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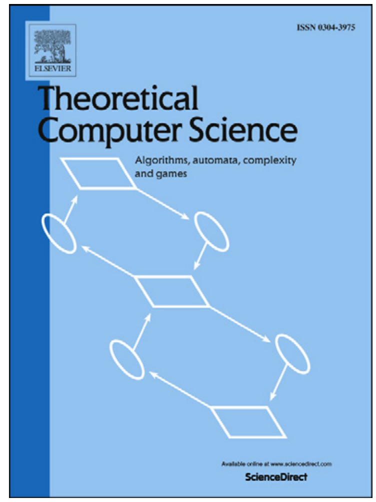

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PII: S0304-3975(14)00356-9
DOI: 10.1016/j.tcs.2014.05.007
Reference: TCS 9733

To appear in: *Theoretical Computer Science*

Received date: 6 January 2014
Revised date: 3 April 2014
Accepted date: 10 May 2014

Please cite this article in press as: R. Sundara Rajan et al., A linear time algorithm for embedding hypercube into cylinder and torus, *Theor. Comput. Sci.* (2014), http://dx.doi.org/10.1016/j.tcs.2014.05.007

A Linear Time Algorithm for Embedding Hypercube into Cylinder and Torus

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Abstract

In this paper we solve two conjectures proposed by Manuel et al. (Discret. Appl. Math. 159(17): 2109–2116, 2011) to obtain exact wirelength of embedding an r-dimensional hypercube into cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$ and torus $C_{2^{r_1}} \times C_{2^{r_2}}$, where $r_1 + r_2 = r$ and $r_1 \leq r_2$. We provide a linear time algorithm to compute the exact wirelength of embedding hypercube into cylinder and torus. Further we extend the result for higher dimensional cylinder and torus.

Keywords: Embedding, wirelength, hypercube, cylinder, torus

1 Introduction

In recent years, among many interconnection networks, the hypercube has been the focus of many researchers due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [1]. Hypercubes are known to simulate other structures such as grids and binary trees [2, 3].

Graph embedding is an important technique that maps a logical graph into a host graph, usually an interconnection network. Many applications can be modeled as graph embedding [4, 5, 6, 7, 8]. The quality of an embedding can be measured by certain cost criteria. One of these criteria is the *wirelength*. The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [6, 9].

Graph embeddings have been well studied for hypercubes into grids [2], meshes into crossed cubes [10], meshes into locally twisted cubes [11], meshes into faulty crossed cubes [12], generalized ladders into hypercubes [13], rectangular grids into hypercubes [14], rectangular grids into hypercubes [15], grids into grids [16], binary trees into grids [17], meshes into möbius cubes [18], tori and grids into twisted cubes [19], hypercube into n-dimensional grid [20].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [21, 22]. But the Congestion Lemma and the Partition Lemma [2, 7] have enabled the computation of exact wirelength for embeddings of various architectures [2, 7, 10, 11, 20, 23]. In fact the technique focuses on specific partitioning of the edge set of the host graph. It is interesting to note that not all host graphs can be partitioned to apply the Partition Lemma. In this paper, we overcome this difficulty partially by retaining a set of edges on which minimum wirelength is computed using Partition Lemma and compute minimum congestion on the rest of the edges using various other procedures.

*This work is supported by National Board of Higher Mathematics(NBHM), Department of Atomic Energy, Government of India.

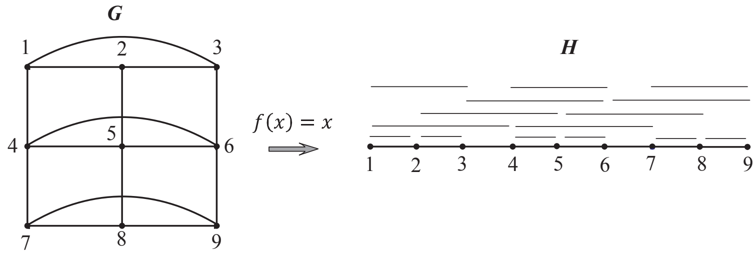

Figure 1: Wiring diagram of a cylinder G into path H with $WL_f(G,H) = 30$

Definition 1.1. [21] *Let G and H be finite graphs. An embedding f of G into H is defined as follows:*

1. *f is a one-to-one map from $V(G) \to V(H)$*
2. *P_f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.*

The *expansion* of an embedding f is the ratio of the number of vertices of H to the number of vertices of G. In this paper, we consider embeddings with expansion one.

Definition 1.2. [21] *The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(e)$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between the vertices $f(u)$ and $f(v)$ in H. In other words,*

$$EC_f(e) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}|$$

where $P_f(u,v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G,H)$ is the minimum, over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [24]. See Figure 1.

Definition 1.3. [2] *The wirelength of an embedding f of G into H is given by*

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} |P_f(u,v)| = \sum_{e\in E(H)} EC_f(e)$$

where $|P_f(u,v)|$ denotes the length of the path $P_f(u,v)$ in H.

The wirelength of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

The *wirelength problem* [2, 17, 21, 22, 24] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$. The isoperimetric problem [25] has been used as a powerful tool in the computation of exact wirelength of graph embeddings. The problem is to determine a subset A of vertices of a graph G such that $\theta_G(A) = \theta_G(m)$, where $\theta_G(A) = \{(u,v) \in E : u \in A, v \notin A\}$ and for a given m, $\theta_G(m) = \min\limits_{B\subseteq V,\ |B|=m} |\theta_G(B)|$. Such subsets are called optimal [25, 28].

The maximum subgraph problem [25] is to find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A \subseteq V,\, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$. The maximum subgraph problem is NP-complete [26]. When G is regular, the isoperimetric problem is equivalent to the maximum subgraph problem.

Lemma 1.4. (Congestion Lemma) [2, 7] *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge (a,b) in G_i, $i = 1,2$, $P_f(a,b)$ has no edges in S.*

(ii) *For every edge (a,b) in G with $a \in V(G_1)$ and $b \in V(G_2)$, $P_f(a,b)$ has exactly one edge in S.*

(iii) *$V(G_1)$ is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = \sum_{e \in S} EC_f(e) = r\,|V(G_1)| - 2\,|E(G_1)|$.

Lemma 1.5. (Partition Lemma) [2, 7] *Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

Definition 1.6. [6] *For $r \geq 1$, let Q^r denote the r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary strings. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit. The vertices of Q^r can also be identified with integers $0, 1, \ldots, n-1$.*

Definition 1.7. [27] *An incomplete hypercube on i vertices of Q^r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i, $1 \leq i \leq 2^r$.*

Theorem 1.8. [28, 29, 30] *Let Q^r be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, L_i is an optimal set on i vertices.*

Lemma 1.9. [23] *For $i = 1, 2, ..., r-1$, $NcutS_i^{2^i} = \{2^i, 2^i + 1, ..., 2^{i+1} - 1\}$ is an optimal set in Q^r.*

Lemma 1.10. [20] *For $1 \leq j < n$ and $i = 1, 2, ..., 2^{r_j}$*

$$\begin{aligned} Lex_j^i \;= \{ \;\; & m + x_{j+1} \cdot 2^{r_1+r_2+\ldots+r_j} + x_{j+2} \cdot 2^{r_1+r_2+\ldots+r_{j+1}} + \cdots + x_n \cdot 2^{r_1+r_2+\ldots+r_{n-1}}, \\ & : 0 \leq m \leq i \cdot 2^{r-(r_j+\ldots+r_n)} - 1,\; 0 \leq x_k \leq 2^{r_k} - 1,\; k = j+1, j+2, ..., n \quad \} \end{aligned}$$

is an optimal set on $i \times 2^{r-r_j}$ vertices in Q^r where $r_1 + r_2 + ... + r_n = r$, $r_1 \leq r_2 \leq \cdots \leq r_n$.

2 Main Results

In this section we compute the exact wirelength of embedding r-dimensional hypercube Q^r into the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$ and the torus $C_{2^{r_1}} \times C_{2^{r_2}}$, where $r_1 + r_2 = r$ and $r_1 \leq r_2$.

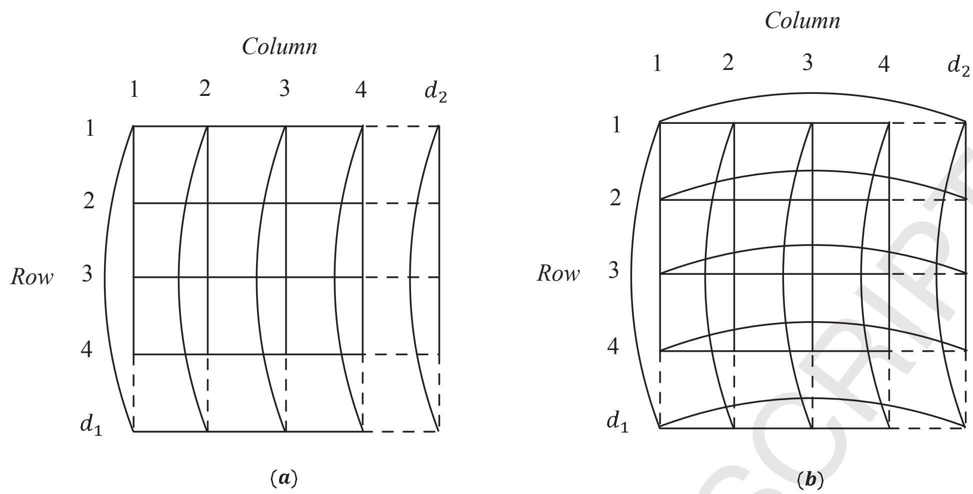

Figure 2: (a) Cylinder $C_{d_1} \times P_{d_2}$ and (b) Torus $C_{d_1} \times C_{d_2}$

2.1 Hypercube into Cylinder

Grid embedding plays an important role in computer architecture. VLSI Layout Problem, Crossing Number Problem, Graph Drawing, and Edge Embedding Problem are all a part of grid embedding. There are very few results in the literature which provide the exact wirelength of embedding grids into other architectures [2]. Cylinder is an extension of the grid network and is defined as follows.

Definition 2.1. [7] The 2-dimensional grid is defined as $P_{d_1} \times P_{d_2}$, where $d_i \geq 2$ is an integer for each $i = 1, 2$. The *cylinder* $C_{d_1} \times P_{d_2}$, where $d_1, d_2 \geq 3$ is a $P_{d_1} \times P_{d_2}$ grid with a wraparound edge in each row.

It is clear that the vertex set of $P_{d_1} \times P_{d_2}$ is $V = \{x_1x_2 : 0 \leq x_i \leq d_i - 1, i = 1, 2\}$ and two vertices $x = x_1x_2$ and $y = y_1y_2$ are linked by an edge, if $|x_1 - y_1| + |x_2 - y_2| = 1$.

Remark 2.2. The cylinder $C_{d_1} \times P_{d_2}$ has d_1d_2 vertices and $2d_1d_2 - d_1$ edges. See Figure 2(a).

Manuel et al. [7] obtained the exact wirelength of embedding an r-dimensional hypercube into the cylinder $C_4 \times P_{2^{r-2}}$, where $r_1 + r_2 = r$ and $r_1 \leq r_2$. Now, we compute the exact wirelength of embedding an r-dimensional hypercube Q^r into the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, where $r_1 + r_2 = r$ and $r_1 \leq r_2$, thereby proving the conjecture proposed in [7].

Lemma 2.3. *Let G be the r-dimensional hypercube Q^r and H be the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, where $r_1 + r_2 = r$ and $r_1 \leq r_2$. Let S_j be an edge cut of H consisting of edges between the columns j and $j+1$ of H, $1 \leq j \leq 2^{r_2} - 1$. Then*

$$\sum_{j=1}^{2^{r_2}-1} EC(S_j) = 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}).$$

Proof. Let C_j denote the cycle induced by the vertices in column j of $C_{d_1} \times P_{d_2}$, where $1 \leq j \leq d_2$ and $d_1, d_2 \geq 3$. Label the vertices of hypercube Q^r using Gray code labeling [3] and label the vertices of C_j, $1 \leq j \leq 2^{r_2}$ in H as $(j-1)2^{r_1}, (j-1)2^{r_1} + 1, \ldots, j\ 2^{r_1} - 1$ from top to bottom and map $f : Q^r \to H$ by $f(x) = x$. See Figure 3.

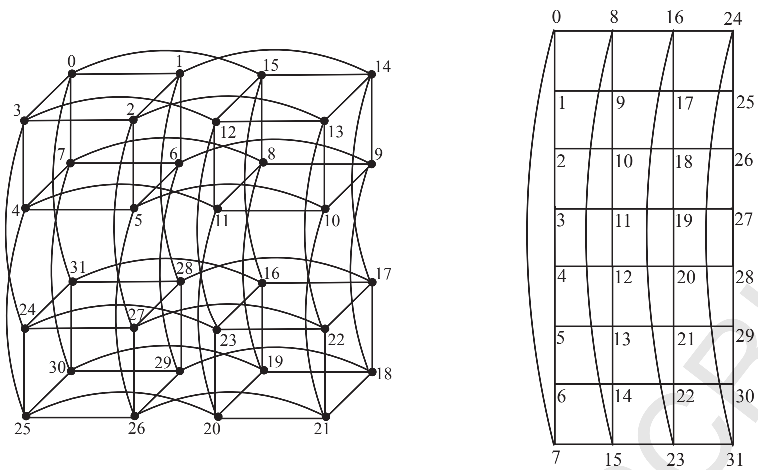

Figure 3: Labeling of Q^5 and $C_8 \times P_4$

Now, S_j disconnects H into two components H_{j1} and H_{j2} where $V(H_{j1}) = \{0, 1, \ldots, j\ 2^{r_1} - 1\}$. See Figure 4. Let G_{j1} and G_{j2} be the inverse images of H_{j1} and H_{j2} respectively under f. By Lemma 1.8, $V(G_{j1})$ is an optimal set in Q^r. By Congestion Lemma, $EC_f(S_j)$ is minimum, $1 \leq j \leq 2^{r_2} - 1$. Thus by Partition Lemma,

$$\sum_{j=1}^{2^{r_2}-1} EC(S_j) = \sum_{j=1}^{2^{r_2}-1} EC_f(S_j) = \sum_{j=1}^{2^{r_2}-1} \theta_G(j \cdot 2^{r_1}) = 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}). \quad \square$$

Lemma 2.4. *Let G be the r-dimensional hypercube Q^r and H be the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, where $r_1 + r_2 = r$ and $r_1 \leq r_2$. Let $f : G \rightarrow H$ be an embedding. If $A \subseteq V(H)$ and $EC_f(\theta_H(A))$ is minimum, then $f^{-1}(A)$ is a maximum subgraph of Q^r.*

Proof. Suppose $EC_f(\theta_H(A))$ is minimum with $|A| = m$. To prove that $B = f^{-1}(A)$ is a maximum subgraph of Q^r on m vertices. Suppose not, there exist $C \subseteq V(Q^r)$ such that $|I_G(B)| < |I_G(C)|$. Since Q^r is r-regular, by Congestion Lemma,

$$\begin{aligned} EC_f(\theta_H(A)) &= r\ m - 2|I_G(B)| \\ &> r\ m - 2|I_G(C)| \\ &= EC_f(\theta_H(f(C))) \end{aligned}$$

which is a contradiction to our assumption that $EC_f(\theta_H(A))$ is minimum. Therefore $f^{-1}(A)$ is a maximum subgraph of Q^r. $\square$

For an embedding of Q^r into C_{2^r}, the cyclic wirelength of Q^r into C_{2^r} has been obtained in [31] and is equal to $3 \cdot 2^{2r-3} - 2^{r-1}$. We have the following result by choosing $A = C_j$, $1 \leq j \leq 2^{r_2}$ in Lemma 2.4.

Lemma 2.5. *The congestion on the edges of C_j in $C_{2^{r_1}} \times P_{2^{r_2}}$ is $3 \cdot 2^{2r_1-3} - 2^{r_1-1}$, $1 \leq j \leq 2^{r_2}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$. Thus $\sum_{j=1}^{2^{r_2}} EC(C_j) = 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1})$.* $\square$

The edge set of cylinder is partitioned into $S_1, S_2, \ldots, S_{2^{r_2}-1}, E(C_1), E(C_2), \ldots, E(C_{2^{r_2}})$. Therefore by Lemmas 2.3 and 2.5, we have the following result.

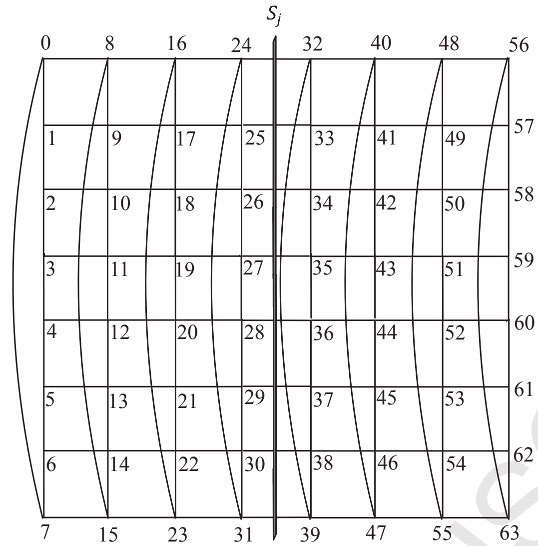

Figure 4: Edge cut of $C_8 \times P_8$

Theorem 2.6. *Let G be an r-dimensional hypercube Q^r and H be the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$. Then the minimum wirelength of embedding G into H satisfies*

$$WL(G, H) \geq 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}). \qquad \square$$

We now proceed to prove that the lower bound obtained in Theorem 2.6 is sharp.

Embedding Algorithm A

Input : The r-dimensional hypercube Q^r and the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$.

Algorithm : Label the vertices of hypercube Q^r using Gray code labeling [3] and label the vertices of C_j, $1 \leq j \leq 2^{r_2}$ in $C_{2^{r_1}} \times P_{2^{r_2}}$ as $(j-1)2^{r_1}, (j-1)2^{r_1} + 1, \ldots, j\ 2^{r_1} - 1$ from top to bottom.

Output : An embedding f of Q^r into $C_{2^{r_1}} \times P_{2^{r_2}}$ given by $f(x) = x$ with minimum wirelength.

Theorem 2.7. *Let G be an r-dimensional hypercube Q^r and H be the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$. Then the minimum wirelength of embedding G into H is given by*

$$WL(G, H) = 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}).$$

Proof. Label the vertices of Q^r and $C_{2^{r_1}} \times P_{2^{r_2}}$ using Embedding Algorithm A. We assume that the labels represent the vertices to which they are assigned. By Lemmas 2.3 and 2.5, we have

(i) $EC_f(C_j) = \theta_G(j \cdot 2^{r_1})$, $1 \leq j \leq 2^{r_2} - 1$ and

(ii) $EC_f(C_k) = 3 \cdot 2^{2r_1-3} - 2^{r_1-1}$, $1 \leq k \leq 2^{r_2}$

Then by Partition Lemma,

$$\begin{aligned} WL(G, H) &= 2\sum_{j=1}^{2^{r_2}-1} \theta_G(j \cdot 2^{r_1}) + \sum_{k=1}^{2^{r_2}} (3 \cdot 2^{2r_1-3} - 2^{r_1-1}) \\ &= 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}). \quad \square \end{aligned}$$

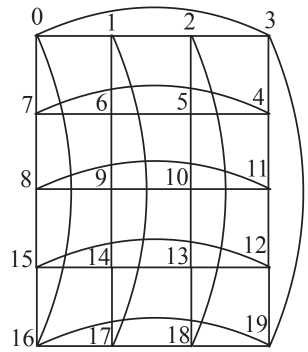

Figure 5: Snake-wise labeling of $C_4 \times C_5$

2.2 Hypercube into Torus

The family of tori is one of the most popular interconnection networks due to its desirable properties such as regular structure, ease of implementation and good scalability. In recent years, the theory of torus embedding has found many applications and has been used in many practical systems such as Cray T3D, Cray T3E, Fujitsu AP3000, Ametak 2010, Intel Touchstone and so on [32].

Definition 2.8. [12] The *torus* $C_{d_1} \times C_{d_2}$, where $d_1, d_2 \geq 3$ is a $P_{d_1} \times P_{d_2}$ grid with a wraparound edge in each row and column.

Remark 2.9. The torus $C_{d_1} \times C_{d_2}$ has $d_1 d_2$ vertices and $2d_1 d_2$ edges. See Figure 2(b).

Notation : Let C^i and C_j denote the cycles induced by the vertices in row i and column j respectively of $C_{d_1} \times C_{d_2}$, where $1 \leq i \leq d_1$, $1 \leq j \leq d_2$ and $d_1, d_2 \geq 3$.

Now, we compute the exact wirelength of embedding an r-dimensional hypercube Q^r into the torus $C_{2^{r_1}} \times C_{2^{r_2}}$, where $r_1 + r_2 = r$ and $r_1 \leq r_2$ respectively.

Lemmas 2.4 and 2.5 yield the following result.

Lemma 2.10. *The congestion on the edges of C^i and C_j of $C_{2^{r_1}} \times C_{2^{r_2}}$ are $3 \cdot 2^{2r_2-3} - 2^{r_2-1}$ and $3 \cdot 2^{2r_1-3} - 2^{r_1-1}$, $1 \leq i \leq 2^{r_1}$, $1 \leq j \leq 2^{r_2}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$.*

From Lemma 2.10, we have the following result.

Theorem 2.11. *Let G be an r-dimensional hypercube Q^r and H be the torus $C_{2^{r_1}} \times C_{2^{r_2}}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$. Then the minimum wirelength of embedding G into H satisfies*

$$WL(G, H) \geq 2^{r_1}(3 \cdot 2^{2r_2-3} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}). \qquad \square$$

Embedding Algorithm B

Input : The r-dimensional hypercube Q^r and the torus $C_{2^{r_1}} \times C_{2^{r_2}}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$.

Algorithm : Label the vertices of hypercube Q^r using Gray code labeling [3] and label the vertices of C^i, $1 \leq i \leq 2^{r_1}$ in H as $(j-1)2^{r_1}, (j-1)2^{r_1}+1, \ldots, j\ 2^{r_1}-1$ using snake-wise labeling beginning with the left most vertex as shown in Figure 5.

Output : An embedding f of Q^r into $C_{2^{r_1}} \times C_{2^{r_2}}$ given by $f(x) = x$ with minimum wirelength.

The proof of the following theorem is an easy consequence of Theorem 2.11 and Embedding Algorithm B.

Theorem 2.12. *Let G be an r-dimensional hypercube Q^r and H be the torus $C_{2^{r_1}} \times C_{2^{r_2}}$, $r_1+r_2 = r$ and $r_1 \leq r_2$. Then the minimum wirelength of embedding G into H is given by*

$$WL(G,H) = 2^{r_1}(3 \cdot 2^{2r_2-3} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}).$$

Proof. Label the vertices of Q^r and $C_{2^{r_1}} \times C_{2^{r_2}}$ using Embedding Algorithm B. We assume that the labels represent the vertices to which they are assigned. By Lemma 2.5, we have

(i) $EC_f(C^i) = 3 \cdot 2^{2r_2-3} - 2^{r_2-1}$, $1 \leq i \leq 2^{r_1}$ and

(ii) $EC_f(C_j) = 3 \cdot 2^{2r_1-3} - 2^{r_1-1}$, $1 \leq j \leq 2^{r_2}$

Then by Partition Lemma,

$$\begin{aligned} WL(G,H) &= 2\sum_{j=1}^{2^{r_2}-1} EC_f(C^i) + \sum_{k=1}^{2^{r_2}} EC_f(C_j) \\ &= 2^{r_1}(3 \cdot 2^{2r_2-3} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}). \quad \square \end{aligned}$$

3 Time Complexity

In computer science, the time complexity of an algorithm quantifies the amount of time taken by an algorithm to run as a function of the size of the input to the problem. An algorithm is said to take linear time, or $O(n)$ time, if its time complexity is $O(n)$. Informally, this means that for large enough input sizes the running time increases linearly with the size of the input [34].

Linear time is often viewed as a desirable attribute for an algorithm. Much research has been invested into creating algorithms exhibiting (nearly) linear time or better. This research includes both software and hardware methods. In the case of hardware, some algorithms which, mathematically speaking, can never achieve linear time with standard computation models are able to run in linear time. There are several hardware technologies which exploit parallelism to provide this. An example is content-addressable memory. This concept of linear time is used in string matching algorithms such as the Boyer-Moore Algorithm and Ukkonen's Algorithm [35, 36].

In this Section, we compute the time complexity of finding the exact wirelength of embedding hypercube into cylinder using Embedding Algorithm B. The algorithm is formally presented as follows.

Time Complexity Algorithm

Input : The r-dimensional hypercube Q^r and the cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$.

Algorithm : Embedding Algorithm A.

Output : The time taken to compute the minimum wirelength of embedding Q^r into $C_{2^{r_1}} \times P_{2^{r_2}}$ is $O(n)$, which is linear.

Method : We know that Q^r contains $n = 2^r$ vertices. For assigning the labels of n vertices, we spend n time units. By Embedding Algorithm A, we have $2^{r_2} - 1$ edge cuts. For each cut, we need one unit of time and hence we need $2^{r_2} - 1$ time units. Again for finding the edge congestion on C_k, $1 \leq k \leq 2^{r_2}$ we need one unit of time. Further, we need 2^{r_2} units of time for finding the wirelength by using Partition Lemma.

$$\begin{aligned} \text{Hence the total time taken is} &= n + 2^{r_2} - 1 + 1 + 2^{r_2} \\ &\leq 2n \end{aligned}$$

Hence, the time taken to compute the exact wirelength of embedding Q^r into $C_{2^{r_1}} \times P_{2^{r_2}}$ is $O(n)$, which is linear. □

Proceeding along the same lines, we can compute the exact wirelength of embedding Q^r into torus $C_{2^{r_1}} \times C_{2^{r_2}}$, $r_1 + r_2 = r$ and $r_1 \leq r_2$ in linear time.

4 Concluding Remarks

In this paper, we prove the conjectures proposed by Manuel et al. [7] on exact wirelength of embedding Q^r into cylinder and torus. We provide a linear time algorithm to compute the exact wirelength of embedding hypercube into cylinder and torus. We extend the results obtained for 2-dimensional cylinder and torus to n-dimensional cylinder [20] and n-dimensional torus [33] and obtain the following results.

Theorem 4.1. *Let G be an r-dimensional hypercube Q^r and H be the n-dimensional cylinder $C_{2^{r_1}} \times P_{2^{r_2}} \times \cdots \times P_{2^{r_n}}$, $r_1 + r_2 + \ldots + r_n = r$ and $r_1 \leq r_2 \leq \cdots \leq r_n$. Then the minimum wirelength of embedding G into H is given by*

$$WL(G,H) = \sum_{i=1}^{n} 2^{r-r_i}(2^{2r_i-1} - 2^{r_i-1}) + \sum_{i=1}^{n} 2^{r-r_i}(3 \cdot 2^{2r_i-3} - 2^{r_i-1}). \quad \square$$

Theorem 4.2. *Let G be an r-dimensional hypercube Q^r and H be the n-dimensional torus $C_{2^{r_1}} \times C_{2^{r_2}} \times \cdots \times C_{2^{r_n}}$, $r_1 + r_2 + \ldots + r_n = r$ and $r_1 \leq r_2 \leq \cdots \leq r_n$. Then the minimum wirelength of embedding G into H is given by*

$$WL(G,H) = \sum_{i=1}^{n} 2^{r-r_i+1}(3 \cdot 2^{2r_i-3} - 2^{r_i-1}). \quad \square$$

Acknowledgements

We thank the anonymous referees for their sugesstions which improved the quality of the paper.

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