International Journal of Pure and Applied Mathematics
Volume 114 No. 4 2017, 867-874
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v114i4.16

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A COMMON FIXED POINT THEOREM UNDER AN AUXILIARY FUNCTION

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Abstract: A generalization of a result of Badshah and Singh [1] was proved in [5] for a pair of compatible maps and dropping the continuity of one of the self-maps. A generalization of the result of [5] is obtained in this paper, by employing an auxiliary function.

AMS Subject Classification: 54H25

Key Words: metric space, auxiliary function, Cauchy sequence, complete metric space, common fixed point

1. Introduction

Badshah and Singh [1] proved the following result for commuting self-maps:

Theorem 1.1. *Let f and g be self-maps on a complete metric space X satisfying the inclusion*

$$f(X) \subset g(X) \tag{1}$$

and the inequality

Received: February 23, 2017
Revised: March 23, 2017
Published: June 7, 2017

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$$[d(fx,fy)]^2 \leq \alpha\,[d(fx,gx)d(fy,gy)+d(fy,gx)d(fx,gy)] \\ +\beta\,[d(fx,gx)d(fx,gy)+d(fy,gx)d(fy,gy)] \\ \text{for all} \quad x,y\in X, \tag{2}$$

where

(a) *α and β are nonnegative constants with $\alpha+2\beta<1$,*

(b) *(f,g) is a commuting pair,*

(c) *f and g are continuous.*

Then f and g have a unique common fixed point.

A generalization of Theorem 1.1 was obtained in [5], by dropping the continuity of f and using a compatible pair[1] (f,g) in (b) with the choice:

$$\lim_{n\to\infty} d(fgx_n, gfx_n)=0 \tag{3}$$

whenever $\langle x_n\rangle_{n=0}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \tag{4}$$

for some $t\in X$.

It is easy to observe that every commuting pair of self-maps is necessarily compatible. Converse is not true. For instance, see [2], [3] and [4].

The generalization proved in [5] is the following:

Theorem 1.2. *Let f and g be self-maps on a complete metric space X satisfying the inclusion* (1) *and the inequality* (2), *where α and β are nonnegative constants with $\alpha+2\beta<1$. If g is continuous, and (f,g) is a compatible pair, then f and g have a unique common fixed point.*

We prove a generalization of Theorem 1.2 by replacing (2) with a general inequality involving an auxiliary function.

[1]Compatible maps was introduced by Gerald Jungck [2] as a generalization of commuting maps

2. Preliminary Notations

Several fixed point theorems in metric space setting have been proved through contraction conditions involving different types of auxiliary functions. Given a positive integer α, a generalized class Φ_α of auxiliary functions was introduced in [6] as follows:

$$\Phi_\alpha = \{\phi : [0,\infty) \to [0,\infty) | \phi(0) = 0, \phi(\alpha t) < t \text{ for } t > 0\}. \tag{5}$$

It is obvious that, for $\alpha = 1$, Φ_α reduces to the class Ψ of all contractive moduli ψ [7] such that $\psi(0) = 0$ and $psi(t) < t$ for $t > 0$.

Definition 2.1. A mapping $\phi \in \Phi_\alpha$ is said to be upper semicontinuous at $t_0 \geq 0$ if $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t_0)$ whenever $\langle t_n \rangle_{n=1}^{\infty}$ is such that $\lim_{n\to\infty} t_n = t_0$, and ϕ is u.s.c if it is u.s.c. at every $t \geq 0$.

Our main result is

Theorem 2.1. *Let f and g be self-maps on a complete metric space X satisfying the inclusion* (1), *and the inequality*

$$\begin{aligned} [d(fx, fy)]^2 \quad \leq \quad & \phi(\max\{d(fx, gx)d(fy, gy) + d(fy, gx)d(fx, gy), \\ & d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy)\}) \\ & \quad for\ all \quad x, y \in X, \end{aligned} \tag{6}$$

where $\phi \in \Phi_2$ is nondecreasing and upper semicontinuous. If g is continuous, and (f, g) is a compatible pair, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary.

In view of (1), we can choose points $x_1, x_2, \ldots, x_n, \ldots$ in X inductively such that

$$fx_{n-1} = gx_n = y_n \quad \text{forall} \quad n \geq 1. \tag{7}$$

Writing $x = x_{n-1}$ and $y = x_n$ in (6) and using (7), we get

$$\begin{aligned} [d(y_n, y_{n+1})]^2 =& [d(fx_{n-1}, fx_n)]^2 \\ \leq& \phi(\max\{d(fx_{n-1}, gx_{n-1})d(fx_n, gx_n) + d(fx_n, gx_{n-1})d(fx_{n-1}, gx_n), \\ & d(fx_{n-1}, gx_{n-1})d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})d(fx_n, gx_n)\}) \\ =& \phi(\max\{d(y_n, y_{n-1})d(y_{n+1}, y_n), d(y_{n+1}, y_{n-1})d(y_{n+1}, y_n)\}) \\ \leq& \phi(d(y_n, y_{n+1})[d(y_n, y_{n-1}) + d(y_{n+1}, y_n)]) \end{aligned} \tag{8}$$

We now prove that

$$d(y_n, y_{n-1}) \geq d(y_{n+1}, y_n) \text{ for } n \geq 2. \tag{9}$$

If possible, suppose that $d(y_m, y_{m-1}) < d(y_{m+1}, y_m)$ for some $m \geq 2$. Then $d(y_{m+1}, y_m) > 0$. Since ϕ is nondecreasing, from (8) it follows that

$$0 < [d(y_{m+1}, y_m)]^2 \leq \phi(2[d(y_m, y_{m+1})]^2) < [d(y_{m+1}, y_m)]^2,$$

which is a contradiction. This proves (9). In other words, $\langle d(y_{n+1}, y_n)\rangle_{n=1}^{\infty}$ is a decreasing sequence of nonnegative real numbers and hence converges to some $t \geq 0$. Now using (9) in (8), we get

$$d(y_{n+1}, y_n) \leq \phi\big(d(y_{n+1}, y_n) + d(y_{n+2}, y_{n+1})\big) \leq \phi(2d(y_{n+1}, y_n)) \text{ for } n \geq 1.$$

Taking the limit superior as $n \to \infty$ in this and then using the upper semicontinuity of ϕ, we obtain that

$$t \leq \phi(2t). \tag{10}$$

If $t > 0$ in (10), then the choice of ϕ implies that $t \leq \phi(2t) < t$, which is a contradiction. Thus

$$t = \lim_{n\to\infty} d(y_{n+1}, y_n) = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \tag{11}$$

We now prove that $\langle y_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in X.

If possible we suppose that $\langle y_n\rangle_{n=1}^{\infty}$ is not Cauchy. Then for some $\epsilon > 0$, we choose sequences $\langle y_{m_k}\rangle_{k=1}^{\infty}$ and $\langle y_{m_k}\rangle_{k=1}^{\infty}$ of positive integers such that $m_k > n_k > k$ and

$$d(y_{m_k}, y_{n_k}) \geq \epsilon \text{ for } k = 1, 2, 3, ... \tag{12}$$

Suppose that m_k is the smallest integer exceeding n_k which satisfies (12). That is

$$d(y_{m_k-1}, y_{n_k}) < \epsilon. \tag{13}$$

Now by triangle inequality of d, we see that

$$\begin{aligned}\epsilon \leq d(y_{m_k}, y_{n_k}) &\leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k})\\ &< d(y_{m_k}, y_{m_k-1}) + \epsilon\end{aligned} \tag{14}$$

and from (11), we see that

$$\lim_{k\to\infty} d(y_{m_k-1}, y_{m_k}) = 0 \tag{15}$$

and

$$\lim_{k\to\infty} d(y_{n_k-1}, y_{n_k}) = 0 \tag{16}$$

Using (15) in (14), we get

$$\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = \epsilon. \tag{17}$$

Again, by the triaangle inequality of d, we get

$$d(y_{n_k-1}, y_{m_k}) \leq d(y_{n_k-1}, y_{n_k}) + d(y_{n_k}, y_{m_k}).$$

As $k \to \infty$ this in view of (16) and (17), gives

$$\lim_{k\to\infty} d(y_{n_k-1}, y_{m_k}) = \epsilon. \tag{18}$$

On the other hand, writing $x = x_{m_k-1}$, $y = x_{n_k-1}$ in (6), we have

$$\begin{aligned}[d(fx_{m_k-1}, fx_{n_k-1})]^2 \leq \phi(\max\{&d(fx_{m_k-1}, gx_{m_k-1})d(fx_{n_k-1}, gx_{n_k-1}) \\ &+ d(fx_{n_k-1}, gx_{m_k-1})d(fx_{m_k-1}, gx_{n_k-1}), \\ &d(fx_{m_k-1}, gx_{m_k-1})d(fx_{m_k-1}, gx_{n_k-1}) \\ &+ d(fx_{n_k-1}, gx_{m_k-1})d(fx_{n_k-1}, gx_{n_k-1})\})\end{aligned}$$

or

$$\begin{aligned}\epsilon^2 &\leq [d(y_{m_k}, y_{n_k})]^2 \\ &\leq \phi(\max\{d(y_{m_k}, y_{m_k-1}))d(y_{n_k}, y_{n_k-1}) + d(y_{n_k}, y_{m_k-1})d(y_{m_k}, y_{n_k-1}), \\ &\quad d(y_{m_k}, y_{m_k-1})d(y_{m_k}, y_{n_k-1}) + d(y_{n_k}, y_{m_k-1})d(y_{n_k}, y_{n_k-1})\})\end{aligned} \tag{19}$$

Since ϕ is nondecreasing, proceeding the limit as $n \to \infty$ in this, and then using upper semicontinuity of ϕ, (13), (15), (16),(17) and (18) we get

$$0 < \epsilon^2 \leq \phi\big(\max\{0 + \epsilon^2, 0\}\big) = \phi(\epsilon^2) \leq \phi(2\epsilon^2) < \epsilon^2,$$

which is a contradiction. Hence $\langle y_n \rangle_{n=1}^{\infty}$ must be a G-Cauchy sequence in X.

Since (X, G) is G-Complete, there exists a point $p \in X$ such that $\langle y_n \rangle_{n=1}^{\infty}$ is G-convergent to p. That is

$$\lim_{n\to\infty} y_{n-1} = \lim_{n\to\infty} y_n = p. \tag{20}$$

Now the compatibility of f and g, and (20) imply that

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0, \tag{21}$$

while the sequential property of the continuiy of g and (20) give

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} g^2 x_n = gz. \tag{22}$$

Hence it follows from (21) and (22), that

$$\lim_{n\to\infty} d(fgx_n, gz) = 0 \quad \text{or} \quad \lim_{n\to\infty} fgx_n = gz. \tag{23}$$

But the use of (6) yields

$$\begin{aligned} [d(fgx_n, fz)]^2 \quad \leq \quad & \phi(\max\{d(fgx_n, g^2x_n)d(fz, gz) + d(fz, g^2x_n)d(fgx_n, gz), \\ & d(fgx_n, g^2x_n)d(fgx_n, gz) + d(fz, g^2x_n)d(fz, gz)\}). \end{aligned}$$

Applying the limit as $n \to \infty$ in this, and using (22) and (23), we obtain that

$$\begin{aligned} [d(gz, fz)]^2 \quad \leq \quad & \phi(\max\{d(gz, gz)d(fz, gz) + d(fz, gz)d(gz, gz), \\ & d(gz, gz)d(gz, gz) + d(fz, gz)d(fz, gz)\}). \end{aligned}$$

or

$$[d(gz, fz)]^2 \leq \phi([d(fz, gz)]^2).$$

If $fz \neq gz$, then the nondecreasing nature of ϕ would lead to a contradiction that

$$0 < [d(gz, fz)]^2 \leq \phi([d(fz, gz)]^2) \leq \phi(2\,[d(fz, gz)]^2) < [d(fz, gz)]^2 .$$

Hence we must have

$$gz = fz. \tag{24}$$

Finally from (6), we see that

$$[d(fx_n, fz)]^2 \leq \phi(\max\{d(fx_n, gx_n)d(fz, gz) + d(fz, gx_n)d(fx_n, gz),$$

$$d(fx_n, gx_n)d(fx_n, gz) + d(fz, gx_n)d(fz, gz).$$

The limiting case of this as $n \to \infty$, (20), and (22) would imply that

$$[d(z, fz)]^2 \leq \phi([d(fz, z)]^2),$$

which with a similar argument as above yields that $d(z, fz) = 0$ or $fz = z$. Thus z is a common fixed point of f and g.

The uniqueness of the common fixed point follows easily from (6). □

Remark 2.1. Theorem 2.1 does not require the continuity of f.

Since every commuting pair is compatible, writing $\phi(t) = qt$ for all $t \geq 0$, where $q < 1/2$, we obtain

Corollary 2.1. *Let f and g be self-maps on a complete metric space X satisfying the inclusion* (1)*, and the inequality*

$$\begin{aligned}[d(fx, fy)]^2 \leq q \max\{&d(fx, gx)d(fy, gy) + d(fy, gx)d(fx, gy), \\ &d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy)\} \\ &for\ all \quad x, y \in X, \end{aligned} \tag{25}$$

If g is continuous, and (f, g) is a commuting, then f and g have a unique common fixed point.

Choosing α and β such that $\alpha + 2\beta < 1/2$, then it is easily seen that the right hand side of (2) is less than or equal to the right hand side of (25), where $r = \alpha + 2\beta$. Thus Theorem 1.2 will become a particular case of Corollary 2.1.

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