IOP Conference Series: Materials Science and Engineering

PAPER • OPEN ACCESS

A numerical scheme for singularly perturbed reaction-diffusion problems with a negative shift via numerov method

To cite this article: S Dinesh Kumar *et al* 2017 *IOP Conf. Ser.: Mater. Sci. Eng.* **263** 042110

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042110 doi:10.1088/1757-899X/263/4/042110

A numerical scheme for singularly perturbed reaction-diffusion problems with a negative shift via numerov method

S Dinesh Kumar[1], R Nageshwar Rao[1] and P Pramod Chakravarthy[2]
[1]Department of Mathematics, School of Advanced Sciences, VIT University, Vellore–632014, India
[2]Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur-440010, India

E-mail: nrao_ragi@yahoo.co.in

Abstract: In this paper, we consider a boundary value problem for a singularly perturbed delay differential equation of reaction-diffusion type. We construct an exponentially fitted numerical method using Numerov finite difference scheme, which resolves not only the boundary layers but also the interior layers arising from the delay term. An extensive amount of computational work has been carried out to demonstrate the applicability of the proposed method.

1. Introduction

A singularly perturbed delay differential equation is a differential equation in which the highest order derivative is multiplied by a small positive parameter ε and which involves at least one delay term. Delay differential equations are prominent in the fields of biology, ecology, medicine, and physics [1-3]. In [4,5], Lange and Miura initiated the asymptotic analysis of singularly perturbed difference-difference equations with small shifts. The numerical analysis of these equations has found considerable growth in recent years due to the applications in several areas, as discussed in [5-11].

The standard discretization methods, when applied to singular perturbation problems, are found to be unstable and also fail to give accurate results for small values of the perturbation parameter ε. Hence, suitable numerical methods are to be developed whose accuracy is independent of ε. Various numerical methods to solve singularly perturbed differential equations can be found in [12–15]. In [16-22], a variety of numerical techniques is discussed for solving second order singularly perturbed differential-difference equation with small shifts. In [23-24], the authors Subburayan and Ramanujan presented initial value techniques for solving second order singularly perturbed boundary value problems with delay. In[25]the authors Amiraliyev and Cimen have presented an exponential fitted difference scheme for singularly perturbed second order boundary value problem with the large delay in the reaction term. Manikandan et.al [26] proposed a first order uniformly convergent numerical method for singularly perturbed differential equations, which exhibit boundary layers at both end points and an internal layer.

In the present paper, we consider a boundary value problem for a singularly perturbed delay differential equation of reaction-diffusion type. We construct an exponentially fitted numerical method using Numerov finite difference scheme, which resolves not only the boundary layers but also the interior layers arising from the delay term. The proposed numerical method converges uniformly with respect toε. The efficiency of the proposed method is discussed with the help of extensive computational work.

2. Statement of the problem

We consider the following boundary value problem for a singularly perturbed delay differential equation of reaction-diffusion type:

$$-\varepsilon y''(x) + a(x)y(x) + b(x)y(x-1) = f(x) \ ,0 < x < 2 \quad (1)$$

subject to the interval and boundary conditions,

$$y(x) = \phi(x); -1 \le x \le 0, \quad y(2) = \beta \tag{2}$$

where $0 < \varepsilon << 1$ and $a(x) > 0,\ b(x) < 0,\ a(x) + b(x) > 2\alpha$ for $\alpha > 0$ and $a(x), b(x),\ f(x)$ are given sufficiently smooth functions on $[0,2]$, $\phi(x)$ is a smooth function on $[-1,0]$ and β is a given constant which is independent of ε.

Assuming $y \in \Im = C^0[0,2] \cap C^1(0,2) \cap C^2\{(0,1) \cup (1,2)\}$.

The problem(1) and (2) can rewritten as

$$L_1 y(x) = -\varepsilon y''(x) + a(x)y(x) = f(x) - b(x)\phi(x-1), 0 \le x \le 1 \tag{3a}$$

$$L_2 y(x) = -\varepsilon y''(x) + a(x)y(x) + b(x)y(x-1) = f(x), 1 \le x \le 2 \tag{3b}$$

and $y(x) = \phi(x)$ on $[-1,0]$, $y(1-) = y(1+)$, $y'(1-) = y'(1+)$, $y(2) = \beta$, where $y(1-)$ and $y(1+)$ denote the left and right limit of y at $x = 1$ respectively. The solution $y(x)$ exhibits boundary layers at $x = 0$ and $x = 2$ and the interior layer $x = 1$ [5].

Throughout the paper, C denotes a generic positive constant that is independent of x as well as ε. In case of discrete problems, C is also independent of the mesh parameter N. $\|.\|$ denotes the global maximum norm over the appropriate domain of the independent variable, i.e., $\|f\| = \max_{x\in[0,2]} |f(x)|$.

The operator L corresponding to equation (1) satisfies the following continuous maximum principle and the stability estimate:

3. Stability result

Lemma 3.1. Let $a(x) > 0,\ b(x) < 0$ satisfy $a(x) + b(x) > 2\alpha$. Let $w \in \Im$ be any function satisfying $w(0) \ge 0$, $w(2) \ge 0$, $Lw(x) \ge 0$ on $(0,2)$, then $w(x) \ge 0$ on $[0,2]$.

Proof: Let x^* be such that $w(x^*) = \min_{x\in[0,2]} w(x)$. If $w(x^*) \ge 0$, there is nothing to prove.

Suppose $w(x^*) < 0$, then we have $x^* \notin [0,2]$. As $w''(x^*) \ge 0$,

$$Lw(x^*) = -\varepsilon w''(x^*) + a(x^*)w(x^*) + b(x^*)w(x^*-1)$$

$$\le -\varepsilon w''(x^*) + \left(a(x^*) + b(x^*)\right)w(x^*) < 0, \text{ as } w(x^*-1) \ge w(x^*)$$

which is a contradiction. Hence our assumption is wrong. Therefore, $w(x^*) \ge 0$, which proves the lemma.

Lemma 3.2. Let $a(x) > 0,\ b(x) < 0$ satisfy $a(x) + b(x) > 2\alpha$. Let $y \in \Im$ is any function, then for all $x \in [0,2]$, we have

$$|y(x)| \le \max\left\{|y(0)|, |y(2)|, \frac{1}{\alpha}\|Ly\|\right\}.$$

Proof: We construct two barrier functions $\psi^{\pm}$ defined by

$$\psi^{\pm}(x) = \max\left\{|y(0)|, |y(2)|, \frac{1}{\alpha}\|Ly\|\right\} \pm y(x).$$

Then we have

$$\psi^{\pm}(0) = \max\left\{y(0), |y(2)|, \frac{1}{\alpha}\|Ly\|\right\} \pm y(0)$$

$$= \max\left\{|y(0)|, |y(2)|, \frac{1}{\alpha}\|Ly\|\right\} \pm \phi_0, \text{ since } y(0) = \phi_0$$

$$\ge 0,$$

$$\psi^{\pm}(2) = \max\left\{|y(0)|, |y(2)|, \frac{1}{\alpha}\|Ly\|\right\} \pm y(2)$$

$$= \max\left\{|y(0)|, |y(2)|, \frac{1}{\alpha}\|Ly\|\right\} \pm \beta, \text{ since } y(2) = \beta$$

$$\ge 0,$$

and we have

$$L\psi^{\pm}(x) = -\varepsilon\left(\psi^{\pm}(x)\right)'' + a(x)\psi^{\pm}(x) + b(x)\psi^{\pm}(x-1)$$

$$= \max\left\{ w(0), |w(2)|, \frac{1}{\alpha}\|Lw\| \right\} \pm Ly(x) \ge 0.$$

Using the condition of $a(x)$ and $b(x)$ and using Lemma 3.1 we get $\psi^{\pm}(x) \ge 0$ on [0,2]. Therefore,

$$|y(x)| \le \max\left\{ w(0), |w(2)|, \frac{1}{\alpha}\|Lw\| \right\}, \quad x \in [0,2].$$

*Lemma 3.3.*Let $a(x) > 0,\ b(x) < 0$ satisfy $a(x) + b(x) > 2\alpha$ .Let y be the solution of (1) and (2). Then, for all $x \in [0,2]$, we have

$$\left|y^{(i)}(x)\right| \le C\varepsilon^{-\frac{i}{2}}\left(\|y\| + \|f\|\right), \text{ for } i = 0, 1$$

and

$$\left|y^{(i)}(x)\right| \le C\varepsilon^{-\frac{i}{2}}\left(\|y\| + \|f\| + \varepsilon^{\frac{(i-2)}{2}}\left\|f^{(i-2)}\right\| \right), \text{ for } i = 2, 3, 4.$$

Proof:

From (1) we have

$$y''(x) = \varepsilon^{-1}\left(a(x)y(x) + b(x)y(x-1) - f(x)\right) \tag{4}$$

and the bound on $y(x)$ follows from lemma 3.2 and bound on $y''(x)$ follows from (4).

To bound $y'(x)$ in the interval $(0,1)$, consider the interval $N = [a, a+\sqrt{\varepsilon}] \subset [0,1]$. Then by mean value theorem, for some $\xi \in N$,

$$y'(\xi) = \frac{y(a+\sqrt{\varepsilon}) - y(a)}{\sqrt{\varepsilon}},$$

$$\Rightarrow |y'(\xi)| \le 2\varepsilon^{-\frac{1}{2}}\|y\|.$$

Now for any $x \in N$, we get

$$y'(x) = y'(\xi) + \int_{\xi}^{x} y''(s)ds$$

$$= y'(\xi) + \int_{\xi}^{x} \left(- f(s) + a(s)y(s) + b(s)\phi(s-1)\right)ds$$

$$|y'(x)| \le |y'(\xi)| + \left(\|f(s)\| + \|a(s)\|\|y(s)\| + \|b(s)\|\|\phi(s-1)\|\right)\int_{\xi}^{x} ds$$

$$\therefore |y'(x)| \le C\varepsilon^{-\frac{1}{2}}\left(\|f\| + \|y\|\right).$$

Similarly, to bound $y'(x)$ in the interval $(1,2)$, consider the interval $N = [a, a+\sqrt{\varepsilon}] \subset (1,2]$ Then by mean value theorem, for some $\xi \in N$, $y'(\xi) = \dfrac{y(a+\sqrt{\varepsilon}) - y(a)}{\sqrt{\varepsilon}} \Rightarrow |y'(\xi)| \le 2\varepsilon^{-\frac{1}{2}}\|y\|$

Then for any $x \in N$, we get

$$\Rightarrow |y'(x)| \le C\varepsilon^{-\frac{1}{2}}\left(\|f\| + \|y(s)\|\right).$$

which follows the required bounds. Similarly differentiating (4) once and twice gives the bounds on $y^{(3)}$ and $y^{(4)}$ follows from those y' and y'' .

4. Numerical algorithm

*Step1.*By setting $\varepsilon = 0$ in equation (1), we get a recurrence relation for the solution of reduced problem as

$$y_0(x) = \frac{f(x) - b(x)y_0(x-1)}{a(x)} \tag{5}$$

whose solution does not satisfy both the conditions (2).

Hence the value of $y(x=1)$ can be obtained by the solution of reduced problem i.e., $y_0(x)$, i.e., $y(1)=\dfrac{f(1)-b(1)y_0(0)}{a(1)}=\gamma$ (say).

Now the problems (3) can be rewritten as

$$L_1 y(x)=-\varepsilon y''(x)+a(x)y(x)=f(x)-b(x)\phi(x-1), 0\le x\le 1 \tag{6}$$

subject to the conditions $y(x)=\phi(x); -1\le x\le 0$, $y(1)=\gamma$

and

$$L_2 y(x)=-\varepsilon y''(x)+a(x)y(x)+b(x)y(x-1)=f(x), 1\le x\le 2 \tag{7}$$

subject to the conditions $y(1)=\gamma$, $y(2)=\beta$.

Step2. The solution of (1)-(2)will be of the form

$$y(x)=y_0+v_0+w_0 \tag{8}$$

where v_0 and w_0 are the left and right boundary layer functions respectively.

v_0, w_0 satisfy the differential equations

$$-\varepsilon(y_o''+v_o''+w_o'')+a(x)(y_o+v_o+w_o)=f(x)-b(x)y(x-1)$$

$$\frac{-d^2v_0(\tau)}{d\eta^2}+b(1)w_0(\tau)=0; \qquad \tau\in(0,\infty) \tag{9}$$

$$\frac{-d^2w_0(\eta)}{d\eta^2}+b(1)w_0(\eta)=0; \qquad \eta\in(0,\infty) \tag{10}$$

with $v_0(\tau=0)+w_0\left(\eta=\dfrac{2}{\sqrt{\varepsilon}}\right)=\phi(0)-y_0(0)$

$$v_0\left(\tau=\frac{1}{\sqrt{\varepsilon}}\right)+w_0(\eta=0)=\beta-y_0(2)$$

$$v_0(\tau=\infty)=w_0(\eta=\infty)=0$$

where $\tau=\dfrac{x}{\sqrt{\varepsilon}}$ and $\eta=\dfrac{2-x}{\sqrt{\varepsilon}}$.

Solutions of (9) and (10) are given by

$$v_0(\tau)=Ae^{-\sqrt{a(0)}\tau} \tag{11}$$

$$w_0(\eta)=Be^{-\sqrt{a(2)}\eta} \tag{12}$$

Therefore, solution of (8) becomes

$$y(x)=y_0(x)+Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}x}+Be^{-\sqrt{\frac{a(2)}{\varepsilon}}(2-x)} \tag{13}$$

Applying the boundary conditions we get A and B as

$$A=\frac{(\phi(0)-y_0(0))-(\beta-y_0(2))e^{-2\sqrt{\frac{a(2)}{\varepsilon}}}}{1-e^{-\frac{\left(\sqrt{2a(0)}+\sqrt{2a(2)}\right)}{\sqrt{\varepsilon}}}}, \tag{14}$$

$$B=\frac{(\beta-y_0(2))-(\phi(0)-y_0(0))e^{-2\sqrt{\frac{a(0)}{\varepsilon}}}}{1-e^{-\frac{\left(\sqrt{2a(0)}+\sqrt{2a(2)}\right)}{\sqrt{\varepsilon}}}}. \tag{15}$$

Step 4.

Now the interval $[0,2]$ is divided into $2N$ equal subintervals of constant step length h. Let $0=x_0,x_1,\ldots x_N(=1),x_{N+1},\ldots,x_{2N}=2$ be the mesh points, such that $x_i=ih; i=0,1,\ldots.2N$. We choose N such that $x_N=1$ and $x_{2N}=2$. The boundary layer lies to the left end of the interval $[0,1]$ and to the right end of the interval $[1,2]$.

At a point $x=x_i$, the equation (6) becomes

$\varepsilon y_i''(x)=g(x_i,y_i)$ where, $g(x_i,y_i)=-f(x_i)+a(x_i)y(x_i)+b(x_i)\phi(x_{i-N})$.

By Numerov method, we have $\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)=\frac{1}{12}(g_{i-1}+10g_i+g_{i+1})$.

$$i.e., \varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)=\frac{1}{12}\begin{pmatrix}(-f_{i-1}+a_{i-1}y_{i-1}+b_{i-1}\phi(x_{i-1-N}))+\\ 10(-f_i+a_iy_i+b_i\phi(x_{i-N}))+\\ (-f_{i+1}+a_{i+1}y_{i+1}+b_{i+1}\phi(x_{i+1+N}))\end{pmatrix}.$$

Therefore, we have

$$\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)-\frac{1}{12}(a_{i-1}y_{i-1}+10a_iy_i+a_{i+1}y_{i+1}) = \frac{-1}{12}\left((f_{i-1}+10f_i+f_{i+1})-(b_{i-1}\phi(x_{i-1-N})+10b_i\phi(x_{i-N})+b_{i+1}\phi(x_{i+1-N}))\right). \tag{16}$$

Now a fitting factor σ is introduced in the above difference scheme as

$$\varepsilon\sigma\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)-\frac{1}{12}(a_{i-1}y_{i-1}+10a_iy_i+a_{i+1}y_{i+1}) = \frac{-1}{12}\left((f_{i-1}+10f_i+f_{i+1})-(b_{i-1}\phi(x_{i-1-N})+10b_i\phi(x_{i-N})+b_{i+1}\phi(x_{i+1-N}))\right)$$

for $i=1,2,..N-1$. (17)

To find σ on the left boundary layer we use the asymptotic solution

$$v_0(x_i)=y_i=Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}x_i} \tag{18}$$

where A is given by (14).

We assume that solution converges uniformly to the solution of (1), then $f_{i-1}+10f_i+f_{i+1}$ is bounded.

As $h\to 0$ equation (17) becomes

$$\lim_{h\to 0}\frac{\sigma}{\rho^2}(y_{i-1}-2y_i+y_{i+1})=\frac{a(0)}{12}\lim_{h\to 0}(y_{i-1}+10y_i+y_{i+1}) \tag{19}$$

where $\rho=\frac{h}{\sqrt{\varepsilon}}$

Substituting (18) in (19) and simplifying, we get the fitting factor as

$$\sigma=\frac{\rho^2 a(0)\left(e^{\sqrt{a(0)}\rho}+e^{-\sqrt{a(0)}\rho}+10\right)}{48Sinh^2\left(\frac{\sqrt{a(0)}\rho}{2}\right)} \tag{20}$$

which is a constant fitting factor. This will be the fitting factor in the interval $[0,1]$.

Substituting the fitting factor (20) in (17), we have the three term recurrence relation as

$$\left(\frac{\varepsilon\sigma}{h^2}-\frac{a_{i-1}}{12}\right)y_{i-1}-\left(\frac{2\varepsilon\sigma}{h^2}+\frac{10}{12}a_i\right)y_i+\left(\frac{\varepsilon\sigma}{h^2}-\frac{a_{i+1}}{12}\right)y_{i+1} = \frac{-1}{12}\left((f_{i-1}+10f_i+f_{i+1})-(b_{i-1}\phi(x_{i-1-N})+10b_i\phi(x_{i-N})+b_{i+1}\phi(x_{i+1-N}))\right)$$

for $i=1,2,..N-1$. (21)

The above tridiagonal system (21) along with the boundary conditions $y_0=\phi(0);\ y_N=\gamma$ can solved by Thomas algorithm.

Step 5.

At a point $x=x_i$, the differential equation (7) can be written as

$\varepsilon y_i''(x)=g(x_i,y_i)$ where, $g(x_i,y_i)=-f(x_i)+a(x_i)y(x_i)+b(x_i)y(x_{i-N})$

Proceeding as in Step 4, we get the three-term recurrence relation

$$\left(\frac{\varepsilon\sigma_1}{h^2}-\frac{a_{i-1}}{12}\right)y_{i-1}-\left(\frac{2\varepsilon\sigma_1}{h^2}+\frac{10}{12}a_i\right)y_i+\left(\frac{\varepsilon\sigma_1}{h^2}-\frac{a_{i+1}}{12}\right)y_{i+1} = \frac{-1}{12}\left((f_{i-1}+10f_i+f_{i+1})-(b_{i-1}y(x_{i-1-N})+10b_iy(x_{i-N})+b_{i+1}y(x_{i+1-N}))\right)$$

for $i=N+1,N+2,...,2N-1$. (22)

The above tridiagonal system (22) along with the boundary conditions $y_N=\gamma;\ y_{2N}=\beta$ can solved by Thomas algorithm.

5. Numerical examples

The numerical method proposed in this paper is applied to four examples to illustrate the $\varepsilon-$uniform convergence. Since the exact solutions for these problems are not available, the maximum absolute errors are calculated using the double mesh principle as follows:

$$E_\varepsilon = \max_{0\le i\le N}\left|y_i^N - y_{2i}^{2N}\right|.$$

For a value of N, the ε-uniform maximum absolute error is calculated by the formula $E^N = \max_{\varepsilon} E_\varepsilon$.

The numerical rate of convergence for all the examples has been calculated by the formula $R^N = \dfrac{\log|E_\varepsilon / E_\varepsilon^{2N}|}{\log 2}$.

Example 1. [24, p. 76]. Consider a constant coefficient boundary value problem
$-\varepsilon y''(x) + 5y(x) - y(x-1) = 1,\ \ y(x) = 1; -1 \le x \le 0,\ y(2) = 2.$

Example 2. [26, p. 87]. Consider a constant coefficient boundary value problem
$-\varepsilon y''(x) + 2y(x) - y(x-1) = 0,\ \ y(x) = 1; -1 \le x \le 0,\ y(2) = 1.$

Example 3. [24, p. 76]. Consider a variable coefficient boundary value problem
$-\varepsilon y''(x) + (x+5)y(x) - y(x-1) = 1,\ \ y(x) = 1; -1 \le x \le 0,\ y(2) = 2.$

Example 4. [24, p. 76]. Consider a boundary value problem with discontinuous source term

$$-\varepsilon y''(x) + 5y(x) - y(x-1) = \begin{cases} 1; & 0 \le x \le 1 \\ -1; & 1 < x \le 2 \end{cases},\ \ y(x) = 1; -1 \le x \le 0,\ y(2) = 2.$$

The maximum point wise errors and the rates of convergence of the boundary value problems in Examples 1-4 are presented in Tables 1-4 respectively. The numerical solutions plotted in Figures 1-4 illustrate the nature of the boundary layers for these problems.

We compared our results with the results available in [26]. It has been observed that the numerical rate of convergence is better than the method proposed in [26].

Table 1.

ε↓ 2N→	512	1024	2048	4096	8192	16384
2^{-9}	2.0501e-004	9.6607e-005	4.6858e-005	2.3073e-005	1.1449e-005	5.7023e-006
2^{-10}	3.0199e-004	1.4003e-004	6.7099e-005	3.2837e-005	1.6242e-005	8.0771e-006
2^{-11}	4.5089e-004	2.0501e-004	9.6607e-005	4.6858e-005	2.3073e-005	1.1449e-005
2^{-12}	6.9788e-004	3.0199e-004	1.4003e-004	6.7099e-005	3.2837e-005	1.6242e-005
2^{-13}	1.0677e-003	4.5089e-004	2.0501e-004	9.6607e-005	4.6858e-005	2.3073e-005
2^{-14}	1.7074e-003	6.9788e-004	3.0199e-004	1.4003e-004	6.7099e-005	3.2837e-005
2^{-15}	2.3379e-003	1.0677e-003	4.5089e-004	2.0501e-004	9.6607e-005	4.6858e-005
R^N	1.0855	1.0438	1.0221	1.0110	1.0056	---

Table 2.

ε↓ 2N→	512	1024	2048	4096	8192	16384
2^{-9}	2.5868e-004	1.2449e-004	6.1054e-005	3.0232e-005	1.5042e-005	7.5028e-006
2^{-10}	3.7669e-004	1.7881e-004	8.7028e-005	4.2926e-005	2.1316e-005	1.0621e-005
2^{-11}	5.5695e-004	2.5868e-004	1.2449e-004	6.1054e-005	3.0232e-005	1.5042e-005
2^{-12}	8.3295e-004	3.7669e-004	1.7881e-004	8.7028e-005	4.2926e-005	2.1316e-005
2^{-13}	1.2725e-003	5.5695e-004	2.5868e-004	1.2449e-004	6.1054e-005	3.0232e-005
2^{-14}	1.8719e-003	8.3295e-004	3.7669e-004	1.7881e-004	8.7028e-005	4.2926e-005
2^{-15}	3.0986e-003	1.2725e-003	5.5695e-004	2.5868e-004	1.2449e-004	6.1054e-005
R^N	1.0551	1.0279	1.0140	1.0071	1.0035	---

Table 3.

ε↓ 2N→	512	1024	2048	4096	8192	16384
2^{-9}	1.8782e-004	8.8565e-005	4.2983e-005	2.1170e-005	1.0505e-005	5.2323e-006
2^{-10}	2.7566e-004	1.2817e-004	6.1432e-005	3.0068e-005	1.4874e-005	7.3967e-006
2^{-11}	4.1067e-004	1.8726e-004	8.8280e-005	4.2842e-005	2.1099e-005	1.0469e-005
2^{-12}	6.2992e-004	2.7512e-004	1.2788e-004	6.1289e-005	2.9997e-005	1.4838e-005
2^{-13}	9.7186e-004	4.1015e-004	1.8697e-004	8.8135e-005	4.2770e-005	2.1063e-005
2^{-14}	1.5026e-003	6.2936e-004	2.7484e-004	1.2774e-004	6.1216e-005	2.9961e-005
2^{-15}	1.9213e-003	9.7137e-004	4.0988e-004	1.8682e-004	8.8062e-005	4.2734e-005
R^N	0.98399	1.0430	1.0217	1.0109	1.0056	---

Table 4.

ε↓ 2N→	512	1024	2048	4096	8192	16384
2^{-9}	2.0501e-004	9.6607e-005	4.6858e-005	2.3073e-005	1.1449e-005	5.7023e-006
2^{-10}	3.0199e-004	1.4003e-004	6.7099e-005	3.2837e-005	1.6242e-005	8.0771e-006
2^{-11}	4.5089e-004	2.0501e-004	9.6607e-005	4.6858e-005	2.3073e-005	1.1449e-005
2^{-12}	6.9788e-004	3.0199e-004	1.4003e-004	6.7099e-005	3.2837e-005	1.6242e-005
2^{-13}	1.0677e-003	4.5089e-004	2.0501e-004	9.6607e-005	4.6858e-005	2.3073e-005
2^{-14}	1.7074e-003	6.9788e-004	3.0199e-004	1.4003e-004	6.7099e-005	3.2837e-005
2^{-15}	2.3379e-003	1.0677e-003	4.5089e-004	2.0501e-004	9.6607e-005	4.6858e-005
R^N	1.0855	1.0438	1.0221	1.0110	1.0056	---

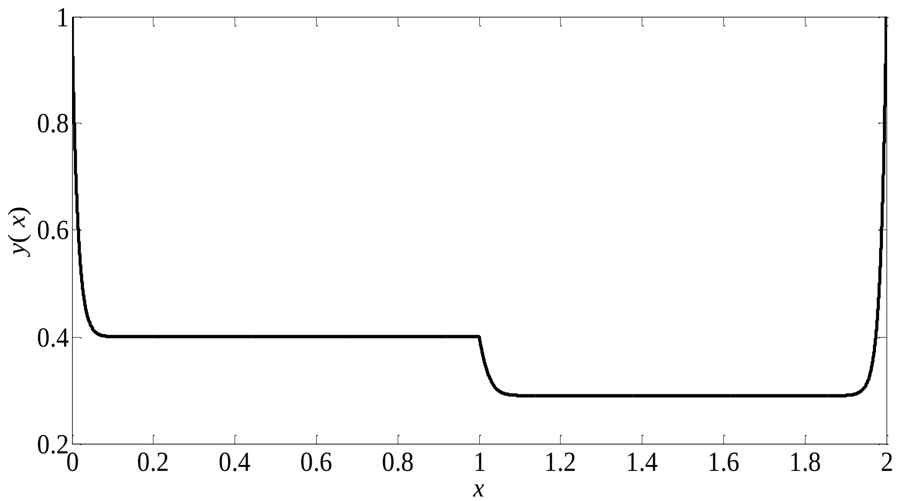

Figure 1. *Graph of the solution with* $\varepsilon = 2^{-10}$ *for Example 1.*

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042110 doi:10.1088/1757-899X/263/4/042110

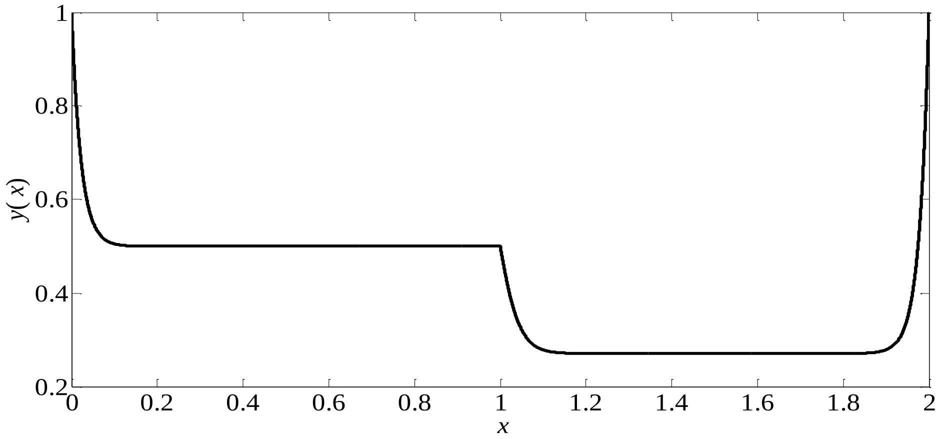

Figure 2. *Graph of the solution with* $\varepsilon = 2^{-10}$ *for Example 2.*

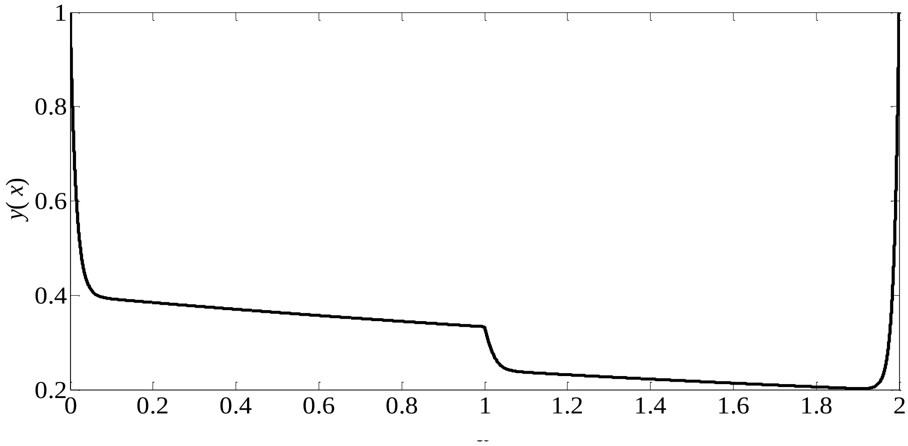

Figure 3. *Graph of the solution with* $\varepsilon = 2^{-10}$ *for Example 3.*

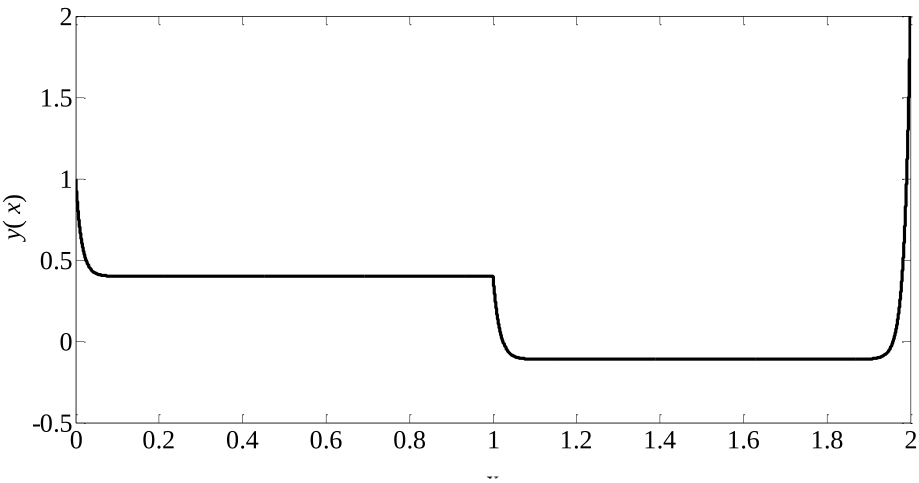

Figure 4. *Graph of the solution with* $\varepsilon = 2^{-10}$ *for Example 4.*

6. Discussion and Conclusions

In this paper, we considered a boundary value problem for a singularly perturbed delay differential equation of reaction-diffusion type. To obtain an approximate solution of this problem, we constructed an exponentially fitted numerical method using Numerov finite difference scheme. The method resolves the boundary layers due to the perturbation parameter as well as the interior layers due to the delay term. The proposed method is ε uniformly convergent order one and has been illustrated in tables 1-4. To illustrate the nature of the boundary layers and the internal layer, graphs are plotted in Figures 1-4 for problems given in Examples 1-4 respectively. By considering several numerical results on a variety of examples, it is concluded that the present method is efficient in solving the singularly perturbed linear differential equations of reaction-diffusion type with delay.

References

[1] Bellman R 1963 Cooke, K.L.,Differential - Difference Equations, Academic press, New York
[2] Driver R D 1977 Ordinary and Delay Differential Equations, Springer, Berlin, Heidelberg, New York.
[3] Bellen, M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford University Press, Oxford, 2003.
[4] Lange C G, Miura R 1994 *J. Applied Mathematics* **54** 249–272
[5] Lange CG, Miura R M 1982 *J. Applied Mathematics* **42**(3) 502-530
[6] Glizer VY 2000 J. *Optim. Theory Appl.***106** 49-85
[7] Glizer V Y 2003 *J. Math. Anal.* **278** 409-433
[8] Derstine M W, Gibbs F A H H M, Kaplan D L 1982 *Phys. Rev. A* **26** 3720-3722
[9] Longtin A, Milton J 1988 *Math. Biosci.* **90** 183-199
[10] Wazewska-Czyzewska M, Lasota A, 1976*Mat. Stos*. **6** 25-40
[11] Mackey MC, Glass L 1977 Oscillations and chaos in physiological control systems, Science **197** 287-289
[12] Doolan E R, Miller J J H, Schilders W H A 1980 *Uniform Numerical methods for problems with initial and boundary layers*, Boole Press, Dublin
[13] Miller JJH, O'Riordan E, Shiskin GI 1996 *Fitted numerical methods for singular perturbation problems*, World scientific, Singapore
[14] Roos H G, Stynes M, Tobiska L 1996 Springer-Verlag, Berlin
[15] Farrell PA, Hegarty A F, Miller J J H, O'Riordan E, Shishkin G I 2000 Robust computational techniques for boundary layers, Chapman & Hall/CRC, New York
[16] Kadalbajoo M K, Sharma K K, 2004 *Applied Mathematics and Computation* **157** 11-28
[17] Kadalbajoo M K, Sharma K K 2006 *Electronic Transactions on Numerical Analysis* **23**180-201
[18] Kadalbajoo M K, Sharma K K A 2008 *Applied Mathematics and Computation* **197** 692-707
[19] Kadalbajoo M K, RameshVP 2007 *Applied Mathematics and Computation* **187** 797–814
[20] Kadalbajoo M K, Kumar D A 2010 *Applied Mathematical Modelling* **34** 2584–2596
[21] Ramos J I 2006 *Applied Mathematics and Computation* **182** 1528–1541
[22] Mohapatra J, Natesan S 2011 *Int. J.Numerical Methods in Biomedical Eng.* **27**(9) 1427–1445
[23] SubburayanV, Ramanujam N J 2013 *Optim Theory Appl.* **158** 234-250
[24] SubburayanV,Ramanujam N 2013 *Novi Sad J. Math*. **43**(2) 67-80
[25] Amiraliyev G M,Cimen E 2010 *Applied Mathematics and Computation* **216** 2351-2359
[26] Manikandan M, Shivaranjani N, Miller J J H, Valarmathi S 2014 *Springer Proceedings in Mathematics & Statistics* **87** 71-88