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A modified Numerov method for solving singularly perturbed differential–difference equations arising in science and engineering

P. Pramod Chakravarthy *, R. Nageshwar Rao

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India

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ABSTRACT

In this paper a modified fourth order Numerov method is presented for singularly perturbed differential–difference equation of mixed type, i.e., containing both terms having a negative shift and terms having positive shift. Similar boundary value problems are associated with expected first exit time problems of the membrane potential in the models for the neuron. To handle the negative and positive shift terms, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization. The proposed finite difference method works nicely when the shift parameters are smaller or bigger to perturbation parameter. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameters on the boundary layer behavior or oscillatory behavior of the solution of the problem.

Introduction

Any system involving feedback control will almost involve time delays. These arise because a finite time is required to sense the information and then react to it. A singularly perturbed differential–difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involves at least one delay term. Such problems arise frequently in the mathematical modeling of various physical and biological phenomena like optically bistable devices [1], description of the human pupil reflex [2], a variety of models for physiological processes or diseases and variational problems in control theory [3,4], the first exit time problem in the modeling of the activation of neuronal variability [5]. Lange and Miura [5,6] gave an asymptotic approach in the study of a class of boundary value problems for linear second order differential–difference equations in which the highest order derivative is multiplied by a small parameter. An extensive numerical work had been initiated by Kadalbajoo et al. [7–11]. In [12] Ramos has presented a variety of exponential methods for the numerical solution of linear ordinary differential–difference equations with a small delay based on piecewise analytical solutions of advection–reaction–diffusion operators. In [13], the authors Jugal Mohapatra, Srinivasan Natesan constructed a numerical method for a class of singularly perturbed differential–difference equations with small delay.

In this paper we modified the fourth order Numerov method and applied to singularly perturbed differential–difference equations of mixed type. To handle the negative and positive shift terms, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization. The proposed finite difference method works nicely when the shift parameters are smaller or bigger to perturbation parameter. An extensive amount of computational work has been carried out to demonstrate the proposed method and to show the effect of shift parameters on the boundary layer behavior and oscillatory behavior of the solution of the problem.

Modified fourth order Numerov method

We consider a linear singularly perturbed differential–difference equation of mixed type i.e., equation containing both the negative and positive shift terms.

$$\varepsilon^2 y''(x) + \alpha(x)y(x-\delta) + \omega(x)y(x) + \beta(x)y(x+\eta) = f(x) \tag{1}$$

on $0 < x < 1$, $0 < \varepsilon << 1$, subject to the interval and boundary conditions

$$y(x) = \phi(x) \quad \text{for} \quad -\delta \leqslant x \leqslant 0 \quad \text{and} \quad y(x) = \psi(x) \quad \text{for} \quad 1 \leqslant x \leqslant 1+\eta \tag{2}$$

where $\alpha(x)$, $\omega(x)$, $\beta(x)$, $f(x)$, $\phi(x)$ and $\psi(x)$ are smooth functions, δ and η are the small shifting parameters. For a function $y(x)$ to constitute a smooth solution to the problem (1), (2) it must be continuous in the interval [0,1] and be continuously differentiable in the interval (0,1). For the shifts δ, η equal to zero and if

* Corresponding author. Tel.: +91 712 2801404; fax: +91 712 2223969/2223230.
E-mail addresses: ppchakravarthy@mth.vnit.ac.in (P. Pramod Chakravarthy), nrao_ragi@yahoo.co.in (R. Nageshwar Rao).

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$\alpha(x) + \omega(x) + \beta(x) < 0$ on the interval [0,1], then the solution exhibits boundary layers at both the ends of the interval [0,1].

We rearrange the differential Eq. (1) and (2) as

$$\varepsilon^2 y''(x) = g(x, y(x), y(x-\delta), y(x+\eta)) \tag{3}$$

where $g(x, y(x), y(x-\delta), y(x+\eta)) = f(x) - \alpha(x)y(x-\delta) - \omega(x)y(x) - \beta(x)y(x+\eta)$

Now, we construct a special type of mesh so that the terms containing the shift parameters lie on the nodal points after discretization. We divide the interval [0, 1] into N equal parts by choosing the mesh parameter h such that $h = \frac{\delta}{k} = \frac{\eta}{\ell}$, where k and ℓ are positive integers chosen such that $1 \leqslant k, \ell \leqslant N$.

At $x = x_i$, the above differential equation can be written as

$$\varepsilon^2 y''(x_i) = g(x_i, y(x_i), y(x_i - \delta), y(x_i + \eta)) = g_i, \tag{4}$$

where $g_i = f_i - \alpha_i y_{i-k} - \omega_i y_i - \beta_i y_{i+\ell}$,

$y_i = y(x_i), f_i = f(x_i), \alpha_i = \alpha(x_i), \omega_i = \omega(x_i), \beta_i = \beta(x_i)$.

Now, we consider the fourth order Numerov finite difference method [14] to solve the Eq. (4) and this equation is approximated by the following finite difference scheme:

$$\frac{\varepsilon^2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = \frac{1}{12}(g_{i-1} + 10g_i + g_{i+1}) \tag{5}$$

Table 1
Numerical solution of example 1 for $\delta = 0.03$, $\eta = 0.07$.

N→ / ε↓	100	200	300	400	500
2^{-1}	6.1000e−007	1.6000e−007	7.0000e−008	4.0000e−008	3.0000e−008
2^{-2}	6.7400e−006	1.6900e−006	7.5000e−007	4.2000e−007	2.7000e−007
2^{-3}	5.0780e−005	1.2710e−005	5.6500e−006	3.1800e−006	2.0300e−006
2^{-4}	2.9686e−004	7.4640e−005	3.3200e−005	1.8690e−005	1.1960e−005
2^{-5}	1.6272e−003	4.1624e−004	1.8578e−004	1.0466e−004	6.7020e−005
2^{-6}	6.6542e−003	1.8089e−003	8.1653e−004	4.6180e−004	2.9629e−004

Table 2
Numerical solution of example 2 for $\delta = 0.07$, $\eta = 0.03$.

N→ / ε↓	100	200	300	400	500
2^{-1}	1.4510e−005	3.6300e−006	1.6100e−006	9.0000e−007	5.8000e−007
2^{-2}	9.3650e−005	2.3420e−005	1.0410e−005	5.8600e−006	3.7500e−006
2^{-3}	5.2693e−004	1.3196e−004	5.8660e−005	3.3000e−005	2.1130e−005
2^{-4}	2.5668e−003	6.4570e−004	2.8731e−004	1.6168e−004	1.0349e−004
2^{-5}	9.9696e−003	2.5626e−003	1.1448e−003	6.4511e−004	4.1321e−004
2^{-6}	3.1132e−002	8.7328e−003	3.9629e−003	2.2454e−003	1.4419e−003

The boundary conditions can be written as

$$y_i = \phi_i; -k \leqslant i \leqslant 0 \quad \text{and} \quad y_i = \psi_i; N \leqslant i \leqslant N + \ell \tag{6}$$

where $\phi_i = \phi(x_i)$ and $\psi_i = \psi(x_i)$.

Using the definition of g_i in Eq. (5), we get the following fourth order finite difference scheme.

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^* y_{i-k-1} + F_i^* y_{i-k} + G_i^* y_{i-k+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} = R_i \tag{7}$$

where

$$E_i = \frac{12\varepsilon^2}{h^2} + \omega_{i-1}, F_i = -\frac{24\varepsilon^2}{h^2} + 10\omega_i, G_i = \frac{12\varepsilon^2}{h^2} + \omega_{i+1},$$
$$E_i^* = \alpha_{i-1}, F_i^* = 10\alpha_i, G_i^* = \alpha_{i+1}, E_i^{**} = \beta_{i-1}, F_i^{**} = 10\beta_i,$$
$$G_i^{**} = \beta_{i+1}, R_i = f_{i-1} + 10f_i + f_{i+1}$$

Using (6), the difference scheme (7) can be written as

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} = R_i - E_i^* \phi_{i-k-1} - F_i^* \phi_{i-k} - G_i^* \phi_{i-k+1} \quad \text{for} \quad 1 \leqslant i \leqslant k-1$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} = R_i - E_i^* \phi_{i-k-1} - F_i^* \phi_{i-k} \quad \text{for} \quad i = k$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} = R_i - E_i^* \phi_{i-k-1} \quad \text{for} \quad i = k+1$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^{**} y_{i+\ell+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i \quad \text{for} \quad k+2 \leqslant i \leqslant N-\ell-2$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + F_i^{**} y_{i+\ell} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i - G_i^{**} \psi_{i+\ell+1} \quad \text{for} \quad i = N-\ell-1$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + E_i^{**} y_{i+\ell-1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i - G_i^{**} \psi_{i+\ell+1} - F_i^{**} \psi_{i+l} \quad \text{for} \quad i = N-\ell$$

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} + G_i^* y_{i-k+1} + F_i^* y_{i-k} + E_i^* y_{i-k-1} = R_i - G_i^{**} \psi_{i+\ell+1} - F_i^{**} \psi_{i+\ell} - E_i^{**} \psi_{i+\ell-1} \quad \text{for} \quad N-\ell+1 \leqslant i \leqslant N-1$$

The above system of equations along with the boundary conditions $y_0 = \phi_0$ and $y_N = \psi_N$ is solved for y_i, i = 0, 1, 2,....., N by Gauss elimination method with partial pivoting. In fact, any numerical method or analytical method can be used to solve the above system of equations for y_i.

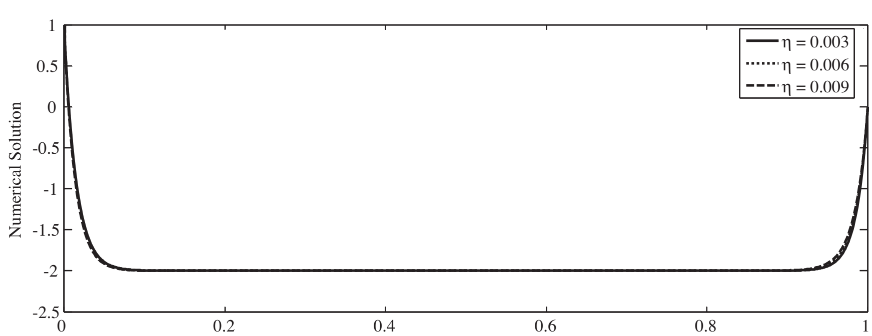

Fig. 1. The numerical solution of example 1 with $\varepsilon = 0.01$ and $\delta = 0.005$ for different values of η.

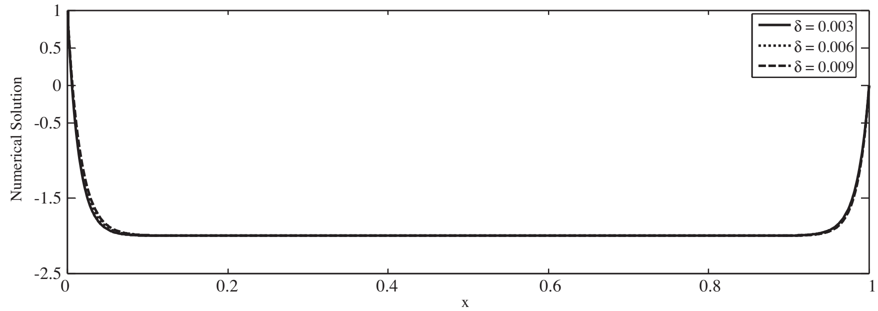

Fig. 2. The numerical solution of example 1 with $\varepsilon = 0.01$ and $\eta = 0.005$ for different values of δ.

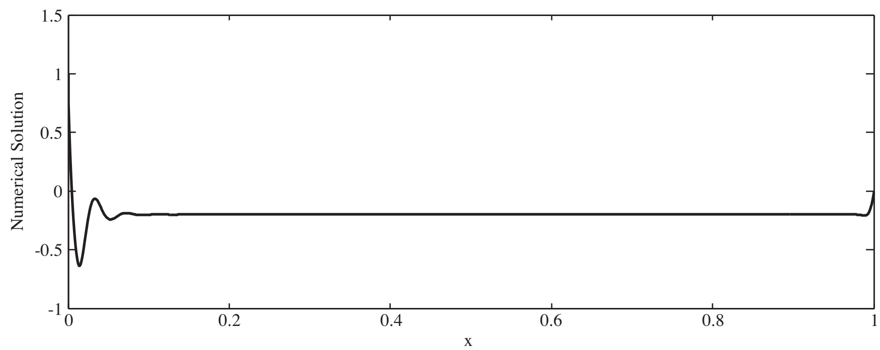

Fig. 3. The numerical solution of example 2 with $\varepsilon = 0.01$ and $\delta = 0.015$, $\eta = 0.007$.

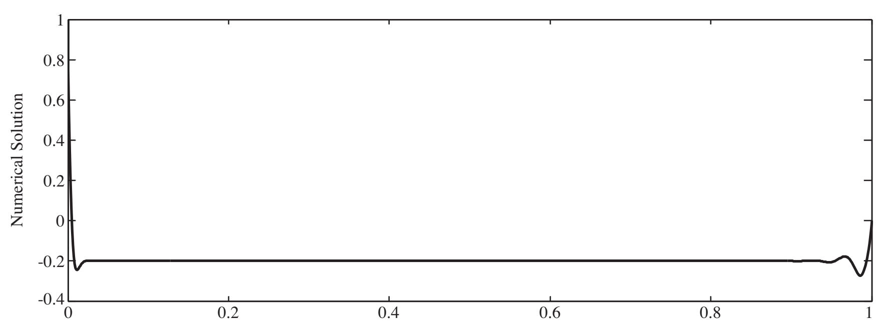

Fig. 4. The numerical solution of example 2 with $\varepsilon = 0.01$ and $\delta = 0.007$, $\eta = 0.015$.

Numerical results

To demonstrate the applicability of the method we consider boundary value problems of singularly perturbed linear differential–difference equations exhibiting boundary layers at both sides of the underlying interval [0, 1]. These examples were discussed in [5,6,9]. Since the exact solutions of the problems for different values of δ and η are not known, the maximum absolute errors for the examples are calculated using the following double mesh principle [15].

$$e_N = \max_{0\leqslant i\leqslant N}|y_i^N - y_{2i}^{2N}|.$$

The maximum absolute error is tabulated in the form of Tables 1 and 2 for the considered examples. The corresponding graphs of the solution for different values of the shift parameters δ and η are plotted in Figs. 1–4 to examine the effect of the shifts on the boundary layer behavior of the solution.

Example 1 [5, p 265] $\varepsilon^2 y''(x) + 0.25y(x-\delta) - y(x) + 0.25y(x+\eta) = 1$, subject to the interval conditions $y(x) = 1; -\delta \leqslant x \leqslant 0$, $y(x) = 0; 1 \leqslant x \leqslant 1+\eta$.

Example 2[5, p 265] $\varepsilon^2 y''(x) - 2y(x-\delta) - y(x) - 2y(x+\eta) = 1$ subject to the interval conditions $y(x) = 1; -\delta \leqslant x \leqslant 0, y(x) = 0; 1 \leqslant x \leqslant 1+\eta$.

Conclusions

A modified fourth order Numerov method is presented for solving singularly perturbed differential–difference equations of mixed type. To handle the shift parameters, we construct a special type of mesh, so that the terms containing shift lie on nodal points after discretization. The proposed finite difference method works nicely when the delay parameter is smaller or bigger to perturbation parameter. It has been observed that the layer behavior is maintained when the shift parameters are smaller than perturbation parameter. It is also observed that the layer behavior of the solution is no longer maintained and the solution exhibits oscillatory behavior when shift parameters are larger than perturbation parameter. From the results, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. On the basis of the extensive numerical work, it is concluded that the present method offers significant advantage for the linear singularly perturbed differential–difference equations of mixed type.

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