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1-Factor Forcing in Certain Interconnection Networks

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Abstract

A effective coloring of the vertices of a graph G starts with an initial subset S of colored vertices, with all residual vertices being uncolored. At each various time interval, a colored vertex with exactly one uncolored adjacent vertex forces this uncolored vertex to be colored. The initial set S is called a forcing set (zero forcing set) of G if, by iteratively applying the forcing process, every vertex in G becomes colored. If the set S has the added property that the subgraph induced by S is a perfect matching or 1-factor, then S is called a 1-factor forcing set of G. The 1-factor forcing number of G, denoted $\zeta_{P_2}(G)$, is the minimum cardinality of a 1-factor forcing set of G. In this paper, we introduce this new parameter namely the 1-factor forcing number and obtain the same for certain interconnection networks.

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Keywords: 1-factor forcing set; triangular graph; Pyrene; Ladder; Cycle of Ladder .

1. Introduction

In 2008 AIM Special Work Group [1] introduced zero forcing problem in graphs. The zero forcing have many applications in electrical power network, quantum physics and logic circuits [2]. In 2015, Davila et. al [3] extended forcing problem to total forcing problem in graphs depending on forcing sets as an induced subgraph. In the same paper they observed the lower bound and upper bound on the total forcing number for any graph G is $2 \leq \zeta_t(G) \leq \zeta_c(G)$[3].

In this paper, we introduce new parameter namely the 1-factor forcing in graphs depending on an induced subgraph as perfect matching. The problem of placing monitoring gadget in a system such that every vertex in the system (including monitoring gadget themselves) is adjacent to a monitor and every monitor is matched with a reinforcement monitor, can be designed by a 1-factor forcing in graphs. In this paper, we study optimal placement of PMUs of the vertices of an independent set of edges which can only force the vertices of the graph G.

For $v \in V(G)$, the open neighbourhood of v, denoted as $N(v)$, is the set of vertices adjacent with v; and the closed neighbourhood of v, denoted by $N[v]$, is $N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighbourhood of S is defined as $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is defined as $N[S] = N(S) \cup S$. Let $G(V, E)$ be a graph and let $S \subseteq V(G)$. We define the sets $M^i(S)$ of vertices monitored by S at level $i, i \geq 0$, inductively as follows:

1. $M^0(S) = N[S]$.
2. $M^{i+1}(S) = M^i(S) \cup \{w : \exists v \in M^i(S), N(v) \cap (V(G) \backslash M^i(S)) = w\}$.

If $M^\infty(S) = V(G)$, then the set S is called a power dominating set of G [7]. The minimum cardinality of a *power dominating set* in G is called the *power domination number* of G written $\gamma_p(G)$ [7].

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A *zero forcing set* of G is a subset of vertices S such that when the vertices in S are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. The zero forcing number, denoted $\zeta(G)$ of G is the minimum cardinality of a zero forcing set of G. If S is a zero forcing set of G with the additional property that the subgraph of G induced by S contains no isolated vertex, then S is a *total forcing set* of G. The total forcing number of G, denoted $\zeta_t(G)$, is the minimum cardinality of a total forcing set of G [3]. A matching M in G is a set of edges in G such that no two edges in M are adjacent. A matching M of G is a perfect matching if every vertex of G is incident to an edge of M. Thus a perfect matching in G is a 1-regular spanning subgraph of G. In the literature it is also known as a 1-factor of G. A zero forcing set S of G is a *1-factor forcing set* of G if the induced subgraph $G[S]$ has a perfect matching. The *1-factor forcing number* of G, denoted $\zeta_{P_2}(G)$, is the minimum number of edges in $G[S]$, where S is minimum.

Definition 1.1. [10] The r-ladder graph L of length r is defined as $P_2 \times P_{r+1}$, where P_{r+1} is a path on r + 1 vertices, r ≥ 1. The graph obtained looks like a ladder having two rails and $r + 1$ rungs between them. The length of the ladder is defined as r.

Definition 1.2. [5] Let T be the tree formed from a $K_{1,n}$ by subdividing any number of its edges any number of times; that is, T has at most one vertex of degree 3 or more. Then T is a spider tree, denoted by $S\ (n)$. See Figure 1(c).

Definition 1.3. [6] The subdivision graph $S\ (G)$ of the graph G is obtained from G by inserting a new vertex of degree 2 on each edge of G. The r-subdivision graph $S_r(G), r \geq 1$ is obtained from G by inserting r new vertices of degree 2 on each edge of G. Thus, $S_0(G) \cong G$ and $S_1(G) \cong S(G)$ and $S_r(G_1 \cup G_2) = S_r(G_1) \cup S_r(G_2),\ r \geq 1$.

2. Main Results

In this section, we obtain the 1-factor forcing number for cycle of ladder, triangular graph and pyrene network. In 2015 Ferrero et al. [7] obtained a relation between zero forcing set and power dominating set.

Stephen et al. [8] have given a lower bound for the power domination number for any graph G. We observe that the lower bound obtained for the power domination number for any graph G by Stephen et al. [8] holds for the forcing problem on any graph G and is quoted below.

Theorem 2.1. [8] $Let\ H_1, H_2, \ldots, H_r$ be pairwise disjoint subgraphs of G satisfying the following conditions.
1. $V(H_j) = V_1(H_j) \cup V_2(H_j)$ where $V_1(H_j) = \{u \in V(H_j)\backslash u \sim v$ for some $v \in V(G) - V(H_j)\}$ and $V_2(H_j) = \{u \in V(H_i)\backslash u \nsim v$ for all $v \in V(G) - V(H_i)\}$.
2. $V_2(H_j) \neq \emptyset$ and for each $u \in V_1(H_i)$, there exists at least 2 vertices in $V_2(H_j)$ which are adjacent to u.

If $V_1(H_j)$ is monitored and if t_j is the minimum number of vertices required to monitoring $V(H_j)$, then $\zeta(G) \geq \sum_{j=0}^{k} t_j$.

Before proceeding with proof of main results we observe the following:

Observations :

The following results hold good:
(a). For $n \geq 2, \zeta_{P_2}(P_n) = 1$.
(b) For $n \geq 3, \zeta_{P_2}(C_n) = 1$.
(c). Let G be a ladder $L_n, n \geq 2$ or Hexagonal chain $H_n, n \geq 2$. Then $\zeta_{P_2}(G) = 1$. See Figure 1(a) and 1(b).
(d). Let H be a subdivision of graph G . Then $\zeta_{P_2}(H) = 1$, where $\Delta(G) \leq 2$ or $G \cong P_n$ or $G \cong C_n$.

Theorem 2.2. Let T be a spider tree $S\ (n), n \geq 3$. Then $\zeta_{P_2}(G) \geq (n - 1)$.

Proof. Choose an edge in $S\ (n)$. Suppose propagation process reaches the hub vertex of degree ≥ 3. To continue the propagation process, one edge is to be selected in each branch except the last one. Then $\zeta_{P_2}(G) \geq (n - 1)$.

1-Factor Forcing Algorithm in Spider Tree
Input: Spider Tree $S(n), n \geq 3$.
Algorithm: Choose all the pendent edges in S . See Figure 1(c).
Output:
$\zeta_{P_2}(S(n)) = n - 1$.
Proof of correctness is obvious.

Theorem 2.3. Let G be a spider tree.
Then $\zeta_{P_2}(G) = n - 1$.

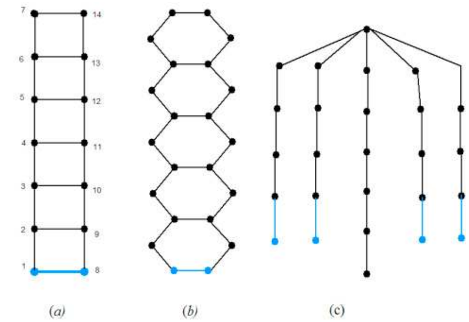

Figure 1. Blue colored edges indicates a minimum 1-factor forcing set of (a) Ladder L_7 (b) Hexagonal chain H_n (c) Spider S (4).

2.1. Cycle-of-Ladder

In 2008, Jywe-Fei Fang introduced a network called cycle-of-ladder and proved that it is a spanning subgraph of the hypercube network, thereby proving that hypercube network is bipancyclic [9].

Definition 2.4. [10] A cycle of ladder is a graph comprising of a cycle C_k of length $2l$ called the spine cycle such that removal of alternate edges on C_k leaves l components $L_1, L_2, \ldots, L_l$, each of which is isomorphic to a ladder. $If\ r_1, r_2, \ldots, r_l$ denote the number of rungs in the ladders $L_1, L_2, \ldots, L_l$ respectively, then the cycle of ladders is denoted by $CL(2l, r_1, r_2, \cdots, r_l)$. Let $R_i^j, 1 \leq i \leq r_l$ denote the number of rungs of L such that the bottom rung R_1^i is the edge of C_k in $L_i, 1 \leq i \leq k$. For brevity, we denote $(r_1, r_2, \cdots, r_k)$ as k and we denote the cycle-of-ladder as $CL(2l, k)$, where l and k represent the number of ladders and the length of each ladder respectively.

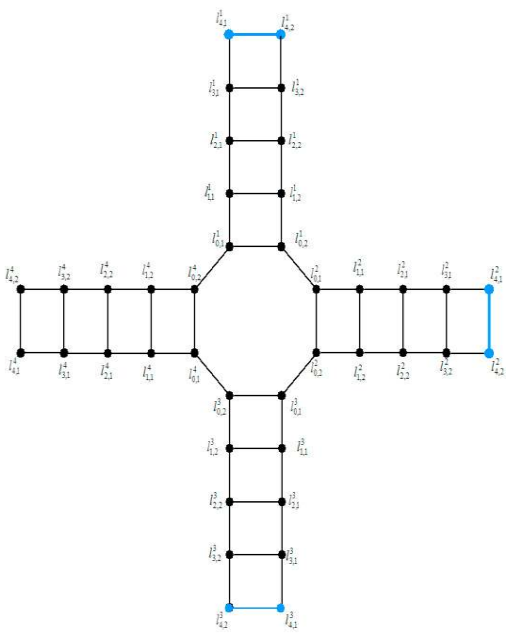

Figure 2. Blue colored edges indicates a minimum 1-factor forcing set of CL(8, 5).

Lemma 2.5. Let G be the cycle-of-ladder graph $CL(2l, k),\ l, k \geq 4$.
Then $\zeta_{P_2}(G) \geq l - 1$.

Proof. To initiate the propagation process, an edge e is chosen with at least one end vertex of e of degree 2. Select e from a ladder, say L_1. In the propagation process, even if both spine cycle vertices are colored as blue, they in turn force two spine vertices of two ladders. These spine vertices are adjacent to two other vertices halting the propagation process. Hence at least one edge from each of these two ladders are to be selected. Continue the process till the last ladder is reached. Thus $\zeta_{P_2}(G) \geq l - 1$.

1-Factor Forcing Set Algorithm in Cycle-of-Ladder CL(2l, k)

Input: Cycle-of-ladder $CL(2l, k),\ l, k \geq 4$.

Algorithm: Select the bottom rung edges from l − 1 ladder. See Figure 2.

Output:

$\zeta_{P_2}\big(CL(2l, k)\big) = l - 1$.

Proof of correctness is obvious.

Theorem 2.6. Let G be an cycle-of-ladder $CL(2l, k),\ l, k \geq 4$.Then $\zeta_{P2}(G) = l - 1$.

2.2. 1-Factor Forcing in Triangle Graph

Definition 2.7. [11] Let r be a positive integer. A triangle graph of order $r \geq 1$, TG_r, is defined as follows: Draw r rows of regular hexagons of the same size within an equilateral triangle (which is called the framework of (TG_r) so that the first row consists of one hexagon, the second row consists of two hexagons, and the r^{th} row consists of r hexagons. Set all the vertices of these hexagons is vertex set of TG_r and set all the sides of these hexagons is the edge set of TG_r.

The following lemma describes a critical subgraph H of G in the sense that H contains at least one edge of any 1-factor forcing set.

Lemma 2.8. Let G be a graph and H be a subgraph G as shown in Figure 3. Then H is a critical subgraph of G.

Proof. Suppose Row i, $1 \leq i \leq r$ does not contain any member of a 1-factor forcing set, then each vertices u_{ij}, $1 \leq j \leq r$ is adjacent to two uncolored vertices in Row i.

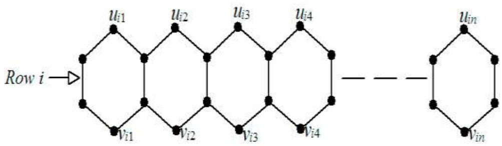

Figure 3. Critical subgraph *H* of G

Lemma 2.9. Let G be a triangle graph TG_r, $r \geq 5$. Then $\zeta_{P_2}(G) \geq r$.

Proof. In TG_r, there are r critical subgraphs, each isomorphic to H as described in Lemma 2.8. By taking H_j;'s f Theorem 2.1 as the subgraph H in G, we conclude $t_j = 1$. Therefore, $\zeta_{P_2}(G) \geq r$.

1-Factor Forcing Algorithm in Triangular Graph

Input: Triangular graph TG_r, $r \geq 5$.

Algorithm: Label the vertices of TG_r, $r \geq 5$ from 1 to $n^2 + 4n + 1$ sequentially from left to right, row wise beginning with the top most row. Consider r hexagons in the outer most layer of the TG_r. Let P^* denote the path induced by edges of the hexagons that are not boundary edges of any other hexagon. Select $\{(2,4),(3,5),(9,12),\ldots,((r+1)^2,(r+1)(r+2))\}$ as r independent edges of P^* in S, which are at distance 1 apart on P^*. See Figure 4(a).

Output: $\zeta_{P_2}(TG_r) = r$.

Proof of Correctness: Let S be the set of vertices labeled $\{2,3,4,5,9,12\ldots,(r+1)^2,(r+1)(r+2)\}$. See Figure 4(a). Then the vertices labeled as $\{2,3,12,20,36,42,\ldots,r(r-1),(r+1)^2,(r+1)(r+2)\}$ in S colored as blue are adjacent to exactly one uncolored vertex say, $\{1,16,\ldots,(r-1)^2,r(r+1),(r+1)^2+1\}$. These vertices can be colored as blue in the propagation step. In the next step vertex labeled say, {4} is adjacent to exactly one uncolored vertex say, $\{6\}$ colored as blue. Proceeding inductively, for every vertex v in TG_r colored as blue is adjacent to exactly one non-colored vertex, at every inductive step $i, i \geq 3$. Now S_r is a 1-factor forcing set of TG_r. This implies that, $\zeta_{P_2}(TG_r) = r$.

Theorem 2.10. Let G be a triangle graph TG_r, $r \geq 5$.
Then $\zeta_{P_2}(G) = r$.

2.3 1-Factor Forcing PyreneNetwork

Pyrene is an fluctuate polycyclic aromatic hydrocarbon (PAH) which includes four fused benzene rings, resulting in a huge flat aromatic system. It is a colorless or pale yellow solid which forms during incomplete combustion of organic materials and therefore can be isolated from coal tar together with a broad range of related compounds[12].

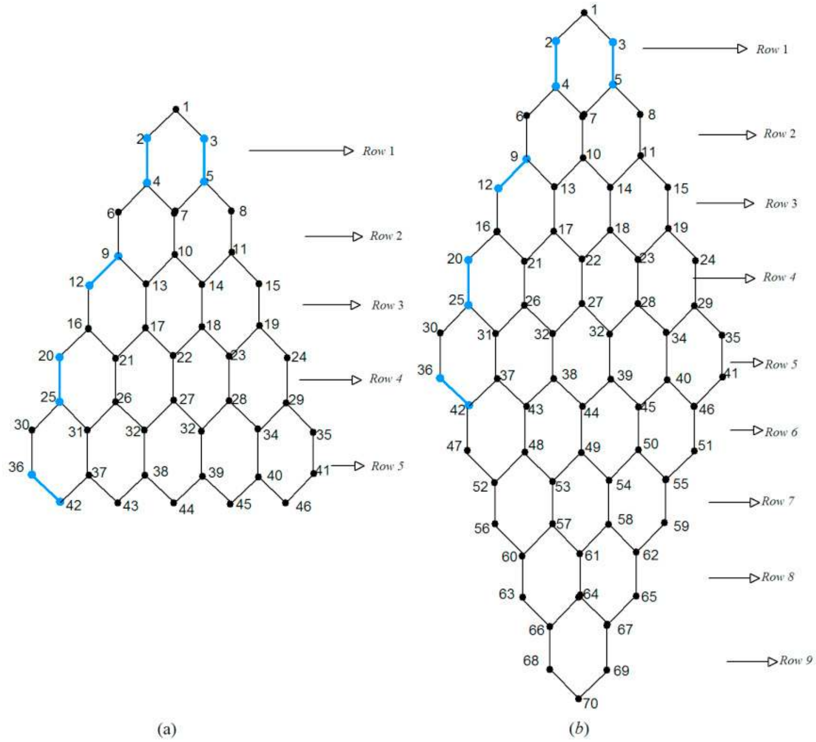

Figure 4. Blue colored edges constitutes (a) a minimum 1-factor forcing set of TG_4 (b) a minimum 1-factor forcing set of PY(4)

Theorem 2.11. Let G be a pyrene network PY(r), r ≥ 1. Then $\zeta_{P_2}(G) = r$.

Proof. The subgraph induced by Rows $r + 1, r + 2, \dots, 2r - 1$ is a mirror image of the subgraph induced by Rows $1, 2, \dots, r - 1$, with a mirror placed along Row r. The 1-factor forcing set of TG_r discussed in Theorem 2.10 is enough to monitor all vertices in PY(r).

3. Conclusion

In this paper, we have obtained the 1-factor forcing number for cycle of ladder, triangular graph and pyrene network.

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