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A GENERALIZED COMMON FIXED POINT THEOREM UNDER AN IMPLICIT RELATION

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Abstract. An extended generalization of recent result of Kikina and Kikina (2011) has been established through the notions of weak compatibility and the property E.A., under an implicit-type relation and restricted orbital completeness of the space. The result of this paper also extends and generalizes that of Imdad and Ali (2007).

1. Introduction

Let (X, d) be a metric space with at least two points. We denote by fx, the image of $x \in X$ under a self-map f on X and by fg, the composition of self-maps f and g on X. Given $x_0 \in X$ and f, g and h self-maps on X, the associated sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice

$$(1.1)\quad x_{3n-2} = fx_{3n-3}, x_{3n-1} = gx_{3n-2}, x_{3n} = hx_{3n-1} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

is an (f, g, h)-*orbit* at x_0. An *associated sequence* involving two self-maps was earlier found in [8]. The metric space X is (f, g, h)-*orbitally complete* [5] if every Cauchy sequence in the (f, g, h)-orbit at each $x_0 \in X$ converges in X.

With this notion, Kikina and Kikina [5] proved the following

Theorem 1.1. *Let f, g and h be self-maps on X satisfying the three conditions:*

$$(1.2)\quad \begin{aligned}[1 + pd(x, y)]d(fx, gy) \le\ & p[d(x, fx)d(y, gy) + d(x, gy)d(y, fx)] \\ & + q \max\Big\{d(x, y), d(x, fx), d(y, gy), \\ & \qquad \frac{1}{2}[d(x, gy) + d(y, fx)]\Big\},\end{aligned}$$

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$$[1 + pd(x,y)]d(gx,hy) \le p[d(x,gx)d(y,hy) + d(x,hy)d(y,gx)] + q\max\left\{d(x,y), d(x,gx), d(y,hy), \frac{1}{2}[d(x,hy) + d(y,gx)]\right\}, \tag{1.3}$$

$$[1 + pd(x,y)]d(hx,fy) \le p[d(x,hx)d(y,fy) + d(x,fy)d(y,hx)] + q\max\left\{d(x,y), d(x,hx), d(y,fy), \frac{1}{2}[d(x,fy) + d(y,hx)]\right\}, \tag{1.4}$$

for all $x, y \in X$, *where* $p > -\dfrac{1}{\max\{d(x,y):\ x,y \in X\}}$ *and* $0 \le q < 1$.
If X *is* (f,g,h)*-orbitally complete, then* f, g *and* h *will have a unique common fixed point.*

It may be noted that if $\max\{d(x,y) : x,y \in X\} = 0$, then X reduces to a singleton space which is against its choice. Thus the choice of p is meaningful.

In this paper, we first extend the notion of orbital completeness of Kikina and Kikina [5] and then prove an extended generalization of Theorem 1.1 through weak compatibility and the property E.A., under certain implicit-type relation and the restricted orbital completeness of the metric space (see the next Section).

2. Preliminaries and notation

As a weaker version of commuting mappings, Gerald Jungck [2] introduced *compatible* self-maps f and r on X, which satisfy the asymptotic condition

$$\lim_{n\to\infty} d(frx_n, rfx_n) = 0, \tag{2.1}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = p \quad \text{for some} \quad p \in X. \tag{2.2}$$

It is interesting to note that if $x_n = x$ for all n, from the compatibility of f and r, it follows that $frx = rfx$ whenever $fx = rx$. That is, the compatible pair (f,r) commute at their coincidence point p. Self-maps which commute at their coincidence points are called *weakly compatible* [3]. However, there can be weakly compatible self-maps which are not compatible [3]. In this context, we see that the noncompatibility of (f,r) ensures the existence of a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X with the choice (2.2) but $\lim_{n\to\infty} d(fx_n, rx_n) \ne 0$ or $+\infty$. Motivated by this idea, Aamri and Moutawakil [1] introduced the notion of *property E.A.* In fact, self-maps f and r on X satisfy the *property E.A.* if (2.2) holds good for some $\langle x_n \rangle_{n=1}^{\infty} \subset X$, where the common limit p is known as a *tangent point*. However, weak compatibility and property E.A. are independent of each other [7], though both are weaker conditions of the compatibility.

As an extension property E.A. to more than two self-maps, Akkouchi and Popa [6] defined a *class C of self-maps satisfying property E.A.* if there is a $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that $\lim_{n\to\infty} fx_n = p$ for some $p \in X$ for each $f \in C$.

Now we extend orbital completeness as follows:

Given $x_0 \in X$ and f, g, h and r self-maps on X, if there exist points $x_1, x_2, x_3, \dots$ in X such that

$$fx_{3n-3} = rx_{3n-2}, gx_{3n-2} = rx_{3n-1}, hx_{3n-1} = rx_{3n} \quad \text{for} \quad n = 1, 2, 3, \dots, \tag{2.3}$$

then the associated sequence $\langle rx_n \rangle_{n=1}^{\infty}$ is an (f, g, h)-*orbit* at x_0 *relative* to r. The space X is (f, g, h)-*orbitally complete* at x_0 *relative* to r if every Cauchy sequence in an (f, g, h)-orbit at x_0 *relative* to r converges in X, and X is (f, g, h)-*orbitally complete* relative to r if it is (f, g, h)-orbitally complete at each $x_0 \in X$ relative to r.

The notion of *implicit-type relations* were fisrt introduced by Popa [10] to cover several contractive conditions and unify fixed point theorems. For instance, $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ is a lower semicontinuous function such that

(C_1) ψ is nonincreasing in the fifth and sixth coordinate variables,
(C_2) there is a constant $0 \leq \omega < 1$ such that for every $l \geq 0, m \geq 0$,

$$\psi(l, m, m, l, l+m, 0) \leq 0 \text{ or } \psi(l, m, l, m, 0, l+m) \leq 0 \;\Rightarrow\; l \leq \omega m, \tag{2.4}$$

and
(C_3) $\psi(l, l, 0, 0, l, l) > 0$, for all $l > 0$.

We shall utilize this without (C_1). Also we note that (2.4) is trivial if $l = 0$ for any $m \geq 0$, while if $m = 0$, (2.4) implies that $l = 0$. Therefore, we modify (C_2) and represent $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ with new labelings as follows:

(P_a) $\psi(l, 0, 0, l, l, 0) > 0$, for all $l > 0$,
(P_b) $\psi(l, 0, l, 0, 0, l) > 0$, for all $l > 0$,
(P_c) $\psi(l, l, 0, 0, l, l) > 0$, for all $l > 0$.

3. Main result and discussion

Our main result is

Theorem 3.1. *Let f, g, h and r be self-maps on X satisfying the property E.A. For all $x, y \in X$, suppose that any two of the following inequalities hold good:*

$$\psi(d(fx,gy),d(rx,ry),d(rx,fx),d(ry,gy),d(rx,gy),d(ry,fx)) \leq 0, \tag{3.1}$$

$$\psi(d(gx,hy),d(rx,ry),d(rx,gx),d(ry,hy),d(rx,hy),d(ry,gx)) \leq 0, \tag{3.2}$$

$$\psi(d(hx,fy),d(rx,ry),d(rx,hx),d(ry,fy),d(rx,fy),d(ry,hx)) \leq 0. \tag{3.3}$$

Suppose that $r(X)$ is (f,g,h)-orbitally complete relative to r. If r is weakly compatible with any one of f, g and h, then all the four maps f, g, h and r will have a common coincidence point, which will also be their common fixed point. Further, the common fixed point is unique.

Proof. Suppose f, g, h and r satisfy the property E.A. Then we can find a $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} rx_n = u, \quad \text{for some} \quad u \in X. \tag{3.4}$$

Since $r(X)$ is (f,g,h)-orbitally complete relative to r, we see that $u \in r(X)$ or

$$u = rp, \quad \text{for some} \quad p \in X. \tag{3.5}$$

Since the assumption that r is weakly compatible with any one of f, g and h involves cyclical invariance, it is enough to prove the result when (f,r) is weakly compatible under any two of the inequalities (3.1)–(3.3). We indeed consider two subcases:

Case (1). Either [(3.1), (3.2)] or [(3.1), (3.3)] hold good:
First we see that

$$fp = rp. \tag{3.6}$$

If possible, we assume that $fp \neq rp$ so that $d(rp,fp) > 0$. Then writing $x = p$ and $y = x_n$ in (3.1), we get

$$\psi(d(fp,gx_n),d(rp,rx_n),d(rp,fp),d(rx_n,gx_n),d(rp,gx_n),d(rx_n,fp)) \leq 0.$$

Applying the limit as $n \to \infty$ and then using (3.4), (3.5) and lower semicontinuity of ψ, we get

$$\psi(d(fp,rp),0,d(rp,fp),0,0,d(rp,fp)) \leq 0.$$

This contradicts the choice (P_b). Therefore (3.6) must hold good.

Since f and r commute at the coincidence point p, it follows that $frp = rfp$ or

$$fu = ru, \tag{3.7}$$

in view of (3.5).

Again, (3.1) with $x = y = u$ and (3.7) gives

$$\psi(d(fu, gu), d(ru, ru), d(ru, fu), d(ru, gu), d(ru, gu), d(ru, fu)) \leq 0,$$

or

$$\psi(d(fu, gu), 0, 0, d(fu, gu), d(fu, gu), 0) \leq 0,$$

which will contradict with (P_a) if $d(fu, gu) > 0$. Hence

$$0 \leq d(fu, gu) \leq 0 \quad \text{or} \quad fu = gu.$$

Suppose that (3.2) holds good. With $x = u = y$, this gives

$$\psi(d(gu, hu), d(ru, ru), d(ru, gu), d(ru, hu), d(ru, hu), d(ru, gu)) \leq 0$$

or that $\psi(d(gu, hu), 0, 0, d(fu, hu), d(ru, hu), 0) \leq 0$, due to (3.7) and $fu = gu$.

This again contradicts (P_a) if $d(gu, hu) > 0$ so that $d(gu, hu) = 0$.

Thus u is a common coincidence point of f, g, h and r, that is

$$fu = gu = hu = ru. \tag{3.8}$$

On the other hand, if (3.3) holds good, then writing $x = y = u$ in this, followed by (3.7) and $fu = gu$, and proceeding as above, we get $gu = hu$ and hence (3.8).

We see below that u is a fixed point of f. In fact, (3.1) with $x = u$ and $y = x_n$ gives

$$\psi(d(fu, gx_n), d(ru, rx_n), d(ru, fu), d(rx_n, gx_n), d(ru, gx_n), d(rx_n, fu)) \leq 0.$$

Applying the limit as $n \to \infty$ and using (3.8) and lower semicontinuity of ψ, we obtain

$$\psi(d(fu, u), d(fu, u), 0, 0, d(fu, u), d(fu, u)) \leq 0. \tag{3.9}$$

This would contradict (P_c) if $d(fu, u) > 0$, proving that $d(fu, u) = 0$ or $fu = u$. This, together with (3.8) implies that u is a common fixed point of f, g, h and r.

Case (2). The inequalities (3.2) and (3.3) hold good:

Writing $x = x_n$ and $y = p$ in (3.3), we get

$$\psi(d(hx_n, fp), d(rx_n, rp), d(rx_n, hx_n), d(rp, fp), d(rx_n, fp), d(rp, hx_n)) \leq 0.$$

Applying the limit as $n \to \infty$ and then using (3.4), (3.5) and the lower semi-continuity of ψ, we get

$$\psi(d(rp, fp), 0, 0, d(rp, fp), d(rp, fp), 0) \leq 0.$$

This gives a contradiction to (P_a) if $d(rp, fp) > 0$. Hence $d(rp, fp) = 0$ or $rp = fp = u$ and (3.7) follows, since (f, r) are weakly compatible.

Again from (3.3) with $x = u = y$ and (3.7), we see that

$$\psi(d(hu, fu), d(ru, ru), d(ru, hu), d(ru, fu), d(ru, fu), d(ru, hu)) \leq 0$$

$$\text{or}\quad \psi(d(hu,fu),0,d(fu,hu),0,0,d(fu,hu)) \le 0,$$

which would be against the choice (P_b) if $d(fu,hu) > 0$.

This shows that $fu = hu$.

But then, (3.2) with $x = u = y$ and (3.7) imply that

$$\psi(d(gu,fu),0,d(fu,gu),0,0,d(fu,gu)) \le 0,$$

which again will contradict (P_b) if $fu \ne gu$.

Thus $fu = gu$ and again (3.8) follows.

Finally with $x = x_n$ and $y = u$, (3.3) becomes

$$\psi(d(hx_n,fu),d(rx_n,ru),d(rx_n,hx_n),d(ru,fu),d(rx_n,fu),d(ru,hx_n)) \le 0.$$

In the limit as $n \to \infty$, this together with (3.8) gives

$$\psi(d(u,fu),d(u,fu),0,0,d(u,fu),d(fu,u)) \le 0,$$

which would be a contradiction to the choice (P_c) if $d(fu,u) > 0$. Hence $d(fu,u) = 0$, that is u is a fixed point of f and hence a common fixed point of f, g, h and r, by virtue of (3.8). ∎

It is well-known that the identity map i on X commutes with every map s on X. Hence (i,s) is weakly compatible. Therefore, taking $r = i$, the identity map on X in Theorem 3.1, we get

Corollary 3.1. *Let f, g and h be self-maps on X satisfying any two of the following inequalities:*

(3.10) $\psi(d(fx,gy),d(x,y),d(x,fx),d(y,gy),d(x,gy),d(y,fx)) \le 0,$

(3.11) $\psi(d(gx,hy),d(x,y),d(x,gx),d(y,hy),d(x,hy),d(y,gx)) \le 0,$

(3.12) $\psi(d(hx,fy),d(x,y),d(x,hx),d(y,fy),d(x,fy),d(y,hx)) \le 0,$

for all $x, y \in X$. If f, g, h and i satisfy the property E.A. and X is (f,g,h)-orbitally complete, then f, g and h will have a unique common fixed point.

Now we show that Corollary 3.1 is a significant generalization of Theorem 1.1:

First we write

$$\psi(l_1,l_2,l_3,l_4,l_5,l_6) = (1+pl_2)l_1 - p(l_3l_4 + l_5l_6) - q\max\left\{l_2,l_3,l_4,\frac{l_5+l_6}{2}\right\},$$

where p and q have the same choice as given in Theorem 1.1.

Then ψ is lower semicontinuous,

$$(P_a)\ \psi(l,0,0,l,l,0) = (1+p\cdot 0)l - p(0\cdot l + l\cdot 0) - q\max\left\{0,0,l,\frac{l+0}{2}\right\}$$
$$= (1-q)l > 0, \quad \text{for all} \quad l > 0,$$

$$(P_b)\ \psi(l,0,l,0,0,l) = (1+p\cdot 0)l - p(l\cdot 0 + 0\cdot l) - q\max\left\{0,l,0,\frac{0+l}{2}\right\}$$
$$= (1-q)l > 0, \quad \text{for all} \quad l > 0,$$

and

$$(P_c)\ \psi(l,l,0,0,l,l) = (1+p\cdot l)l - p(0\cdot 0 + l\cdot l) - q\max\left\{l,0,0,\frac{l+l}{2}\right\}$$
$$= (1-q)l > 0, \quad \text{for all} \quad l > 0.$$

Thus (1.1)–(1.1) are particular cases of the relations (3.10)–(3.12).

Let $x_0 \in X$ be arbitrary. From the proof of Theorem 1.1, it follows that $\langle x_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is (f,g,h)-orbitally complete, $x_n \to z$ for some $z \in X$. That is,

$$\lim_{n\to\infty} fx_{3n-3} = \lim_{n\to\infty} gx_{3n-2} = \lim_{n\to\infty} hx_{3n-1} = z.$$

Now let $\lim_{n\to\infty} fx_{3n-2} = \xi$. Writing $x = y = x_{3n-2}$ in (1.1), we get

$$\begin{aligned}
&[1 + pd(x_{3n-2},x_{3n-2})]d(fx_{3n-2},gx_{3n-2}) \\
&\le p[d(x_{3n-2},fx_{3n-2})d(x_{3n-2},gx_{3n-2}) + d(x_{3n-2},gx_{3n-2})d(x_{3n-2},fx_{3n-2})] \\
&\quad + q\max\Big\{d(x_{3n-2},x_{3n-2}), d(x_{3n-2},fx_{3n-2}), d(x_{3n-2},gx_{3n-2}), \\
&\qquad\qquad \frac{1}{2}[d(x_{3n-2},gx_{3n-2}) + d(x_{3n-2},fx_{3n-2})]\Big\}.
\end{aligned}$$

Applying the limit as $n \to \infty$, using the choice of ξ and then simplifying, we get $d(\xi,z) \le qd(z,\xi)$ so that $\xi = z$.

Similarly, if $\lim_{n\to\infty} hx_{3n-2} = \tau$, using (1.1) with $x = y = x_{3n-2}$ in the limit as $n \to \infty$ gives $\tau = z$. In other words,

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} hy_n = z,$$

where $y_n = x_{3n-2}$, proving that the triad (f,g,h) satisfies the property E.A., and a unique common fixed point can be ensured by Corollary 3.1.

It is remarkable that Theorem 1.1 employs all the three conditions (1.1)–(1.1), while Corollary 3.1 uses only two out of three at a time.

Corollary 3.2. *Let f and r be self-maps on X satisfying the property E.A. and the inequality*

$$\psi(d(fx,fy), d(rx,ry), d(rx,fx), d(ry,fy), d(rx,fy), d(ry,fx)) \le 0, \tag{3.13}$$

for all $x, y \in X$. If $r(X)$ is f-orbitally complete relative to r, then f and r will have a coincidence point. Further, if (f,r) is weakly compatible, then f and r will have a unique common fixed point.

Proof. We set $h = g = f$ in Theorem 3.1, we get a particular case of each of (3.1)–(3.3) as (3.13). Also the space X reduces to *f-orbitally complete* relative to r [9] in the sense that every Cauchy sequence in the (f, r)-orbit $O_{f,r}(x_0)$ at each x_0 converges in X, where $O_{f,r}(x_0)$ has the choice : $fx_{n-1} = rx_n$ for $n = 1, 2, 3, \ldots$. ∎

Since every complete metric space is f-orbitally complete relative to r [9], we immediately have

Corollary 3.3. (Theorem 3.1, [4]) *Let f and r be self-maps on X satisfying the property E.A. and the inequality* (3.13)*. If $r(X)$ is complete, then f and r will have a coincidence point. Further, f and r will have a unique common fixed point, provided (f, r) is weakly compatible.*

Imdad and Ali [4] asserted that the completeness of $r(X)$ is necessary to obtain a coincidence point for f and r through the following example:

Example 3.1. (Example 5.2, [4]) Let

$$F(l_1, l_2, l_3, l_4, l_5, l_6) = l_1^2 - al_2^2 - \frac{bl_5 l_6}{l_3^2 + l_4^2 + 1},$$

where $a = 1/2$ and $b = 1/4$. Set $X = \left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$ with the usual metric d. Define $f, r : X \to X$ by

$$f0 = \frac{1}{2^2}, f\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^{n+1}} \quad \text{and} \quad r0 = \frac{1}{2}, r\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n},$$

for $n = 1, 2, 3, \ldots$. Then (f, r) satisfies the property E.A. and

$$r(X) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}.$$

For $x_0 = 0$, choose $x_1 = \frac{1}{2}, x_2 = \frac{1}{2^2}, x_3 = \frac{1}{2^3}, \ldots$ so that

$$O_{f,r}(x_0) = \left\{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\right\},$$

while for $x_0 = \frac{1}{2^{n-1}}$, we have

$$O_{f,r}(x_0) = \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \ldots\right\},$$

for each $n = 1, 2, 3, \ldots$. In either case, $O_{f,r}(x_0)$ converges to $0 \notin r(X)$.

Thus $r(X)$ is not orbitally complete at each x_0. As such, the maps f and r do not have a coincidence point, even though X is complete.

In view of this example, it is more appropriate to assert that the orbital completeness of $r(X)$, rather than its completeness, is necessary for the existence of a coincidence point for f and r. In other words, orbital completeness of $r(X)$ is necessary for the existence of a coincidence point for f and r in Corollary 3.2.

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References

[1] M. A. Aamri, D. El. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. 270 (2002), 181–188.
[2] G. Jungck, *Compatible maps and common fixed points*, Internat J. Math. Math. Sci. 9 (1986), 771–779.
[3] G. Jungck, B. E. Rhoades, *Fixed point for set valued functions without continuity*, Indian J. Pure Appl. Math. 29 (1998), 227–238.
[4] M. Imdad, J. Ali, *Jungck's common fixed point theorem and* E.A. *property*, Acta Math. Sinica (English Ser.) 24 (2008), 87–94.
[5] L. Kikina, K. Kikina, *Fixed points in k-complete metric spaces*, Demonstratio Math. 44 (2011), 349–357.
[6] M. Akkouchi, V. Popa, *Well-posedness of a common fixed point problem for three mappings under strict contractive conditions*, Buletin. Univers. Petrol-Gaze din Ploiesti, Seria Math. Inform. Fiz. 61 (2009), 1–10.
[7] H. K. Pathak, Y. J. Cho, S. M. Kang, *Remarks on R-weakly commuting mappings and common fixed point theorems*, Bull. Korean Math. Soc. 17 (1997), 247–257.
[8] T. Phaneendra, *Contractive modulus and common fixed point*, Math. Ed. 38(4) (2004), 210–212.
[9] T. Phaneendra, *Coincidence points of two weakly compatible self-maps and common fixed point theorem through orbits*, Indian J. Math. 46(2–3) (2004), 173–180.
[10] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math. 32 (1999), 157–163.

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