to N for all $N > N_0$ [2]. This result was the first of the kind where an upper bound was obtained.

The circle method was first used by Hardy and Ramanujan to attack the partition problem. This reduces to understanding the generating function of the partition function $p(n)$ given by

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}. \tag{2.8}$$

The key observation that led to the development of the circle method was that $p(n)$ could be written as

$$\frac{1}{2\pi\mathrm{i}}\int_C \frac{1}{(1-z)(1-z^2)(1-z^3)\dots}z^{-n-1}\,\mathrm{d}z, \tag{2.9}$$

where C is the circle $|z| = r$ and $r < 1$. Hardy and Ramanujan observed that the right hand side of (2.8) is (apart from an innocuous factor) the Dedekind eta function which satisfies a modular transformation law. This transformation law allows one to determine the residue at each of the singularities occurring in the integrand of (2.9). Namely, the singularities are the roots of unity and one needs to take r close to 1 near each of the singularities. Their epic paper gave rise to the celebrated circle method.

We would like to remark that clues that the roots of unity play an important role in additive number theoretic problems are available in the work of several authors, see ([13], [11]), in which a simple minded partial fraction expansion yields

$$\begin{aligned} &\frac{1}{(1-x)(1-x^2)(1-x^3)}\\ &\quad= \frac{1}{6(1-x)^3}+\frac{1}{4(1-x)^2}+\frac{17}{72(1-x)}+\frac{1}{8(1+x)}+\frac{1}{9(1-\omega x)}+\frac{1}{9(1-\omega^2 x)}, \end{aligned} \tag{2.10}$$

where ω and ω^2 denote the two complex cube roots of unity. If

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = 1 + \sum_{n=1}^{\infty} r(n)x^n, \tag{2.11}$$

then

$$r(n) = \frac{(n+3)^2}{12} - \frac{7}{72} + \frac{(-1)^n}{8} + \frac{2}{9}\cos\left(\frac{2n\pi}{3}\right). \tag{2.12}$$

Hardy and Ramanujan (page 275, [16]) obtained the asymptotic formula for the partition function

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\alpha\sqrt{n}} k^{1/2} A_k(n) \frac{\mathrm{d}}{\mathrm{d}n} \frac{\exp \frac{\pi}{k}\sqrt{\frac{2}{3}}\lambda_n}{\lambda_n} + O(n^{-1/4}), \tag{2.13}$$

and Rademacher (page 274, [16]) obtained an exact formula

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} k^{1/2} A_k(n) \frac{\mathrm{d}}{\mathrm{d}n} \frac{\sinh \frac{\pi}{k}\sqrt{\frac{2}{3}}\lambda_n}{\lambda_n}, \tag{2.14}$$

where $\lambda_n = \sqrt{n - \frac{1}{24}}$, $A_k(n) = \sum\limits_{\substack{h \bmod k \\ (h,k)=1}} \omega_{hk} \mathrm{e}^{-2\pi \mathrm{i} hn/k}$, $\omega_{hk} = \mathrm{e}^{\mathrm{i}\pi s(h,k)}$ and $s(h,k) = \sum\limits_{\mu \bmod k} ((\mu/k))((h\mu/k))$, where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases} \tag{2.15}$$

In both the formulae, one can see that the roots of unity play an important role.

After Ramanujan's untimely death, Hardy along with Littlewood went on attacking the other problems in the additive number theory like Waring's problem and the Goldbach problem using the circle method. In the circle method, the unit circle is divided into major arcs and minor arcs. The major arcs are the union of segments of the unit circle centering around the points $\mathrm{e}^{2\pi \mathrm{i} k/q}$, where $1 \leqslant k \leqslant q$, $(k,q) = 1$ and q's are small and the remaining parts form the minor arcs. The circle method was successful in solving the ternary Goldbach problem and Waring's problem for large integers, where the major arc contribution could be shown to dominate the minor arc contribution. However, the circle method is not successful in giving a proof of the binary Goldbach conjecture and the twin prime conjecture (even assuming the

Riemann hypothesis). Hardy and Littlewood made their conjectures (like Conjecture D) from the major arc contribution which can be expressed in terms of what is called singular series.

While Hardy and Littlewood shifted their focus to the complex analytic aspects of the circle method, Ramanujan wrote a paper in which he introduced the concept of what is now called the Ramanujan-Fourier expansion for arithmetical functions which we will describe below.

2.2. Ramanujan-Fourier series. In [17], Ramanujan showed that many important arithmetical functions $a(n)$ have an expansion of the form

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n), \tag{2.16}$$

where

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \mathrm{e}^{2\pi i n k/q} \tag{2.17}$$

is called the Ramanujan sum and the a_q's are known as the Ramanujan-Fourier coefficients. He obtained such expansions for $d(n)$, $\sigma(n)$, $\varphi(n)$ and so on where $d(n)$ denotes the number of divisors of n, $\sigma(n)$ denotes the sum of divisors of n and $\varphi(n)$ denotes the number of positive integers less than n and coprime to n. For example, he showed that

$$d(n) = -\sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n), \tag{2.18}$$

$$\sigma(n) = \frac{\pi^2 n}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}. \tag{2.19}$$

Ramanujan proved these formulae by elementary methods. He used finite algebra and simple general theorems concerning infinite series.

In [10], Hardy proved that the Ramanujan sum is a multiplicative function of q, that is

$$c_{qq'}(n) = c_q(n) c_{q'}(n) \quad \text{if } (q, q') = 1. \tag{2.20}$$

Also, if p is prime, then

$$c_p(n) = \begin{cases} -1, & \text{if } p \nmid n, \\ p-1, & \text{if } p \mid n. \end{cases} \tag{2.21}$$

Using the properties of $c_q(n)$ and the theory of analytic functions Hardy rederived many of Ramanujan's formulae like (2.19). He also obtained the Ramanujan-Fourier expansion for $(\varphi(n)/n)\Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } \ n = p^k, \ p \text{ is prime and } k \text{ any positive integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.22}$$

That is,

$$\frac{\varphi(n)}{n}\Lambda(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} c_q(n), \tag{2.23}$$

where $\mu(q)$ is the Möbius function defined as follows:

$$\mu(q) = \begin{cases} (-1)^k, & \text{if } \ q = p_1 p_2 \dots p_k, \ p_i\text{'s are distinct primes}, \\ 0\,, & \text{otherwise}. \end{cases} \tag{2.24}$$

We would like to make a remark here as to why rational points dominate in the circle method. It is simply because of the fact that the $a(n)$'s have Ramanujan-Fourier expansions which give rise to simple poles at all rational points on the unit circle. For,

$$\sum_{n=1}^{\infty} a(n) x^n = \sum_{n=1}^{\infty}\sum_{q=1}^{\infty} \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_q \mathrm{e}^{2\pi \mathrm{i} nk/q} x^n \tag{2.25}$$

$$= \sum_{q=1}^{\infty} \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_q \sum_{n=1}^{\infty} (\mathrm{e}^{2\pi \mathrm{i} k/q} x)^n \tag{2.26}$$

$$= \sum_{q=1}^{\infty} \sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_q \frac{\mathrm{e}^{2\pi \mathrm{i} k/q} x}{1 - \mathrm{e}^{2\pi \mathrm{i} k/q} x}. \tag{2.27}$$

The Goldbach problem and the twin prime problem correspond to $a(n) = (\varphi(n)/n) \times \Lambda(n)$ and $a_q = \mu(q)/\varphi(q)$, see (2.23).

However, neither Ramanujan nor Hardy gave a formula for finding the Ramanujan-Fourier coefficients which are the backbone of Fourier analytic approach to such questions. This was done later by Carmichael [3]. He proved orthogonality relations for $c_q(n)$ and this led to a method of evaluating the Ramanujan-Fourier coefficients.

Let $M(f)$ denote the mean value of an arithmetical function f, that is

$$M(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{n \leqslant N} f(n). \tag{2.28}$$

For $1 \leqslant k \leqslant q$, $(k,q)=1$, let $e_{k/q}(n) = \mathrm{e}^{2\pi \mathrm{i} nk/q}$ $(n \in \mathbb{N})$. If $a(n)$ is an arithmetical function with expansion (2.16) then

$$a_q = \frac{1}{\varphi(q)} M(ac_q) = \frac{1}{\varphi(q)} \lim_{N\to\infty} \frac{1}{N} \sum_{n \leqslant N} a(n) c_q(n). \tag{2.29}$$

Also,

$$M(e_{k/q}\overline{e_{k'/q'}}) = \begin{cases} 1, & \text{if } \dfrac{k}{q} = \dfrac{k'}{q'}, \\ 0, & \text{if } \dfrac{k}{q} \neq \dfrac{k'}{q'}. \end{cases} \tag{2.30}$$

Now we state a theorem which is used in electrical engineering and the theory of probability.

2.3. The Wiener-Khintchine theorem. The Wiener-Khintchine theorem ([6], [19], [14]) basically says that if

$$f(t) = \sum_n f_n \mathrm{e}^{\mathrm{i}\lambda_n t}, \tag{2.31}$$

then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau)\overline{f(t)}\,\mathrm{d}t = \sum_n |f_n|^2 \mathrm{e}^{\mathrm{i}\lambda_n \tau}. \tag{2.32}$$

The left hand side of (2.32) is called an autocorrelation function. The right hand side is nothing but the power spectrum. It is used practically to extract hidden periodicities in seemingly random phenomena [15]. For recent historical comments on this topic, see [20].

3. Main result

In this section we give our main result and its proof.

3.1. Statement of main result. Let $\Lambda_1(n) = \varphi(n) n^{-1} \Lambda(n)$. Also let

$$\Psi_{(a,b,l)}(N) = \sum_{n \leqslant N} \Lambda_1(n) \Lambda_1\Big(\frac{bn+l}{a}\Big).$$

Then, up to interchange of certain limits (which we cannot at present justify),

$$\lim_{N\to\infty} \frac{1}{N}\Psi_{(a,b,l)}(N) = \begin{cases} \dfrac{2C}{a} \displaystyle\prod_{\substack{p|abl\\ p>2}} \dfrac{p-1}{p-2}, & \text{if } (a,l)=(b,l)=1 \\ & \text{and exactly one of } a,\ b,\ l \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Note that (1.2) (that is, Conjecture D) follows immediately from (3.1). See for example [7]. We hope our approach can be developed along more rigorous lines into a viable theory.

In [7], we showed that the twin prime problem is related to autocorrelation and hence to the Wiener-Khintchine theorem. The key idea in this paper can be extended to prove (3.1).

3.2. Outline of the proof. Our approach uses the following tools.

1. The Ramanujan-Fourier series for $\Lambda_1(n)$.

2. Carmichael's formula for getting the Ramanujan-Fourier coefficients for arithmetical functions.

3. The Wiener-Khintchine theorem for arithmetical functions which we state below.

For an arithmetical function $a(n)$ having the Ramanujan-Fourier series

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n),$$

the Wiener-Khintchine formula can be stated as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} a(n)a(n+h) = \sum_{q=1}^{\infty} a_q^2 c_q(h) \tag{3.2}$$

up to a certain convergence criterion.

4. Multiplicative property of $c_q(n)$. If a_q's are also multiplicative, then the singular series $\sum_{q=1}^{\infty} a_q^2 c_q(h)$, if it converges, can be expressed as an Euler product. That is,

$$\sum_{q=1}^{\infty} a_q^2 c_q(h) = \prod_p (1 + a_p^2 c_p(h) + \ldots). \tag{3.3}$$

Unfortunately, we are not able to prove the Wiener-Khintchine theorem for $\Lambda_1(n)$ as its Ramanujan-Fourier series is not uniformly and absolutely convergent. Hence our proof remains heuristic as an interchange of certain limits has to be justified.

However, numerical evidence is given for various choices of a and b which shows remarkable accuracy of the conjecture.

In [8], Solomon Golomb introduced the Lambda method to study the twin prime problem. As in our argument below, Golomb runs into the same problem of an unjustified interchange of summation.

3.3. "Proof" of main result. Using the results in Section 2.2, we prove (3.1). Since

$$\frac{1}{a}\sum_{j=0}^{a-1} \mathrm{e}^{2\pi i m j/a} = \begin{cases} 1, & \text{if } \ a \mid m, \\ 0, & \text{if } \ a \nmid m, \end{cases} \tag{3.4}$$

and noting that

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \mathrm{e}^{2\pi i n k/q} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \mathrm{e}^{2\pi i n k/q}, \tag{3.5}$$

we write

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N}\Psi_{(a,b,l)}(N) &= \lim_{N\to\infty} \frac{1}{N}\sum_{n\leqslant N} \Lambda_1(n)\Lambda_1\Big(\frac{bn+l}{a}\Big)\frac{1}{a}\sum_{j=0}^{a-1} \mathrm{e}^{-2\pi i (j/a)(bn+l)} \\
&\overset{?}{=} \left(\frac{1}{a}\sum_{j=0}^{a-1}\sum_{q=1}^{\infty}\sum_{\substack{k=1 \\ (k,q)=1}}^{q}\sum_{q'=1}^{\infty}\sum_{\substack{k'=1 \\ (k',q')=1}}^{q'} \frac{\mu(q)}{\varphi(q)}\frac{\mu(q')}{\varphi(q')}\mathrm{e}^{-2\pi i (k'/q'+j)l/a}\right) \\
&\qquad \times \left(\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqslant N} \mathrm{e}^{2\pi i (k/q-(k'/q')(b/a)-j(b/a))n}\right) &(3.6) \\
&= \frac{1}{a}\sum_{j=0}^{a-1}\sum_{q=1}^{\infty}\sum_{\substack{k=1 \\ (k,q)=1}}^{q}\sum_{q'=1}^{\infty}\sum_{\substack{k'=1 \\ (k',q')=1 \\ k/q=(k'/q'+j)b/a}}^{q'} \frac{\mu(q)}{\varphi(q)}\frac{\mu(q')}{\varphi(q')}\mathrm{e}^{-2\pi i (k'/q'+j)l/a} &(3.7) \\
&= S \text{ (say)},
\end{aligned}$$

where we have used (2.23), freely interchanged the sums and limits to obtain (3.6), and then used (2.30) to get (3.7). We prove that S is equal to the right-hand side of (3.1) as a lemma.

Lemma 1. *If* $(a,b)=1$, *then*

$$S=\begin{cases}\dfrac{2C}{a}\displaystyle\prod_{\substack{p\mid abl\\ p>2}}\dfrac{p-1}{p-2}, & \text{if } (a,l)=(b,l)=1\\ & \text{and exactly one of } a,\ b,\ l \text{ is even,}\\ 0, & \text{otherwise.}\end{cases}\tag{3.8}$$

P r o o f. Since $(k',q')=1$ and $0\leqslant j\leqslant a-1$, we have

$$\frac{k}{q}=\Big(\frac{k'}{q'}+j\Big)\frac{b}{a}=\frac{k'+jq'}{q'a}b=\frac{k_1}{q'a}b,\quad \text{where } 1\leqslant k_1\leqslant q'a \text{ and } (k_1,q')=1.\tag{3.9}$$

Now, (3.9) can happen if and only if

$$qd_2=q'd_1\tag{3.10}$$

for some divisors d_1 and d_2 of a and b, respectively, and

$$k=\frac{k_1}{q'a}bq=\frac{k_1(b/d_2)qd_2}{q'd_1(a/d_1)}=k_1\frac{b/d_2}{a/d_1}\tag{3.11}$$

is an integer. Since $(a,b)=1$, this can happen if and only if a/d_1 divides k_1. Also, from (3.10), $d_1\mid qd_2$ and since $(a,b)=1$, $d_1\mid q$. So we write $q=d_1q_1$ where $q_1\geqslant 1$ is an integer. Similarly $q'=d_2q_2$ where $q_2\geqslant 1$ is an integer. Thus from (3.10), $q_1=q_2$. Also $\mu(q)=\mu(d_1q_1)\neq 0$ if and only if $(d_1,q_1)=1$. Similarly we have $(d_2,q_2)=1$. □

Let us write $k_1=(a/d_1)k_2$. Since $(k,q)=1$ and $d_1\mid q$, we have $(d_1,k)=1$ and from (3.11), $k=k_2(b/d_2)$ and thus $(k_2,d_1)=1$. So we can write

$$S=\frac{1}{a}\sum_{d_1\mid a}\sum_{d_2\mid b}\sum_{\substack{q_2=1\\(d_1,q_2)=1\\(d_2,q_2)=1}}^{\infty}\frac{\mu(d_1)}{\varphi(d_1)}\,\frac{\mu(d_2)}{\varphi(d_2)}\,\frac{\mu^2(q_2)}{\varphi^2(q_2)}\sum_{\substack{k_2=1\\(k_2,q_2d_1d_2)=1}}^{q_2d_1d_2}\mathrm{e}^{-2\pi\mathrm{i}(k_2/q_2d_1d_2)l}.\tag{3.12}$$

That is,

$$S=\frac{1}{a}\sum_{d_1\mid a}\sum_{d_2\mid b}\sum_{\substack{q_2=1\\(d_1,q_2)=1\\(d_2,q_2)=1}}^{\infty}\frac{\mu(d_1)}{\varphi(d_1)}\,\frac{\mu(d_2)}{\varphi(d_2)}\,\frac{\mu^2(q_2)}{\varphi^2(q_2)}c_{(q_2d_1d_2)}(l),\tag{3.13}$$

by the definition of the Ramanujan sum (2.17). Now by the multiplicative property (2.20) of the Ramanujan sum and as q_2, d_1, d_2 are pairwise relatively prime we obtain

$$
\begin{aligned}
S &= \frac{1}{a}\sum_{d_1|a}\sum_{d_2|b}\sum_{\substack{q_2=1\\(d_1,q_2)=1\\(d_2,q_2)=1}}^{\infty}\frac{\mu(d_1)}{\varphi(d_1)}\frac{\mu(d_2)}{\varphi(d_2)}\frac{\mu^2(q_2)}{\varphi^2(q_2)}c_{q_2}(l)c_{d_1}(l)c_{d_2}(l) \\
&= \frac{1}{a}\sum_{\substack{q_2=1\\(a,q_2)=1\\(b,q_2)=1}}^{\infty}\frac{\mu^2(q_2)}{\varphi^2(q_2)}c_{q_2}(l)\sum_{d_1|a}\frac{\mu(d_1)}{\varphi(d_1)}c_{d_1}(l)\sum_{d_2|b}\frac{\mu(d_2)}{\varphi(d_2)}c_{d_2}(l) \\
&= \frac{1}{a}\sum_{\substack{q_2=1\\(ab,q_2)=1}}^{\infty}\frac{\mu^2(q_2)}{\varphi^2(q_2)}c_{q_2}(l)\sum_{d_1|a}\frac{\mu(d_1)}{\varphi(d_1)}c_{d_1}(l)\sum_{d_2|b}\frac{\mu(d_2)}{\varphi(d_2)}c_{d_2}(l),
\end{aligned}
\tag{3.14}
$$

as $(a,b)=1$. Writing the series and sums in (3.14) as Euler products, we get

$$
S = \frac{1}{a}\prod_{p\nmid ab}\Big(1+\frac{c_p(l)}{(p-1)^2}\Big)\prod_{p|a}\Big(1-\frac{c_p(l)}{p-1}\Big)\prod_{p|b}\Big(1-\frac{c_p(l)}{p-1}\Big). \tag{3.15}
$$

By the property (2.21) of the Ramanujan sum, if $u>1$ is an integer, then

$$
\prod_{p|u}\Big(1-\frac{c_p(l)}{p-1}\Big) = \begin{cases} \prod_{p|u}\frac{p}{p-1}, & \text{if } p\nmid l, \\ 0, & \text{if } p \mid l. \end{cases} \tag{3.16}
$$

Hence we will assume that $(a,l)=1$ and $(b,l)=1$ so that

$$
S = \frac{1}{a}\prod_{p\nmid abl}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p\nmid ab\\p|l}}\frac{p}{p-1}\prod_{\substack{p|a\\p\nmid l}}\frac{p}{p-1}\prod_{\substack{p|b\\p\nmid l}}\frac{p}{p-1}. \tag{3.17}
$$

If none of a, b, or l is even, then the product vanishes:

$$
\prod_{p\nmid abl}\Big(1-\frac{1}{(p-1)^2}\Big) = 0. \tag{3.18}
$$

So we will assume that one of a, b or l is even. But $(a,b)=1$, $(a,l)=1$ and $(b,l)=1$ and therefore exactly one of a, b or l is even. We can therefore write the infinite product

$$
\prod_{p\nmid abl}\Big(1-\frac{1}{(p-1)^2}\Big) = \prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p|abl\\p>2}}\frac{(p-1)^2}{p(p-2)}. \tag{3.19}
$$

Thus $S=0$ unless $(a,l)=1$, $(b,l)=1$ and exactly one of a,b or l is even, but if these conditions are satisfied, then the value of S as given in (3.8) is obtained by simplifying (3.17) using (3.19). □

4. Experimental evidence

We give now the compelling numerical evidence of the main result (3.1) by varying a and b. We have taken the value of $C \sim 0.660161816$ and the ratio is defined by right-hand side of (3.1) divided by $\Psi_{(a,b,l)}(N)/N$.

Example 1. We take $a = 1$, $b = 2$, $l = 1$ which corresponds to Sophie Germain primes. In this case, the right-hand side of (3.1) is $2C = 1.320323632$.

N	$\Psi_{(1,2,1)}(N)$	$\frac{\Psi_{(1,2,1)}(N)}{N}$	Ratio
50000	66130.966133	1.322619	0.998264
100000	132886.401744	1.328864	0.993573
150000	200755.416380	1.338369	0.986517
200000	265612.706085	1.328064	0.994172
250000	331585.551940	1.326342	0.995462
300000	394316.641234	1.314389	1.004515
350000	459668.599011	1.313339	1.00531
400000	521496.993567	1.303742	1.012718
450000	588393.432192	1.307541	1.009776
500000	652614.182933	1.305228	1.011565

Table 1.

Example 2. We take $a = 1$, $b = 10$, $l = 1$. In this case, the right-hand side of (3.1) is $8C/3 = 1.760431509$.

N	$\Psi_{(1,10,1)}(N)$	$\frac{\Psi_{(1,10,1)}(N)}{N}$	Ratio
10000	17107.791529	1.710779	1.029023
20000	34210.057148	1.710503	1.029189
30000	51939.100560	1.731303	1.016824
40000	70219.348038	1.755484	1.002818
50000	89934.594398	1.798692	0.978729
60000	106902.836342	1.781714	0.988055
70000	123796.944818	1.768528	0.995422
80000	141470.265879	1.768378	0.995506
90000	159287.348829	1.769859	0.994673
100000	177824.093558	1.778241	0.989985

Table 2.

Example 3. We take $a = 3$, $b = 5$, $l = 2$. In this case, the right-hand side of (3.1) is $16C/9 = 1.173621006$.

N	$\Psi_{(3,5,2)}(N)$	$\dfrac{\Psi_{(3,5,2)}(N)}{N}$	Ratio
60000	69649.061665	1.160837	1.011013
120000	140371.214304	1.169770	1.003292
180000	211924.646933	1.177366	0.996819
240000	282504.323361	1.177106	0.997039
300000	355072.360724	1.183578	0.991587
360000	423152.712312	1.175427	0.998463
420000	496296.973007	1.181662	0.993195
480000	568659.361599	1.184709	0.990640
540000	642488.622118	1.189796	0.986405
600000	712048.221861	1.186749	0.988938

Table 3.

5. m-tuple conjecture

Let $a_1, a_2, \dots, a_{m-1}$ be distinct integers and let us now count the number of groups $n, n+a_1, n+a_2, \dots, n+a_{m-1}$ between 1 and x and consisting of primes.

Consider

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqslant N}\Lambda_1(n)\Lambda_1(n+a_1)\dots\Lambda_1(n+a_{m-1})$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n\leqslant N}\sum_{q}\sum_{q_1}\dots\sum_{q_k}\frac{\mu(q)}{\varphi(q)}\sum_{\substack{k=1\\(k,q)=1}}^{q}\mathrm{e}^{-2\pi\mathrm{i}(k/q)n}$$

$$\times\frac{\mu(q_1)}{\varphi(q_1)}\sum_{\substack{k_1=1\\(k_1,q_1)=1}}^{q_1}\mathrm{e}^{2\pi\mathrm{i}(k_1/q_1)(n+a_1)}\dots\frac{\mu(q_{m-1})}{\varphi(q_{m-1})}\sum_{\substack{k_{m-1}=1\\(k_{m-1},q_{m-1})=1}}^{q_{m-1}}\mathrm{e}^{2\pi\mathrm{i}(k_{m-1}/q_{m-1})(n+a_{m-1})}$$

$$(5.1)\quad \stackrel{?}{=}\sum_{q}\sum_{q_1}\dots\sum_{q_{m-1}}\frac{\mu(q)}{\varphi(q)}\prod_{i=1}^{m-1}\frac{\mu(q_i)}{\varphi(q_i)}\sum_{k_1}\dots\sum_{k_{m-1}}\mathrm{e}^{2\pi\mathrm{i}((k_1/q_1)a_1+\dots+(k_{m-1}/q_{m-1})a_{m-1})}$$

$$\times\lim_{N\to\infty}\frac{1}{N}\sum_{n\leqslant N}\mathrm{e}^{2\pi\mathrm{i}n(-k/q+k_1/q_1+\dots+k_{m-1}/q_{m-1})}$$

$$(5.2)\quad =\sum_{q_1}\dots\sum_{q_{m-1}}\frac{\mu(q)}{\varphi(q)}\prod_{i=1}^{m-1}\frac{\mu(q_i)}{\varphi(q_i)}\sum_{k_1}\dots\sum_{k_{m-1}}\mathrm{e}^{2\pi\mathrm{i}((k_1/q_1)a_1+\dots+(k_{m-1}/q_{m-1})a_{m-1})},$$

where $k/q = k_1/q_1 + \dots + k_{m-1}/q_{m-1}$. Also we have used (3.5). Here again we have freely interchanged the limits and applied (2.30). The right hand side of (5.2) is the sum

$$(5.3)\qquad S_{m-1} = \sum_{q_1,q_2,\dots,q_{m-1}} A(q_1, q_2, \dots, q_{m-1})$$

in the notation of Hardy and Littlewood, see ((5.625I), page 55 of [12]). Using the multiplicative property of $A(q_1, q_2, \ldots, q_m)$ and mathematical induction, they showed that

$$S_{m-1} = A(q_1, q_2, \ldots, q_{m-1}) = \prod_{p \geqslant 2} \left(\frac{p}{p-1}\right)^{m-1} \frac{p-\nu}{p-1}, \tag{5.4}$$

where $\nu = \nu_m = \nu\ (p; 0, a_1, a_2, \ldots, a_{m-1})$ is the number of distinct residues of $0, a_1, a_2, \ldots, a_{m-1}$ to modulus p.

Thus

$$\sum_{n \leqslant N} \Lambda_1(n)\Lambda_1(n+a_1)\ldots\Lambda_1(n+a_{m-1}) \sim S_{m-1}N \tag{5.5}$$

and hence

$$\sum_{n \leqslant N} \Lambda(n)\Lambda(n+a_1)\ldots\Lambda(n+a_{m-1}) \sim S_{m-1}N, \tag{5.6}$$

from which the m-tuple conjecture of Hardy and Littlewood follows which we state below.

Conjecture. Let $a_1, a_2, \ldots, a_{m-1}$ be $m-1$ distinct integers and $P(x; a_1, \ldots, a_{m-1})$ the number of groups $n, n+a_1, \ldots, n+a_{m-1}$ between 1 and x and consisting wholly of primes. Then

$$P(x; a_1, \ldots a_{m-1}) \sim S_{m-1}\mathrm{Li}_m(x), \tag{5.7}$$

where $\mathrm{Li}_m(x) = \int_2^x \mathrm{d}u/(\log u)^m$. Note that Hardy and Littlewood state this conjecture in a more symmetrical form at page 61 of [12].

6. Conclusion

If the step (3.6) could be proved rigorously, which involves justification of interchange of certain limits, then a whole class of outstanding problems including the twin prime problem and the Sophie Germain prime problem could be solved completely. Similarly, if the step (5.1) is proved rigorously, then the m-tuple conjecture will be solved completely. We may say that in a precise sense the Ramanujan-Fourier series for the (refined) von Mangoldt function traps the fluctuations in the distribution of primes. It is hoped that the theory of Ramanujan-Fourier series could be developed to study various properties of arithmetical functions. Numerical agreement between conjecture and experiment means that this technique could become a common tool and lead to further developments in number theory.

Acknowledgement. We wish to thank Professor M. Ram Murty for encouragement given and also for drawing our attention to the interesting reference of Golomb [8]. We are grateful to the referee for suggesting several changes in the paper that have improved the presentation of the paper.

References

[1] *M. Agrawal, N. Kayal, N. Saxena*: PRIMES is in P. Ann. Math. *160* (2004), 781–793. zbl MR

[2] *V. Brun*: The sieve of Eratosthenes and the theorem of Goldbach. The Goldbach Conjecture (Y. Wang, ed.). Series in Pure Mathematics 4, World Scientific, Singapore, 2002, pp. 99–136. zbl MR

[3] *R. D. Carmichael*: Expansions of arithmetical functions in infinite series. Proc. Lond. Math. Soc., II. Ser. *34* (1932), 1–26. zbl MR

[4] *A. C. Cojocaru, M. R. Murty*: An Introduction to Sieve Methods and their Applications. London Mathematical Society Lecture Note Series 66, Cambridge University Press, Cambridge, 2006. zbl MR

[5] *L. E. Dickson*: History of the Theory of Numbers, Vol. I: Divisibility and Primality. Reprint of the 1919 original. Chelsea Publishing, New York, 1966. zbl MR

[6] *A. Einstein*: Method for the determination of the statistical values of observations concerning quantities subject to irregular fluctuations. Arch. Sci. Phys. et Natur. *37* (1914), 254–256.

[7] *H. G. Gadiyar, R. Padma*: Ramanujan-Fourier series, the Wiener-Khintchine formula and the distribution of prime pairs. Phys. A *269* (1999), 503–510. MR

[8] *S. W. Golomb*: The lambda method in prime number theory. J. Number Theory *2* (1970), 193–198. zbl MR

[9] *G. H. Hardy*: Goldbach's Theorem. (A lecture to the Math. Soc. of Copenhague on 6. October 1921). Mat. Tidsskr. B *1922* (1922), 1–16. zbl

[10] *G. H. Hardy*: Note on Ramanujan's trigonometrical function $c_q(n)$ and certain series of arithmetical functions. Cambr. Phil. Soc. Proc. *20* (1921), 263–271. zbl

[11] *G. H. Hardy*: Some Famous Problems of the Theory of Numbers and in Particular Waring's Problem. An inaugural lecture delivered before the University of Oxford. Clarendon Press, Oxford, 1920. zbl

[12] *G. H. Hardy, J. E. Littlewood*: Some problems of "Partitio numerorum" III: On the expression of a number as a sum of primes. Acta Math. *44* (1923), 1–70. zbl MR

[13] *G. H. Hardy, S. Ramanujan*: Asymptotic formulae in combinatory analysis. Lond. M. S. Proc. *17* (1918), 75–115; Collected Papers of Srinivasa Ramanujan. AMS Chelsea Publ., Providence (2000), 276–309. zbl MR

[14] *A. Khintchine*: Korrelationstheorie der stationären stochastischen Prozesse. Math. Ann. *109* (1934), 604–615. (In German.) zbl MR

[15] *C. Kittel*: Elementary Statistical Physics. John Wiley, New York, 1958. MR

[16] *H. Rademacher*: Topics in Analytic Number Theory (E. Grosswald et al., eds.). Die Grundlehren der mathematischen Wissenschaften. Band 169, Springer, New York, 1973. zbl MR

[17] *S. Ramanujan*: On certain trigonometrical sums and their applications in the theory of numbers. Trans. Cambridge Philos. Soc. *22* (1918), 259–276; Collected Papers of Srinivasa Ramanujan. AMS Chelsea Publ., Providence (2000), 179–199. zbl MR

[18] *D. R. Stinson*: Cryptography. Theory and Practice. Series on Discrete Mathematics and its Applications, CRC Press, Boca Raton, 1995. zbl MR

[19] *N. Wiener*: Generalized harmonic analysis. Acta Math. *55* (1930), 117–258. zbl MR

[20] *A. M. Yaglom*: Einstein's 1914 paper on the theory of irregularly fluctuating series of observations. ASSP Magazine, IEEE *4* (1987), 7–11.
[21] *Y. Zhang*: Bounded gaps between primes. Annals of Math. *179* (2014), 1121–1174.

Authors' address: H. Gopalakrishna Gadiyar, Ramanathan Padma, School of Advanced Sciences, V. I. T. University, Vellore 632014, India, e-mail: gadiyar@vit.ac.in, gopigadiyar@gmail.com, rpadma@vit.ac.in, rpadma@hotmail.com.