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Abstract. Let G_1 and G_2 be two undirected graphs, (v w) be an edge of G_1, and (x y) be an edge of G_2. Hajos construction forms a new graph H, that combines the two graphs by identifying vertices v and x into a single vertex, removing the two edges (v w) and (x y), and adding a new edge (w y). In this paper we have discussed the properties of graphs G_1 and G_2 that are Hajos stable and not Hajos stable graphs. We have proved that, if G_1 and G_2 are Euler graphs, then the Hajos graph is also Euler. We have obtained conditions whether the Hajos stable graphs are Hamiltonian and not Hamiltonian graphs. Also we have provided on NG type result for Hajos stable graphs. We have proved that, if G_1 and G_2 are just excellent and Hajos stable then H is not just excellent. Also we have shown that very excellent graphs are not Hajos stable graphs.

1. Introduction

GyorgyHajos was a great mathematician who worked in graph theory, group theory etc. He was a member of the Hungarian Academy of Sciences, initially he was a corresponding member in 1948 and at later stage he became a full member in 1958. He was chosen to the Romanian Academy of Sciences in 1965, and in 1967 to the German Academy of Sciences Leopoldina. In 1942, he obtained the Gyula Konig prize and he won the Kossuth prize in 1951 and 1962 [1].

In graph theory, the Hajos construction is one of thebinaryoperation on graphs named after Gyorgy Hajos. Catlin has provided a conjecture for Hajos graph coloring [2]. Brown et al. have proved that the Hajos construction of two amenable k – critical graphs need be amenable for any $k \geq 5$ [3].

An analogue of Hajos theorem for the circular chromatic number was proved by Zhu [4]. Kral has studied about an analogue of Hajos theorem for list coloring. Also one of the operations of Hajos sum was introduced by Kral [5]. Hajos join construction was introduced by Liu [6].

Yamuna et al have introduced the following stable graphs. A graph G is domination dot stable (DDS), if $\gamma (G.uv) = \gamma (G)$, $\forall$ u, v $\in$V (G), u $\perp$ v [7]. A graph G is domination subdivision stable (DSS), if $\gamma (G_{sd}uv) = \gamma (G)$, $\forall$ u, v $\in$V (G), u $\perp$ v [8]. A graph G is domatic subdivision stable (dss), if d (G) = d (G_{sd}uv), $\forall$ u, v $\in$V (G), u $\perp$ v [9]. A graph G is γ - stable graph if $\gamma (G_{xy}) = \gamma (G)$, $\forall$, x, y $\in$ V (G), x is not adjacent to y [10].

A graph G is said to be total domination dot – stable if dotting any pair of adjacent vertices leaves the total domination number unchanged [11]. Desormeaux et al., have studied about the total domination stable graphs upon edge removal [12]. Also they have studied about the total domination vertex removal changing and stable graphs [13].

Graph operations produce new graphs from initial one. Binary operations are in general tough, since they involve more than one graph. Hajos construction is a binary operation involving three operations edge removal, edge addition and vertex merging. In this paper, we have studied the results on

Hajosstable graphs and not Hajos stable graphs. Also we have obtained results using general graph properties with the Hajos stable graphs.

2. Materials and methods

We consider only simple connected undirected graphs G = (V, E). The open neighborhood of vertex v ∈V (G) is denoted by N (v) = { u ∈ V (G) / (u v) ∈ E (G) } while its closed neighborhood is the set N [v] = N (v) ∪ { v }. A cut vertex (edge) of a graph G is a vertex (edge) whose deletion increases the number of components. We write G – v or G – S for the subgraph obtained by deleting a vertex v or set of vertices S.

An Euler path is a path that uses every edge of a graph exactly once. An Euler circuit is a circuit that uses every edge of a graph exactly once. A Hamiltonian path is a path in an undirected or directed graph that visits each vertex exactly once. A Hamiltonian cycle is a Hamiltonian path that is a cycle. We denote u adjacent to v by u ⊥ v. For details of on graph theory we refer to [14].

A set of vertices D, in a graph G = (V, E) is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set. The cardinality of any minimum dominating set for G is called the domination number of G and it is denoted by γ(G). γ - set denotes a dominating set for G with minimum cardinality.

A vertex v is said to be good if ∃ a γ - set of G containing v. If ∃ no γ - set of G containing v, then v is said to be bad vertex. A vertex v is said to be a, down vertex if γ (G – u) <γ (G), level vertex if γ (G – u) = γ (G), up vertex if γ (G – u) >γ (G). A vertex v is said to be selfish in the γ - set D, if v is needed only to dominate itself. A vertex in V – D is k – dominated if it is dominated by at least k – vertices in D that is | N (v) ∩ D | ≥ k. The private neighborhood of v ∈ D is denoted by pn [v, D], is defined by pn [v, D] = N (v) – N (D − { v }). For details of on domination we refer to [15].

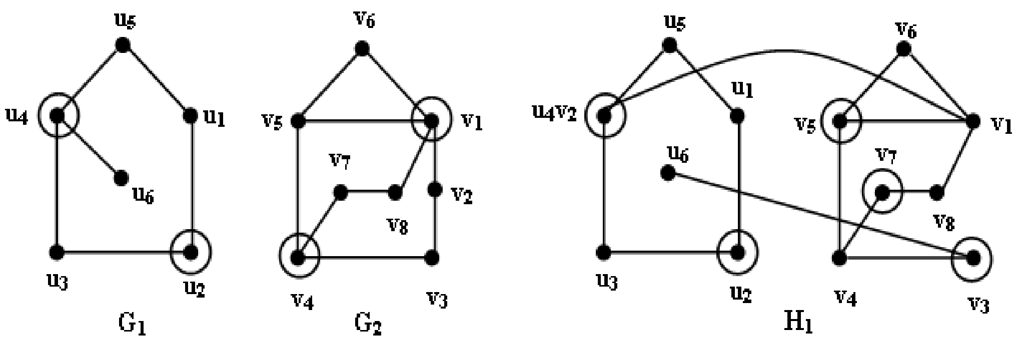
Hajos construction

Let G_1 and G_2 be two graphs, (u_1 v_1) be an edge of G_1, and (u_2 v_2) be an edge of G_2. Then the Hajos construction produce a different graph H that combines the 2 graphs by merging vertices u_1 and u_2 into a single vertex u_{12}, eliminating the two edges (u_1 v_1) and (u_2 v_2), and adding a new edge (v_1 v_2) [16].

Hajos stable graphs

Let G_1 and G_2 be any two graphs. Let E (G_1) = { e_{11}, e_{12}, …, e_{1p} } and E (G_2) = { e_{21}, e_{22}, …, e_{2q} }. Let M = E (G_1) × E (G_2) = { (e_{1i} e_{2j}) | e_{1i}∈ E (G_1), e_{2j}∈ E (G_2) }, that is M is the cartesian product between sets E (G_1) and E (G_2). Let | M | = k. Let H_1, H_2, …, H_{4k} be the Hajos graphs generated by applying Hajos construction 4k times. If γ (H_i) = γ (G_1) + γ (G_2), ∀ i = 1, 2, …, 4k, then G_1 and G_2 are said to be Hajos stable graphs [17].

Example



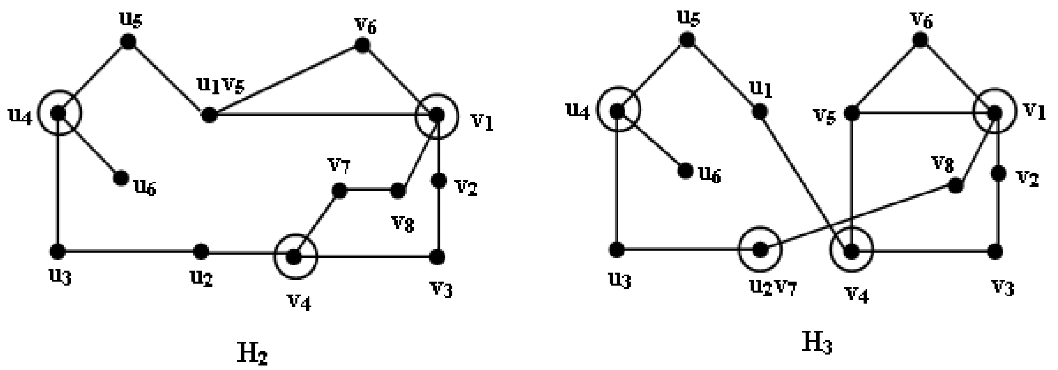


Figure 1.

Fig. 1, H_1, H_2 and H_3 are the Hajos graph obtained from G_1 and G_2 using the edge pairs { (u_4 u_6), (v_2 v_3) }, { (u_1 u_2), (v_4 v_5) }, { (u_1 u_2), (v_4 v_7) } respectively. $\gamma(H_1) > \gamma(G_1) + \gamma(G_2)$, $\gamma(H_2) < \gamma(G_1) + \gamma(G_2)$, $\gamma(H_3) = \gamma(G_1) + \gamma(G_2)$.

The results R1 and R2 were proved in [17].

R1.

Let G_1 and G_2 be any two graphs. Let D_1 and D_2 be γ - sets for G_1 and G_2 respectively. Let H be the Hajos graph. Then $\gamma(H) < \gamma(G_1) + \gamma(G_2)$ if and only if either

1. $\exists$ some (u_i v_i) $\in D_i$ such that $u_i \perp v_i$, i = 1, 2, or
2. $\exists$a selfish vertex in G_i, i = 1, 2, or
3. both G_1 and G_2 have 2 – dominated vertices simultaneously together, or
4. ifpn [u_i, D_i] = v_i in G_i, then G_j has 2 – dominated vertices, where i, j = 1, 2, i $\neq$ j.

R2

Let G_1 and G_2 be any two graphs. Let D_1 and D_2 be γ - sets for G_1 and G_2 respectively. Let H be the Hajos graph. Then $\gamma(H) > \gamma(G_1) + \gamma(G_2)$ if and only if either

1. if u_i is an up vertex, u_j, v_i, v_j are bad vertices, then
   a. v_i is not a 2 – dominated vertex with respect to every D_i in G_i and
   b. v_j is not a good vertex in $C_j - N[u_j]$, $\forall\gamma$ - sets D_3 for $C_j - N[u_j]$ such that $|D_3| = |D_j|$,
   where i, j = 1, 2, i $\neq$ j, or
2. ifu_i are bad vertices, v_i are up vertices, then $u_i \in$pn [v_i, D_i], $\forall$ possible γ - sets in G_i, i = 1, 2.

R3. If R1 and R2 are not satisfied, then G_1 and G_2 are said to be Hajos stable graphs.

3. Results and discussions

We shall use the following notations throughout the paper.

Notations

Throughout this paper

1. G_1 and G_2 are any two graphs.
2. H is the Hajos graph generated from G_1 and G_2.
3. $e_i = (u_i\ v_i)$, i = 1, 2 are any two random edges from G_1 and G_2 respectively.
4. While creating a Hajos graph using any $e_i = (u_i\ v_i) \in G_i$, i = 1, 2, the vertex obtained by merging vertices u_1, u_2 is labeled as u_{12}.
5. Whenever $\gamma(H) = \gamma(G_1) + \gamma(G_2)$, let R be a γ - set for H. In this case let $A_i = G_i - \{e_i\}$. Let $X_i = R \cap V(G_i)$, $Y_i = X_i \cup \{u_i\}$, i = 1, 2.

Theorem 1

If H is the Hajos graph, then $\gamma(H) \geq 2$.

Proof

Since G_1 and G_2 are connected graphs, $\gamma(G_1)$, $\gamma(G_2) \geq 1$. u_{12} is not adjacent v_1, v_2 and vice – versa. Any $x \in V(G_1)$, $x \neq v_1$ does not dominate any vertex in G_2. Similarly any $y \in V(G_2)$, $y \neq v_2$ does not dominates any vertex in G_1. There is no vertex in H adjacent to the remaining vertices, implies$\gamma(H) \geq 2$.

Theorem 2

If H is the Hajos graph, then both G_1 and G_2can not have cut edges simultaneously.

Proof

If G_1 and G_2 have cut edges, say $e_1 = (u_1 v_1)$, $e_2 = (u_2 v_2)$. Let
$G_1 – e_1 = G_{11}, G_{12}$, such that $u_1\in G_{11}$, $v_1\in G_{12}$ and
$G_2 – e_2 = G_{21}, G_{22}$, such that $u_1\in G_{21}$, $v_1\in G_{22}$.
Since there is no connectivity between G_{11} and G_{12} and G_{21}, G_{22}, H is disconnected with two components $G_{11}\cup G_{21}$ and $G_{12}\cup G_{22}$, which is not possible.

Theorem 3

If G_1 and G_2 are Euler graphs, then the Hajos graph H is also Euler.

Proof

Let G_1 and G_2 be Euler graphs. Let H be Hajos graph. By removing edges in G_1 and G_2, we create two odd degree vertices in each G_i, where i = 1, 2. By merging u_1 and u_2 and adding an edge between v_1 and v_2. Every vertex in H is of even degree and hence Euler.

Theorem 4

If G_1 and G_2 are Hamiltonian graph, then

1. H is Hamiltonian,$\forall(u_i v_i) \in E (G_i)$ such that $(u_i v_i)$ is in some Hamiltonian circuit of G_i.
2. H is not Hamiltonian,$\forall(u_i v_i) \in E (G_i)$ such that $(u_i v_i)$ is not in Hamiltonian circuit of G_i.

Proof

Let G_1 and G_2 be Hamiltonian including edges $(u_i v_i) \in E (G_i)$, i = 1, 2. Let H_i be the Hamiltonian circuit in G_i including $(u_i v_i)$. In A_i, there is a Hamiltonian path from u_i to v_i, i = 1, 2. In graph H start from vertex u_{12} trace H_2 to reach vertex v_2, then v_2 to v_1 and back to vertex u_{12} through H_1. This generates a Hamiltonian circuit in H.

Let $(u_i v_i)$ be edges in G_i not included in any Hamiltonian circuit for G_i. If H is Hamiltonian, then there is a Hamiltonian circuit for H say H_3. Since $(v_1 v_2)$ is the only edge in H such that $v_1\in G_1$ and $v_2\in G_2$, $G_i\cap H_3$ generates a Hamiltonian path from u_i to v_i. Adding an edge between u_i, v_i generates a Hamiltonian circuit in G_i including edge $(u_i v_i)$, a contradiction. Hence H is not Hamiltonian.

Theorem 5

If G_1 and G_2 are the Hajos stable graphs, then

1. $\gamma (H) + \gamma (\bar{H}) = \gamma (G_1) + \gamma (G_2) + 2$.
2. $\gamma (H) \gamma (\bar{H}) = 2 (\gamma (G_1) + \gamma (G_2))$.

Proof

In H, v_1 is not adjacent to u_{12}, v_1 not adjacent to any $y \neq v_2\in G_2$. Similarly v_2 is not adjacent to u_{12}, v_2 not adjacent to any $x \neq v_1\in G_1$. So, v_1, v_2 collectively not adjacent to the remaining vertices of H, implies v_1, v_2 are together adjacent to the remaining vertices of$\bar{H}$, implies $\gamma (\bar{H}) = 2$.

Hence $\gamma (H) + \gamma (\bar{H}) = \gamma (G_1) + \gamma (G_2) + 2$.

$\gamma (H) \gamma (\bar{H}) = 2 (\gamma (G_1) + \gamma (G_2))$.

Hajos stable and DSS graph

A subdivision of any graph G is the another graph generating from the subdivision of edges in G. Consider any edge e with endpoints { u, v } generating a new graph which contains one new vertex say w. Also an edge set e supplanting by 2 new edges$e_1 = (u w)$ and $e_2 = (w v)$.

A graph G is domination subdivision stable (DSS), if $\gamma (G_{sd}uv) = \gamma (G)$, $\forall u, v \in V (G)$, $u \perp v$.Subdividing any edge (uv) of a graph G denotes $G_{sd}uv$. Let w be a vertex introduced by subdividing(uv) and denote this by $G_{sd}uv = w$ [8].

Theorem 6
If G_1 and G_2 are DSS, then H is not Hajos stable.
Proof
If G_i is DSS, then we know that, for every (u_i v_i) $\in$ E (G_i), either there is some u_i, $v_i \in D_i$, $u_i \perp v_i$ or there is some $u_i \in D_i$, such that either, v_i is either pn [u_i, D_i] = { v_i } or v_i is 2 – dominated.
If G_1 and G_2 has to be Hajos stable, then

i. u_i, $v_i \in D_i$ is not possible, by R1.
ii. Both G_1, G_2 containing 2 – dominated vertices is not possible, by R1.
iii. G_1 having a 2 – dominated vertex and v $\in$pn [u, D_2] are simultaneously not possible together, by R1.

If G_1 and G_2 has to be Hajos stable, the only possibility is there is some $v_1 \in G_1$, $v_2 \in G_2$ such that pn [u_1, D_1] = { v_1 }, pn [u_2, D_2] = { v_2 }.
If pn [u_1, D_1] = { v_1 }, then any other $x_i \in$ N [u_1, D_1], is either 2 – dominated or it belongs to D_1. But (u_1 x_1) $\in D_1$ is not possible, by R1.
So, the only possibility is x_i is 2 – dominated, v_1, v_2 private neighbors, which is not possible, by R1. Hence both G_1 and G_2 cannot have single private neighbors simultaneously together for any γ - set for G_1 and G_2. Hence if G_1 and G_2 are DSS, G_1 and G_2 are not Hajos stable graphs.
Hajos stable and excellent graph
A graph G is excellent if every vertex of G is good. In [18] M. Yamuna and N. Sridharan, had defined a graph G to be Just excellent (JE), if it to every u$\in$V (G), $\exists$a unique γ - set of G containing u.
An excellent graph G is very excellent (VE), if $\exists$a γ - set D of G, such that to every vertex vin V – D,$\exists$ oneuin D, such that D – { v } $\cup$ { u } is a γ - set of a graph G. A γ - set D of G is said to be a very excellent γ - set if D nourishing the above property [19].
Theorem 7
If G_1 and G_2 are JE and Hajos stable graphs, then H is not JE.
Proof
Since the Hajos graph is stable, γ (H) = γ (G_1) + γ (G_2). | V (H) | = | V (G_1) | + | V (G_2) | – 1. If H is JE, then there should exist a domatic partition for H such that the size of every partition is equal to γ (G_1) + γ (G_2), which is not possible. Hence H is not JE.
Theorem 8
If G_1 and G_2 are VE, then G_1 and G_2 are not Hajos stable graphs.
Let v $\in$V (G_1), v $\in$V (G_1) – D_1. Let u dominate v.

1. If there is some x $\in D_1$ such that D_1 – { x } $\cup$ { v } is a γ – set for G_1, then $\exists$ a γ – set D_1' for G_1 such that { v, x } $\in D_1'$.
2. So, every vertex in V – D can be interchanged only with vertices dominating it. This means that every v $\in$V (G_1) – D_1 is a private neighbor. So $\langle$pn [u, D_1] $\rangle$ is complete for every u $\in D_1$. Let v_i, $w_j \in$ V (G_1) – D_1 such that $v_i \perp w_j$. Since γ (G_1) $\geq$ 2, there is some u_1, $u_2 \in D_1$. Let pn [u_1, D_1] = { v_1, v_2, …, v_k } and pn [u_2, D_2] = { w_1, w_2, …, w_p }. Since $\langle$pn [u_1, D_1] $\rangle$ and $\langle$pn [u_2, D_2] $\rangle$ are complete, D_1 – { u_1, u_2 } $\cup$ { v_i, w_j } is a γ – set for G_1 such that $v_i \perp w_j$. So we conclude that there is some γ – sets for G_i such that $u_i \perp u_j$ where u_i, $u_j \in D_i$. Hence if G_1 and G_2 are VE, then G_1 and G_2 are not Hajos stable graphs.

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