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A study on M/G/1 feedback retrial queue with subject to server breakdown and repair under multiple working vacation policy

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Abstract In this paper, we consider a single server feedback retrial queueing system with multiple working vacations and vacation interruption. An arriving customer may balk the system at some particular times. As soon as orbit becomes empty at regular service completion instant, the server goes for a working vacation. The server works at a lower service rate during working vacation (WV) period. After completion of regular service, the unsatisfied customer may rejoin into the orbit to get another service as feedback customer. The normal busy server may get to breakdown and the service channel will fail for a short interval of time. The steady state probability generating function for the system size is obtained by using the supplementary variable method. Some important system performance measures are obtained. Finally, some numerical examples and cost optimization analysis are presented.

1. Introduction

In queueing theory, vacation queues and retrial queues have been intensive research topics for long time. We can find general models in vacation queues from Ke et al. [1] and in retrial queues from Artalejo and Gomez-Corral [2]. In retrial queueing system, retrial queues with repeated attempts are characterized by the fact that an arriving customer finds the server busy upon arrival is requested to leave the service area and join a retrial queue called orbit. After some time the customer in the orbit can repeat their request for service. An arbitrary customer in the orbit who repeats the request for service is independent of the rest of the customers in the orbit. Such queues play a special role in computer and telecommunication systems.

The concept of balking (customers decide not to join the line at all if he finds the server is unavailable upon arrival) was first studied by Haight in 1957. There are many situations where the customers may be impatient, such as impatient telephone switchboard customers, web access, including call centers and computer systems. Ke [3] studied an $M^{[X]}/G/1$ queue with variant vacations and balking. Some of the authors

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like Wang and Li [4], and Gao and Wang [5] discussed about the concept balking.

One additional feature that has been widely discussed in retrial queueing systems is the Bernoulli feedback of customers. Many queueing situations have the feature that the customers may be served repeatedly for a certain reason. When the service of a customer is unsatisfied, it may be retried again and again until a successful service completion. These queueing models arise in the stochastic modeling of many real-life situations. For example, in data transmission, a packet transmitted from the source to the destination may be returned and it may continue this process until the packet is finally transmitted. The retrial queue with feedback can be used to model the Automative Repeat re-Quest (ARQ) protocol in a high frequency communication network. In ARQ, if the sender does not receive an acknowledgment before timeout, it usually re-transmits the frame/packet until the sender receives an acknowledgment or exceeds a predefined number of retransmissions. These types of retransmissions are called feedback. Ke and Chang [6] have discussed Modified vacation policy for M/G/1 retrial queue with balking and feedback. Some of the authors like Baruah et al. [7], Krishnakumar et al. [8] and Rajadurai et al. [9] have discussed the concept of feedback.

In the queueing literature, mostly it is assumed that the server is available all the time on permanent basis. But this is unrealistic due to heavy influence of breakdowns on the server. Thus, it is important to study retrial queue with breakdowns to implement retrial queueing models practically. Choudhury and Ke [10] have studied the batch arrival retrial queue with Bernoulli vacations and delaying repair. Recently authors like Choudhury and Deka [11], Yang et al. [12], Rajadurai et al. [13–15] and Dimitriou [16,17] discussed about the retrial queueing systems with the concept of breakdown and repair.

In working vacation period (WV), the server gives service to customer at lower service rate, but the server stops the service completely in the normal vacation period. This queueing system has major applications in providing services such as network service, web service, file transfer service and mail service. In 2002, Servi and Finn [18] have introduced an M/M/1 queueing system with working vacations. Wu and Takagi [19] have extended the M/M/1/WV queue to an M/G/1/WV queue. Very recently, Arivudainambi et al. [20] have introduced the single working vacation concept in the M/G/1 retrial queue. Chandrasekaran et al. [21] have presented a short survey on working vacation queueing models. A retrial queueing system with non-persistent customers and working vacations has been discussed by Liu and Song [22].

Furthermore, during the working vacation period, if there are customers at a service completion instant, the server can stop the vacation and come back to regular busy state. This policy is called vacation interruption. Li and Tian [23] have presented an M/M/1 queueing model with working vacations and vacation interruption. Some of the authors like Gao and Liu [24], Gao et al. [25], Zhang and Liu [26], Zhang and Hou [27] have analyzed a single server retrial queue with working vacations and vacation interruptions.

In this paper, we have generalized the work of Gao et al. [25] and Arivudainambi et al. [20] by incorporating the concept of balking, feedback and subject to server breakdown and repair. To the author's best of knowledge, there is no work published in the queueing literature with the combination of retrial queueing system with general retrial times, feedback, balking, multiple working vacations and vacation interruption, where the server subjects to breakdown and repair by using the method of supplementary variable technique. The mathematical results and theory of queues of this model provide a specific and convincing application in Simple Mail Transfer Protocol (SMTP) mail system which uses to deliver the messages between mail servers and computer processing system. Our model is helpful to managers who can design a system with economic management.

The rest of this work is given as follows. The detailed mathematical description and practical applications of our model are given in Section 2. The steady state joint distribution of the server state and the number of customers in the system and its orbit are obtained in Section 3. Some system performance measures and reliability measures are obtained in Section 4. In Section 5, conditional stochastic decomposition is shown good for our model. Important special cases of this model are given in Section 6. Cost optimization analysis is discussed in Section 7. In Section 8, the effects of various parameters on the system performance are analyzed numerically. Summary of the work and some future directions are presented in Section 9.

2. Description of the model and its applications

In this paper, we consider a single server retrial queueing system with balking and feedback under multiple working vacation policy, where the busy server is subjected to breakdown and repair. The detailed description of model is given as follows:

The arrival Process: The primary customers arrive at the system according to a Poisson process of rate λ.

The retrial process: We assume that there is no waiting space and therefore if an arriving customer finds that the server is free, the customer occupies it immediately. Otherwise, the server is busy or working vacation or breakdown; the arrivals either leave the system with probability $(1-b)$ or join pool of blocked customers called an orbit with probability b in accordance with FCFS discipline, which means that only the customer at the head of the orbit is allowed to access the server. We assume that inter-retrial times follow a general random variable R with an arbitrary distribution $R(t)$ having corresponding Laplace Stieltjes Transform (LST) $R^*(\vartheta)$.

The working vacation process: The server begins a working vacation each time when the orbit becomes empty and the vacation time follows an exponential distribution with parameter θ. If any customers arrive in a vacation period, the server continues to work at a lower rate. The working vacation period is an operation period at a lower speed. If any customers in the orbit are at a service completion instant in the vacation period, the server will stop the vacation and come back to the normal busy period which means vacation interruption happens. Otherwise, it continues the vacation. When a vacation ends, if there are customers in the orbit, the server switches to the normal working level. Otherwise, the server begins another vacation. During the working vacation period, the service time follows a general random variable S_v with distribution function (d.f.) $S_v(t)$, LST $S_v^*(x)$ and $S_v^{*\prime}(\theta)=\int_0^\infty xe^{-\theta x}dS_v(x)$.

The regular service process: Whenever a new customer or retry customer arrives at the server idle state then the server immediately starts normal service for the arrivals. The service time follows a general distribution and it is denoted by the random variable S_b with distribution function $S_b(t)$ having LST $S_b^*(\vartheta)$ and the first and second moments are $\beta^{(1)}$ and $\beta^{(2)}$.

The feedback rule: After completion of regular service for each customer, the unsatisfied customers may rejoin into the orbit as a feedback customer for receiving another service with probability $p(0 \leqslant p \leqslant 1)$ or may leave the system with complement probability $q(=1-p)$.

The breakdown process: The regular busy server may breakdown at any instance and the service channel will fail for a short interval of time. i.e., server is down for a short interval of time. The breakdown, i.e., server's life times, is generated by exogenous Poisson processes with rates α which we may call some sort of disaster during regular busy period.

The repair process: As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the primary customers till service channel is repaired. The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by G) distribution of the server is assumed to be arbitrarily distributed with d.f. $G(t)$, having LST $G^*(\vartheta)$ and the first and second moments are $g^{(1)}$ and $g^{(2)}$.

We assume that inter-arrival times, retrial times, service times, working vacation times and repair times are mutually independent.

2.1. Practical application of the model

The proposed model has potential application in the transfer model of an email system. In Simple Mail Transfer Protocol (SMTP) mail system uses to deliver the messages between mail servers for relaying. The mail transfer program contacts a server on a remote machine; it forms a Transmission Control Protocol (TCP) connection over which it communicates. When the TCP is connected, SMTP allows the sender to identify it and specify the recipients and then transfers an email message. For receiving messages, client applications usually use the Internet Message Access Protocol (IMAP) to access their mail box accounts on a mail server. Typically, contacting messages arrive at the mail server following the Poisson stream.

At the arrival epoch, the arriving message starts its service immediately if the server is free or else may join the buffer. In the buffer, each message waits for some amount of time and retries the service again. The mail server employs a spam filter service in a low service rate to prevent spam mails. This is done to filter the incoming message via the normal mail receiving service. The target server is the same as sender's mail server and the sending message will be possibly retransmitted to the server to request the receiving service from the buffer one more time. The mail server may subject to electronic fail during service period and receive repair immediately.

To keep the mail server functioning well, virus scan is an important maintenance activity for the mail server. It can be performed when the mail server is idle. During the period of virus scan, the server can still provide its service, but with lower processing speed. When the virus scan is done, there is no mail in the buffer and the server takes another maintenance activity like power savings or again virus scanning. At the end of the period of virus scan or lower processing service, if there are any mails in the buffer, the mail server changes the service rate and comes back to the normal service level. In this queueing scenario, the buffer in the sender mail server, the receiver mail server IMAP, the retransmission policy, the electronic fail and the maintenance activities correspond to the orbit, the server, the retrial, the breakdown and the working vacation policy respectively.

This model finds the another practical applications in the telephone consultation in medical service systems, in the manufacturing system, in performance analysis of Small Cell Networks, in a packet switched network to forward the packets within a network for transmission, in computer networking systems and communication systems.

3. Analysis of the steady state probabilities

In this section, we first develop the steady state equations for the retrial system by treating the elapsed retrial times, the elapsed times of normal service and lower speed service, and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states, the PGF for number of customers in the system and orbit.

3.1. The steady state equations

In steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_b(0)=0$, $S_b(\infty)=1$, $S_v(0)=0$, $S_v(\infty)=1$ are continuous at $x=0$ and $G(0)=0$, $G(\infty)=1$ are continuous at $y=0$. So that the function $a(x)$, $\mu_b(x)$, $\mu_v(x)$ and $\xi(y)$ are the conditional completion rates for retrial, normal service, lower rate service, and repair respectively.

$$a(x)dx=\frac{dR(x)}{1-R(x)},\ \mu_b(x)dx=\frac{dS_b(x)}{1-S_b(x)},$$
$$\mu_v(x)dx=\frac{dS_v(x)}{1-S_v(x)}\ \text{and}\ \xi(y)dy=\frac{dG(y)}{1-G(y)}.$$

In addition, let $R^0(t), S_b^0(t), S_v^0(t)$ and $G^0(t)$ be the elapsed retrial times, elapsed normal service times, elapsed lower rate service times and elapsed repair times respectively at time t. Further, introduce the random variable,

$$C(t)=\begin{cases}0, & \text{if the server is in working vacation period at time } t \text{ and the server is free,}\\ 1, & \text{if the server is in normal service period at time } t \text{ and the server is free,}\\ 2, & \text{if the server is in normal service period at time } t \text{ and the server is busy,}\\ 3, & \text{if the server is in working vacation period at time } t \text{ and the server is busy,}\\ 4, & \text{if the server is under regular repair period at time } t.\end{cases}$$

We also note that the states of the system at time t can be described by the bivariate Markov process $\{C(t), N(t); t \geqslant 0\}$ where $C(t)$ denotes the server state $(0,1,2,3,4)$ depending on the server is free or busy on both normal service and working vacation periods and repair. $N(t)$ denotes the number of customers in the orbit. If $C(t)=1$ and $N(t)>0$, then $R^0(t)$ represent the elapsed retrial time, if $C(t)=2$ and $N(t)\geqslant 0$ then $S_b^0(t)$ corresponding to the elapsed time of the customer being served in normal busy period. If $C(t) = 3$ and $N(t) \geqslant 0$, then $S_v^0(t)$ corresponding to the elapsed time of the customer being served in lower rate service period. If $C(t) = 4$ and $N(t) \geqslant 0$, then $G^0(t)$ corresponding to the elapsed time of the server being repaired.

Let $\{t_n, n \in N\}$ be the sequence of epochs at which a normal service or a lower speed service completion occurs or a repair period ends. The sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. It follows from Appendix A that $\{Z_n; n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda)$ for our system to be stable, where $\rho = p + \lambda b\beta^{(1)}(1 + \alpha g^{(1)})$.

For the process $\{N(t), t \geqslant 0\}$ we define the probabilities $P_0(t) = P\{C(t) = 0, N(t) = 0\}$ and the probability densities

$$\begin{aligned}
&\psi_n(x,t)dx = P\{C(t)=1, N(t)=n, x \leqslant R^0(t) < x+dx\}, && \text{for } t \geqslant 0,\ x \geqslant 0 \text{ and } n \geqslant 1,\\
&\Pi_{b,n}(x,t)dx = P\{C(t)=2, N(t)=n, x \leqslant S_b^0(t) < x+dx\}, && \text{for } t \geqslant 0,\ x \geqslant 0 \text{ and } n \geqslant 0,\\
&\Pi_{v,n}(x,t)dx = P\{C(t)=3, N(t)=n, x \leqslant S_v^0(t) < x+dx\}, && \text{for } t \geqslant 0,\ x \geqslant 0 \text{ and } n \geqslant 0,\\
&R_n(x,y,t)dy = P\{C(t)=4, N(t)=n, y \leqslant G^0(t) < y+dy/S_b^0(t)=x\}, && \text{for } t \geqslant 0,\ (x,y) \geqslant 0 \text{ and } n \geqslant 0.
\end{aligned}$$

The following probabilities are used in sequent sections:

$P_0(t)$ is the probability that the system is empty at time t and the server is in working vacation.
$\psi_n(x, t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed retrial time of the test customer undergoing retrial is x.
$\Pi_{b,n}(x, t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed normal service time of the test customer undergoing service is x.
$\Pi_{v,n}(x, t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed lower rate service time of the test customer undergoing service is x.
$R_n(x, y, t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed normal service time of the test customer undergoing service is x and the elapsed repair time of server is y.

We assume that the stability condition is fulfilled and so that we can set for $t \geqslant 0$, $x \geqslant 0$ and $n \geqslant 1$.

$$P_0 = \lim_{t\to\infty} P_0(t),\ \psi_n(x) = \lim_{t\to\infty}\psi_n(x,t),\ \Pi_{b,n}(x) = \lim_{t\to\infty}\Pi_{b,n}(x,t),$$
$$\Pi_{v,n}(x) = \lim_{t\to\infty}\Pi_{v,n}(x,t),\ \text{and}\ R_n(x,y) = \lim_{t\to\infty}R_n(x,y,t).$$

Using the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavior,

$$(\lambda+\theta)P_0 = q\int_0^\infty \Pi_{b,0}(x)\mu_b(x)dx + q\int_0^\infty \Pi_{v,0}(x)\mu_v(x)dx + \theta P_0 \tag{3.1}$$

$$\frac{d\psi_n(x)}{dx} + [\lambda + a(x)]\psi_n(x) = 0, \quad n \geqslant 1 \tag{3.2}$$

$$\frac{d\Pi_{b,0}(x)}{dx} + [\lambda+\alpha+\mu_b(x)]\Pi_{b,0}(x) = \lambda(1-b)\Pi_{b,0}(x), \quad n=0, \tag{3.3}$$

$$\begin{aligned}\frac{d\Pi_{b,n}(x)}{dx} + [\lambda+\alpha+\mu_b(x)]\Pi_{b,n}(x) &= \lambda(1-b)\Pi_{b,n}(x)\\ &+ \lambda b\Pi_{b,n-1}(x) + \int_0^\infty R_n(x,y)\xi(y)dy, \quad n \geqslant 1,\end{aligned} \tag{3.4}$$

$$\frac{d\Pi_{v,0}(x)}{dx} + [\lambda+\theta+\mu_v(x)]\Pi_{v,0}(x) = \lambda(1-b)\Pi_{v,0}(x), \quad n=0, \tag{3.5}$$

$$\begin{aligned}&\frac{d\Pi_{v,n}(x)}{dx} + [\lambda+\theta+\mu_v(x)]\Pi_{v,n}(x)\\ &= \lambda(1-b)\Pi_{v,n}(x) + \lambda b\Pi_{v,n-1}(x),\ n \geqslant 1,\end{aligned} \tag{3.6}$$

$$\frac{dR_0(x,y)}{dy} + [\lambda+\xi(y)]R_0(x,y) = \lambda(1-b)R_0(x,y), \quad n=0, \tag{3.7}$$

$$\begin{aligned}&\frac{dR_n(x,y)}{dy} + [\lambda+\xi(y)]R_n(x,y)\\ &= \lambda(1-b)R_n(x,y), +\lambda bR_{n-1}(x,y), \quad n \geqslant 1,\end{aligned} \tag{3.8}$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$\begin{aligned}\psi_n(0) &= p\int_0^\infty \Pi_{b,n-1}(x)\mu_b(x)dx + q\int_0^\infty \Pi_{b,n}(x)\mu_b(x)dx\\ &+ p\int_0^\infty \Pi_{v,n-1}(x)\mu_v(x)dx + q\int_0^\infty \Pi_{v,n}(x)\mu_v(x)dx\end{aligned} \tag{3.9}$$

$$\Pi_{b,0}(0) = \int_0^\infty \psi_1(x)a(x)dx + \theta\int_0^\infty \Pi_{v,0}(x)dx, \quad n=0, \tag{3.10}$$

$$\begin{aligned}\Pi_{b,n}(0) &= \int_0^\infty \psi_{n+1}(x)a(x)dx + \lambda\int_0^\infty \psi_n(x)dx\\ &+ \theta\int_0^\infty \Pi_{v,n}(x)dx, \quad n \geqslant 1,\end{aligned} \tag{3.11}$$

$$\Pi_{v,n}(0) = \begin{cases}\lambda P_0, & n=0\\ 0, & n \geqslant 1\end{cases} \tag{3.12}$$

$$R_n(x,0) = \alpha\Pi_b(x),\ n \geqslant 1. \tag{3.13}$$

The normalizing condition is

$$P_0+\sum_{n=1}^{\infty}\int_0^{\infty}\psi_n(x)dx+\sum_{n=0}^{\infty}\left(\int_0^{\infty}\Pi_{b,n}(x)dx+\int_0^{\infty}\Pi_{v,n}(x)dx+\int_0^{\infty}\int_0^{\infty}R_n(x,y)dxdy\right)=1 \quad (3.14)$$

3.2. The steady state solutions of the model

The probability generating function technique is used here to obtain the steady state solution of the retrial queueing model. To solve the above equations, we define the generating functions for $|z|\leqslant 1$, as follows:

$$\psi(x,z)=\sum_{n=1}^{\infty}\psi_n(x)z^n;\ \psi(0,z)=\sum_{n=1}^{\infty}\psi_n(0)z^n;\ \Pi_b(x,z)=\sum_{n=0}^{\infty}\Pi_{b,n}(x)z^n;\ \Pi_b(0,z)=\sum_{n=0}^{\infty}\Pi_{b,n}(0)z^n;\ \Pi_v(x,z)=\sum_{n=0}^{\infty}\Pi_{v,n}(x)z^n;\ \Pi_v(0,z)=\sum_{n=0}^{\infty}\Pi_{v,n}(0)z^n;$$

$$R(x,y,z)=\sum_{n=0}^{\infty}R_n(x,y)z^n \text{ and } R(x,0,z)=\sum_{n=0}^{\infty}R_n(x,0)z^n$$

Multiplying the steady state equation and steady state boundary condition (3.1)–(3.13) by z^n and summing over n, $(n=0,1,2,\ldots)$

$$\frac{\partial\psi(x,z)}{\partial x}+[\lambda+a(x)]\psi(x,z)=0 \quad (3.15)$$

$$\frac{\partial\Pi_b(x,z)}{\partial x}+[\lambda b(1-z))+\alpha+\mu_b(x)]\Pi_b(x,z)=\int_0^{\infty}R(x,y,z)\xi(y)dy \quad (3.16)$$

$$\frac{\partial\Pi_v(x,z)}{\partial x}+[\lambda b(1-z))+\theta+\mu_v(x)]\Pi_v(x,z)=0. \quad (3.17)$$

$$\frac{\partial R(x,y,z)}{\partial y}+[\lambda b(1-z)+\xi(y)]R(x,y,z)=0 \quad (3.18)$$

$$\psi(0,z)=(pz+q)\int_0^{\infty}\Pi_b(x,z)\mu_b(x)dx+(pz+q)\int_0^{\infty}\Pi_v(x,z)\mu_v(x)dx-\lambda P_0 \quad (3.19)$$

$$\Pi_v(0,z)=\lambda P_0 \quad (3.20)$$

$$\Pi_b(0,z)=\frac{1}{z}\int_0^{\infty}\psi(x,z)a(x)dx+\lambda\int_0^{\infty}\psi(x,z)dx+\theta\int_0^{\infty}\Pi_v(x,z)dx, \quad (3.21)$$

$$R(x,0,z)=\alpha\Pi_b(x,z) \quad (3.22)$$

Solving the partial differential Eqs. (3.15)–(3.18), it follows that

$$\psi(x,z)=\psi(0,z)[1-R(x)]\exp\{-\lambda x\} \quad (3.23)$$

$$\Pi_b(x,z)=\Pi_b(0,z)[1-S_b(x)]\exp\{-A_b(z)x\} \quad (3.24)$$

$$\Pi_v(x,z)=\Pi_v(0,z)[1-S_v(x)]\exp\{-A_v(z)x\} \quad (3.25)$$

$$R(x,y,z)=R(x,0,z)[1-G_b(y)]\exp\{-b(z)y\} \quad (3.26)$$

where $A_b(z)=b(z)+\alpha(1-G^*(b(z))), A_v(z)=\theta+\lambda b(1-z)$ and $b(z)=\lambda b(1-z)$.

Inserting Eqs. (3.20), (3.23), (3.25) in (3.21) and make some manipulation, finally we get,

$$\Pi_b(0,z)=\frac{\psi(0,z)}{z}[R^*(\lambda)+z(1-R^*(\lambda))]+\lambda P_0V(z) \quad (3.27)$$

where $V(z)=\frac{\theta}{\theta+\lambda b(1-z)}[1-S_v^*(A_v(z))]$.

Using (3.24), (3.25) in (3.19), we get

$$\psi(0,z)=(pz+q)\Pi_b(0,z)S_b^*(A_b(z))+(pz+q)\Pi_v(0,z)S_v^*(A_v(z))-\lambda P_0 \quad (3.28)$$

Using (3.28) and (3.27), we get

$$(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))\Pi_b(0,z)=\lambda P_0\left[\left((pz+q)S_v^*(A_v(z))-1\right)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right] \quad (3.29)$$

From the above equation, we know that the key element for obtaining $\psi(0,z)$ is to find the zeros of $f(z)=z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))=0$ in the range $0<z<1$ for the equation $f(z)=0$ (from Gao et al. [25]). To this end, we give the following lemma.

Lemma 3.1. *If $\rho<R^*(\lambda)$, the equation $z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))=0$ has no roots in the range $0<z<1$ and has the minimal nonnegative root $z=1$.*

Proof. We only need to prove that

$$u(z)\triangleq(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))$$

is a probability generating function of the number of customers that arrive in the system. Denote by U the time period from the epoch a service completion occurs, leaving the orbit non-empty, to the next service completion epoch, by N_U the number of primary customers that arrive during U and define $u_j(t)dt=P(t<U\leqslant t+dt,N(U)=j)$.

Then,

$$u_j(t)=e^{-\lambda t}\alpha(t)*a_j(t)+(1-\delta_{j,0})\lambda e^{-\lambda t}(1-R(t))*a_{j-1}(t),\quad j=0,1,2,\ldots$$

where * means convolution, $\alpha(t)$ is the p.d.f. of inter-retrial times, $b(t)$ is the p.d.f. of normal service times and $a_j(t)dt=e^{-\lambda t}\frac{(\lambda t)^j}{j!}b(t)$. Denote by $N_U(z)$ the probability generating function of N_U, we have that

$$\begin{aligned}N_U(z)&=\sum_{j=0}^{\infty}z^j\int_0^{\infty}u_j(t)dt\\&=\sum_{j=0}^{\infty}z^j\int_0^{\infty}\left(e^{-\lambda t}\alpha(t)*a_j(t)+(1-\delta_{j,0})\lambda e^{-\lambda t}(1-R(t))*a_{j-1}(t)\right)dt\\&=(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))\\&=u(z),\end{aligned}$$

which proves the expected result that $u(z) \triangleq (pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))$ is exactly a probability generating function. From assumption $\rho < R^*(\lambda)$, we have $E[N_u] = \frac{d}{dz}u(z)|_{z=1} = 1-(R^*(\lambda)-\rho) < 1$ and the convex function $u(z)$ is a monotonically increasing function of z for $0 \leqslant z \leqslant 1$, and $u(0) = P(N_U = 0) < 1, u(1) = 1$. So we can easily prove the expected result of Lemma 3.1. □

Then for $\rho < R^*(\lambda)$, $z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))$ never vanishes in the range $0 < z < 1$, and from (3.20) and (3.27) in (3.28), we obtain that

$$\psi(0,z) = \left\{\frac{z\lambda P_0[(pz+q)\left(S_v^*(A_v(z)) + V(z)S_b^*(A_b(z))\right)-1]}{z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))}\right\} \tag{3.30}$$

From (3.29), we get

$$\Pi_b(0,z) = \left\{\frac{\lambda P_0[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)]}{z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))}\right\} \tag{3.31}$$

Using (3.24), (3.31) and (3.22), we get

$$R(x,0,z) = \left\{\frac{\lambda\alpha P_0\left[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right][1-S_b(x)]\exp\{-A_b(z)x\}}{z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))}\right\} \tag{3.32}$$

Using (3.20), (3.30)–(3.32) in (3.23)–(3.26), then we get the limiting probability generating functions $\psi(x,z)$, $\Pi_b(x,z)$, $\Pi_v(x,z)$ and $R(x,z)$.

Next we are interested in investigating the marginal orbit size distributions due to system state of the server in following theorem.

Theorem 3.1. *Under the stability condition $\rho < R^*(\lambda)$, the stationary distributions of the number of customers in the system states are given by*

$$\psi(x,z) = \left\{\frac{z\lambda P_0\left[(pz+q)\left(S_v^*(A_v(z))+V(z)S_b^*(A_b(z))\right)-1\right](1-R(x))e^{-\lambda x}}{z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))}\right\} \tag{3.33}$$

$$\Pi_b(x,z) = \left\{\frac{\lambda P_0\left[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right][1-S_b(x)]\exp\{-A_b(z)x\}}{z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))}\right\} \tag{3.34}$$

$$\Pi_v(x,z) = \lambda P_0[1-S_v(x)]\exp\{-A_v(z)x\} \tag{3.35}$$

$$R(x,z) = \left\{\frac{\lambda\alpha P_0\left(1-G_b^*(b(z))\right)\left[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right][1-S_b(x)]\exp\{-A_b(z)x\}}{b(z)(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))}\right\} \tag{3.36}$$

where $A_b(z) = b(z)+\alpha(1-G^*(b(z)))$, $A_v(z) = \theta+\lambda b(1-z)$ and $b(z) = \lambda b(1-z)$.

$$V(z) = \frac{\theta}{\theta+\lambda b(1-z)}[1-S_v^*(A_v(z))]$$

From Theorem 3.1, we focus on the marginal generating functions of the orbit size and the server states given in the following Corollary 3.1.

Corollary 3.1. *Under the stability condition $\rho < R^*(\lambda)$, the stationary joint distributions of the number of customers in the orbit when server being idle, busy, on vacation and under repair are given by*

$$\begin{aligned}\psi(z) &= \int_0^\infty \psi(x,z)dx \\ &= \left\{\frac{z(1-R^*(\lambda))P_0\left[(pz+q)\left(S_v^*(A_v(z))+V(z)S_b^*(A_b(z))\right)-1\right]}{z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))}\right\}\end{aligned} \tag{3.37}$$

$$\Pi_b(z) = \int_0^\infty \Pi_b(x,z)dx = \left\{ \frac{\lambda P_0(1-S_b^*(A_b(z)))\left[\left((pz+q)S_v^*(A_v(z))-1\right)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right]}{A_b(z)\left(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))\right)} \right\} \tag{3.38}$$

$$\Pi_v(z) = \int_0^\infty \Pi_v(x,z)dx = \left\{ \frac{\lambda P_0 V(z)}{\theta} \right\} \tag{3.39}$$

$$R(z) = \int_0^\infty R(x,z)dx = \left\{ \frac{\lambda\alpha P_0\left(1-S_b^*(A_b(z))\right)\left(1-G_b^*(b(z))\right)\left[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right]}{(A_b(z)\times b(z))(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))} \right\} \tag{3.40}$$

Applying the normalizing condition $P_0 + \psi(1) + \Pi_b(1) + \Pi_v(1) + R(1) = 1$ *and using the equations by setting* $z = 1$ in (3.37)–(3.40), *we get*

$$P_0 = \left\{ \frac{R^*(\lambda) - p - \lambda b\beta^{(1)}(1+\alpha g^{(1)})}{qR^*(\lambda) + \frac{\lambda}{\theta}(1-S_v^*(\theta))[(b-p)+(1-b)R^*(\lambda)] + \lambda\beta^{(1)}(1+\alpha g^{(1)})\left((p-b)S_v^*(\theta)+(1-b)(1-S_v^*(\theta))R^*(\lambda)\right)} \right\} \tag{3.41}$$

Corollary 3.2. *Under the stability condition* $\rho < R^*(\lambda)$.

The probability generating function of the number of customers in the system

$$K_s(z) = \frac{Nr_s(z)}{Dr(z)} = P_0 + \psi(z) + z(\Pi_b(z) + \Pi_v(z) + R(z)) \tag{3.42}$$

$$Nr_s(z) = P_0 \left\{ \begin{array}{c} b(1-z)\left[\begin{array}{c} (z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda))) \\ S_b^*(A_b(z)))((\lambda zV(z)/\theta)+1) \\ +z(1-R^*(\lambda))\left[(pz+q)(S_v^*(A_v(z)) \\ +V(z)S_b^*(A_b(z)))-1\right] \end{array} \right] \\ +z(1-S_b^*(A_b(z)))\left[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda) \\ +z(1-R^*(\lambda)))+zV(z)\right] \end{array} \right\}$$

$$Dr(z) = b(1-z)\left(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))\right)$$

The probability generating function of the number of customers in the orbit

$$K_o(z) = \frac{Nr_o(z)}{Dr(z)} = P_0 + \psi(z) + \Pi_b(z) + \Pi_v(z) + R(z) \tag{3.43}$$

$$Nr_o(z) = P_0 \left\{ \begin{array}{c} b(1-z)\left[\begin{array}{c} \left(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))\right)((\lambda V(z)/\theta)+1) \\ +z(1-R^*(\lambda))\left[(pz+q)(S_v^*(A_v(z))+V(z)S_b^*(A_b(z)))-1\right] \end{array} \right] \\ +\left(1-S_b^*(A_b(z))\right)\left[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right] \end{array} \right\}$$

where P_0 *is given in* Eq. (3.41).

4. System performance measures of the model

In this section, we derive some system probabilities, mean number of customers in the system and its orbit, reliability analysis, mean busy period and mean busy cycle of this model.

4.1. System state probabilities

If the system satisfies the stability condition $\rho < R^*(\lambda)$, then from Eqs. (3.34)–(3.37), by setting $z \to 1$ and applying l'Hôpital's rule whenever necessary, we get the following results

(i) Let ψ be the steady state probability that the server is idle during the retrial time

$$\psi = \left\{ \frac{P_0(1-R^*(\lambda))\left(p+\left(1-S_v^*(\theta)\right)((\lambda b/\theta)+\lambda b\beta^{(1)}(1+\alpha g^{(1)}))\right)}{R^*(\lambda)-p-\lambda b\beta^{(1)}(1+\alpha g^{(1)})} \right\} \tag{4.1}$$

(ii) Let Π_b be the steady-state probability that the server is busy,

$$\Pi_b = \left\{ \frac{\lambda P_0 \beta^{(1)} \left(p S_v^*(\theta) + \left(1 - S_v^*(\theta)\right) \left((\lambda b/\theta) + R^*(\lambda) \right) \right)}{R^*(\lambda) - p - \lambda b \beta^{(1)} (1 + \alpha g^{(1)})} \right\} \tag{4.2}$$

(iii) Let Π_v be the steady-state probability that the server is on working vacation,

$$\Pi_v = \left\{ \lambda P_0 \left(1 - S_v^*(\theta)\right) / \theta \right\} \tag{4.3}$$

(iv) Let R be the steady state probability that the server is under repair

$$R = \left\{ \frac{\alpha \lambda P_0 g^{(1)} \beta^{(1)} \left(p S_v^*(\theta) + \left(1 - S_v^*(\theta)\right) \left((\lambda b/\theta) + R^*(\lambda) \right) \right)}{R^*(\lambda) - p - \lambda b \beta^{(1)} (1 + \alpha g^{(1)})} \right\} \tag{4.4}$$

4.2. Mean system size and orbit size

If the system in steady state condition $\rho < R^*(\lambda)$,

(i) The expected number of customers in the orbit (L_q) is obtained by differentiating (3.43) with respect to z and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z \to 1} \frac{d}{dz} K_o(z) = P_0 \left[\frac{Nr_q'''(1) Dr_q''(1) - Dr_q'''(1) Nr_q''(1)}{3 \left(Dr_q''(1) \right)^2} \right] \tag{4.5}$$

$$\begin{aligned} Nr_q''(1) = &-2b \left(\frac{\lambda b}{\theta} (1-p)(1 - S_v^*(\theta)) + (1-b)(p - R^*(\lambda)) \right) \\ &- 2\lambda b \beta^{(1)} (1 + \alpha g^{(1)}) ((p-b) S_v^*(\theta) \\ &+ b(1 - R^*(\lambda))(1 - S_v^*(\theta))) \end{aligned}$$

$$\begin{aligned} Nr_q'''(1) = &-3(\lambda b)^2 (\beta^{(2)} (1 + \alpha g^{(1)})^2 + \alpha \beta^{(1)} g^{(2)}) ((p-b) S_v^*(\theta) \\ &+ (1-b) R^*(\lambda)(1 - S_v^*(\theta))) \\ &- 3\lambda b \beta^{(1)} (1 + \alpha g^{(1)}) \\ &\times \begin{pmatrix} 2(\lambda b/\theta)(q + \lambda b)(1 - (1-\theta) S_v^*(\theta)) \\ + S_v^*(\theta)(1 - R^*(\lambda))(p + (\lambda - 2)b) \\ + 2b(1 - R^*(\lambda))((1-\theta) - (\lambda + b S_v^*(\theta))) \end{pmatrix} \\ &- 6(\lambda b/\theta)^2 (1 - S_v^*(\theta) + \theta S_v^{*\prime}(\theta))((b-p) \\ &+ (1-b) R^*(\lambda) + b\beta^{(1)} (1 + \alpha g^{(1)})(1 - R^*(\lambda) - \lambda)) \\ &- 6b((1 - R^*(\lambda))/\theta)(\lambda(1 - S_v^*(\theta))(p(b - \theta) + b) - p) \end{aligned}$$

$$Dr_q''(1) = -2b(R^*(\lambda) - p - \lambda b \beta^{(1)} (1 + \alpha g^{(1)}))$$

$$\begin{aligned} Dr_q'''(1) = &3b[\lambda b \beta^{(1)} (q - R^*(\lambda))(1 + \alpha g^{(1)}) + 2p(1 - R^*(\lambda)) \\ &+ (\lambda b)^2 (\beta^{(2)} (1 + \alpha g^{(1)})^2 + \alpha \beta^{(1)} g^{(2)})] \end{aligned}$$

(ii) The expected number of customers in the system (L_s) is obtained by differentiating (3.42) with respect to z and evaluating at $z = 1$

$$L_s = K_s'(1) = \lim_{z \to 1} \frac{d}{dz} K_s(z) = P_0 \left[\frac{Nr_s'''(1) Dr_q''(1) - Dr_q'''(1) Nr_q''(1)}{3 \left(Dr_q''(1) \right)^2} \right]$$

$$\begin{aligned} Nr_s'''(1) = &Nr_q'''(1) - 6\lambda b \beta^{(1)} (1 + \alpha g^{(1)}) \left(p S_v^*(\theta) \right. \\ &\left. + R^*(\lambda)(1 - S_v^*(\theta)) \right) - 6(\lambda b/\theta)(1 - R^*(\lambda)) \left(1 - S_v^*(\theta)\right) \end{aligned} \tag{4.6}$$

(iii) The average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) are found by using the Little's formula

$$W_s = \frac{L_s}{\lambda} \text{ and } W_q = \frac{L_q}{\lambda}. \tag{4.7}$$

4.3. Reliability measures

In the retrial queueing system with unreliable server, the reliability measures will provide the information which is required for the improvement of the system. To justify and validate the analytical results of this model, the availability measure (A_v) is obtained as follows:

(i) The steady state availability A_v, which is the probability that the server is either working for a primary customer or in an idle period such that the steady state availability of the server is given by

$$\begin{aligned} A_v &= 1 - \lim_{z \to 1} (R(z)) = 1 - R(1) \\ &= 1 - \left\{ \frac{\alpha \lambda P_0 g^{(1)} \beta^{(1)} \left(p S_v^*(\theta) + (1 - S_v^*(\theta))((\lambda b/\theta) + R^*(\lambda)) \right)}{R^*(\lambda) - p - \lambda b \beta^{(1)} (1 + \alpha g^{(1)})} \right\} \end{aligned}$$

4.4. Mean busy period and busy cycle

Let $E(T_b)$ and $E(T_c)$ be the expected length of busy period and busy cycle under the steady state conditions. The results follow directly by applying the argument of an alternating renewal process [10] which leads to

$$\begin{aligned} &P_0 = \frac{E(T_0)}{E(T_b) + E(T_0)}; \quad E(T_b) = \frac{1}{\lambda} \left(\frac{1}{P_0} - 1 \right) \text{ and} \\ &E(T_c) = \frac{1}{\lambda P_0} = E(T_0) + E(T_b) \end{aligned} \tag{4.8}$$

where T_0 is length of the system in empty state and $E(T_0) = (1/\lambda)$. Substituting Eq. (3.39) into (4.8) and using the above results, then we can get

$$E[T_b] = \frac{1}{\lambda}\left\{\frac{p(1-R^*(\lambda)) + \frac{\lambda}{\theta}\left(1-S_v^*(\theta)\right)[(b-p)+(1-b)R^*(\lambda)] + \lambda\beta^{(1)}(1+\alpha g^{(1)})\left((p-b)S_v^*(\theta)+(1-b)(1-S_v^*(\theta))R^*(\lambda)-b\right)}{R^*(\lambda)-p-\lambda b\beta^{(1)}(1+\alpha g^{(1)})}\right\} \quad (4.9)$$

$$E[T_c] = \frac{1}{\lambda}\left\{\frac{qR^*(\lambda) + \frac{\lambda}{\theta}\left(1-S_v^*(\theta)\right)[(b-p)+(1-b)R^*(\lambda)] + \lambda\beta^{(1)}(1+\alpha g^{(1)})\left((p-b)S_v^*(\theta)+(1-b)(1-S_v^*(\theta))R^*(\lambda)\right)}{R^*(\lambda)-p-\lambda b\beta^{(1)}(1+\alpha g^{(1)})}\right\} \quad (4.10)$$

5. Conditional stochastic decomposition

In this section, we will study the conditional stochastic decomposition property of the number of customers in the orbit. The number of customers in the orbit is distributed as the sum of two independent random variables. Let N_b be the conditional orbit size of our retrial queuing system given that server is busy and N_0 be the conditional orbit size of the M/G/1 feedback retrial queueing system with balking given that the server is busy, which is discussed in Theorem 5.1.

Theorem 5.1. *The conditional orbit size N_b given that the server is busy can be decomposed into the sum of two independent random variables $N_b = N_0 + N_c$.*

where N_0 has the generating function $N_0(z)$ as follows,

$$N_0(z) = \left\{\frac{\left(S_b^*(A_b(z))-1\right)\times(R^*(\lambda)-p-\lambda b\beta^{(1)}(1+\alpha g^{(1)}))}{(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))\times(\lambda\beta^{(1)}(1+\alpha g^{(1)}))}\right\}$$

$$N_0(z) = \frac{\Phi(z)}{\Phi(1)} = \left\{\frac{\left(S_b^*(A_b(z))-1\right)\times(R^*(\lambda)-p-\lambda b\beta^{(1)}(1+\alpha g^{(1)}))}{(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))\times(\lambda\beta^{(1)}(1+\alpha g^{(1)}))}\right\}$$

and N_c is the additional queue length due to vacations with the probability generating function $N_c(z)$ as follows,

$$N_c(z) = \frac{\left\{\begin{array}{c}\lambda b(z)V(z)(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))\\ +\theta(1-S_b^*(A_b(z)))\left[((pz+q)S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))\right]\end{array}\right\}\times\lambda\beta^{(1)}(1+\alpha g^{(1)})}{b(z)\Big((1-S_v^*(A_v(z)))(R^*(\lambda)+p)+\theta\beta^{(1)}(1+\alpha g^{(1)})\left((1-S_v^*(A_v(z)))R^*(\lambda)+pS_v^*(A_v(z))\right)\Big)\times\left(1-S_b^*(A_b(z))\right)}.$$

Proof. The mathematical version of the stochastic decomposition law is $N_b(z) = N_0(z)\, N_c(z)$.

We know that the M/G/1 feedback retrial queuing system with balking and service breakdown, the marginal function of the number of customers in the orbit when the server is busy is given by

$$\Phi(z) = \left\{\frac{R^*(\lambda)-p-\lambda b\beta^{(1)}(1+\alpha g^{(1)})}{qR^*(\lambda)+\lambda\beta^{(1)}(1+\alpha g^{(1)})(p-b)}\right\} \times \left\{\frac{R^*(\lambda)\left(S_b^*(A_b(z))-1\right)}{b(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))}\right\}$$

and the probability that the server is busy is given by $\Phi(1) = \left\{\frac{R^*(\lambda)(\lambda\beta^{(1)}(1+\alpha g^{(1)}))}{b(qR^*(\lambda)+\lambda\beta^{(1)}(1+\alpha g^{(1)})(p-b))}\right\}$, then for the generating function $N_0(z)$, we have

From Eqs. (3.38)–(3.40), we know that for our retrial system the generating function of N_b is given by

$$\begin{aligned} N_b(z) &= \frac{\Pi_b(z)+\Pi_v(z)+R(z)}{\Pi_b(1)+\Pi_v(1)+R(1)} \\ &= \frac{\left\{\begin{array}{c}\lambda b(z)V(z)(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))\\ +\theta(1-S_b^*(A_b(z)))\left[\left((pz+q)S_v^*(A_v(z))-1\right)(R^*(\lambda)+z(1-R^*(\lambda)))\right]\end{array}\right\}}{b(z)\left(\left(1-S_v^*(A_v(z))\right)(R^*(\lambda)+p)+\theta\beta^{(1)}(1+\alpha g^{(1)})\left(\left(1-S_v^*(A_v(z))\right)R^*(\lambda)+pS_v^*(A_v(z))\right)\right)} \\ &\quad\times\frac{(R^*(\lambda)-p-\lambda b\beta^{(1)}(1+\alpha g^{(1)}))}{\left(z-(pz+q)(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z))\right)} \\ N_b(z) &= N_0(z)\times N_c(z) \end{aligned}$$

From above stochastic decomposition law, we observe that $N_b(z) = N_0(z)\ N_c(z)$ which conform the decomposition result of Gao et al. [25], also valid for this special vacation system.

This completes the proof. □

6. Special cases

In this section, we analyze some special cases of our model, which are consistent with the existing literature.

Case 1: No balking, No feedback and No breakdown.

Let $b = 1$ and $p = \alpha = 0$. Our model can be reduced to a single server retrial queueing system with working vacations. In this case, $K_s(z)$ can be simplified to the following expressions.

$$K_s(z) = P_0 \frac{\left\{ \begin{array}{l} (1-z)\left[\begin{array}{c} (z-(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))((\lambda V(z)/\theta)+1) \\ +z(1-R^*(\lambda))\left[\left(S_v^*(A_v(z))+V(z)S_b^*(A_b(z))\right)-1\right] \end{array} \right] \\ +(1-S_b^*(A_b(z)))\left[(S_v^*(A_v(z))-1)(R^*(\lambda)+z(1-R^*(\lambda)))+zV(z)\right] \end{array} \right\}}{(1-z)(z-(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(A_b(z)))}$$

This coincides with the result of Gao et al. [25].

Case 2: No balking, No feedback, No vacation interruption and No breakdown.

Let $(b, p, \alpha, \theta) \to (1, 0, 0, 0)$ our model can be reduced to M/G/1 retrial queue with single working vacation.

$$K_s(z) = \left\{ \frac{S_v^*(\lambda)(R^*(\lambda)-\lambda\beta^{(1)})}{\lambda E(S_v) - R^*(\lambda)S_v^*(\lambda)} \right\} \left\{ \frac{\left[\left(S_v^*(\lambda-\lambda z)-1\right)(R^*(\lambda)+z(1-R^*(\lambda)))+(1-z)R^*(\lambda)S_v^*(\lambda)\right]S_b^*(\lambda-\lambda z)}{S_v^*(\lambda)(z-(R^*(\lambda)+z(1-R^*(\lambda)))S_b^*(\lambda-\lambda z))} \right\}$$

This coincides with the result of Arivudainambi et al. [20].

Case 3: No retrial, No balking, No feedback and No breakdown.

Let $(p, b, \alpha) \to (0, 1, 0)$ and $R^*(\lambda) \to 1$. Suppose that there is no retrial time in the system then we get an M/G/1 queue with working vacations.

$$K_s(z) = \left\{ \frac{(1-\lambda\beta^{(1)})\{(1-z)\left[(z-S_b^*(A_b(z)))((\lambda zV(z)/\theta)+1)\right]+z\left(1-S_b^*(A_b(z))\right)\left[S_v^*(A_v(z))+zV(z)-1\right]\}}{(1+(\lambda/\theta)\left(1-S_v^*(\theta)\right)-\lambda\beta^{(1)}S_v^*(\theta))(1-z)\left(z-S_b^*(A_b(z))\right)} \right\}$$

This coincides with the result of Zhang and Hou [27].

Case 4: No Vacation, No balking, No feedback and No breakdown.

Let we assume $(p, \alpha, b, \mu_v, \theta) \to (0, 0, 1, 0, 0)$. Then we get a single server retrial queueing system with general retrial times.

$$K_s(z) = \frac{\left\{[R^*(\lambda)-\lambda\beta^{(1)}][z-1]S_b^*[\lambda-\lambda z]\right\}}{\left\{z-[R^*(\lambda)+z(1-R^*(\lambda))]S_b^*[\lambda-\lambda z]\right\}};$$

$$L_q = \frac{\lambda^2\beta^{(2)} + 2\lambda\beta^{(1)}[1-R^*(\lambda)]}{2[R^*(\lambda)-\lambda\beta^{(1)}]}$$

These coincide with the result of Gomez-Corral [28].

7. Cost optimization analysis

In order to carry out cost analysis, the optimum design of a retrial queueing system is to determine the optimal system parameters, such as optimal mean service rate or optimal number of servers (see in [29,30]). In this section, the optimal design of the single server feedback retrial queue with subject to server breakdown and repair under multiple working vacations is addressed. Based on the definitions of cost elements (C_h, C_o, C_s and C_a) and cost structure listed below, the total expected cost function per unit time is given by

$$\begin{aligned} TC &= C_h L_s + C_o\frac{E(T_b)}{E(T_c)} + C_s\frac{1}{E(T_c)} + C_a\frac{E(T_0)}{E(T_c)} \\ &= C_h L_s + C_o(1-P_0) + C_s\lambda + C_a P_0 \end{aligned}$$

where C_h is the holding costs per unit time for each customer present in the system, C_o is the cost per unit time for keeping the server on and in operations, C_s is setup cost per busy cycle and C_a is the startup cost per unit time for the preparation work of the server before starting the service.

If we assume exponential retrial times, service times, working vacation times and repair times then for the following values of the cost elements and other parameters such as: $\lambda = 1$; $\mu_b = 5$; $\mu_v = 3$; $a = 2$; $b = 0.5$; $\xi = 2$; $\theta = 1$; $\alpha = 0.3$; $p = 0.5$; $C_h = \$\ 5$, $C_o = \$\ 100$, $C_s = \$\ 1000$ and $C_a = \$\ 100$, we find the total expected cost per unit of time ***TC*** = $ **200.1620**.

Moreover, we can examine the behavior of the expected cost function under different values of the cost parameters. Let us fix the system parameters values as follows: $\lambda = 1$; $\mu_b = 5$; $\mu_v = 3$; $a = 2$; $b = 0.5$; $\xi = 2$; $\theta = 1$; $\alpha = 0.3$; $p = 0.5$. Tables 1–3 illustrate the effects of (C_h, C_o), (C_o, C_a)

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Table 1 Effects of (C_h, C_o) on the expected cost function TC with C_s = \$1000 and C_a = \$100.

(C_h, C_o)	(5, 100)	(5, 110)	(5, 120)	(10, 100)	(15, 100)
TC	200.1620	201.1797	202.1974	200.3239	200.4859

Table 2 Effects of (C_o, C_a) on the expected cost function TC with C_h = \$5 and C_s = \$1000.

(C_o, C_a)	(100, 100)	(125, 100)	(150, 100)	(100, 110)	(100, 120)
TC	200.1620	202.7062	205.2505	209.1443	218.1265

Table 3 Effects of (C_a, C_s) on the expected cost function TC with C_h = \$5 and C_o = \$100.

(C_a, C_s)	(100, 1000)	(110, 1000)	(120, 1000)	(100, 1050)	(100, 1100)
TC	200.1620	209.1443	218.1265	205.1620	210.1620

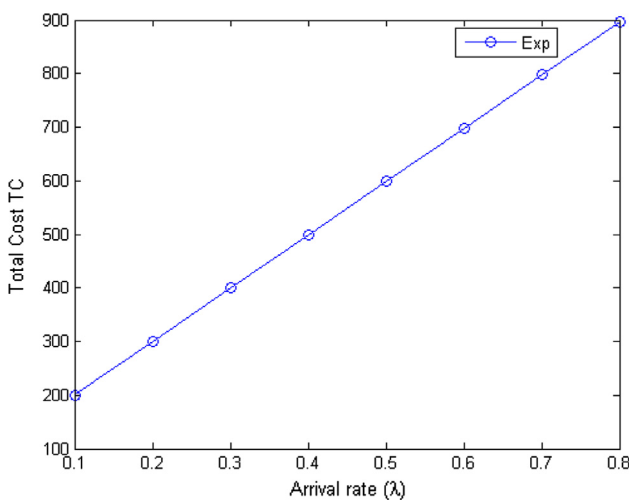

Figure 1 TC versus λ.

and (C_s, C_a) on the expected cost function, respectively. It can be see that the expected cost function shows a linearly increasing trend with increasing cost parameters.

Similarly, a sensitivity analysis of some of the parameters on the system can be conducted. Fixing the base values given above, one parameter can be varied at a time and the corresponding objective function value can be computed. The graphs (from Figs. 1–3) show the effect of some of the system parameters (λ, p, a) on the total expected cost per unit of time.

8. Numerical examples

In this section, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. Without loss of generality, we assume that the retrial times, service times, working vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed with the parameters a, μ_v, μ_b and ξ. The arbitrary values to the parameters are so chosen such that they satisfy the stability condition. The following

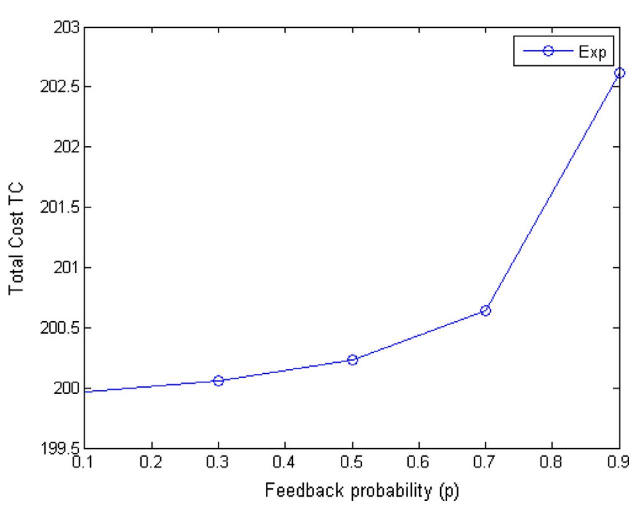

Figure 2 TC versus p.

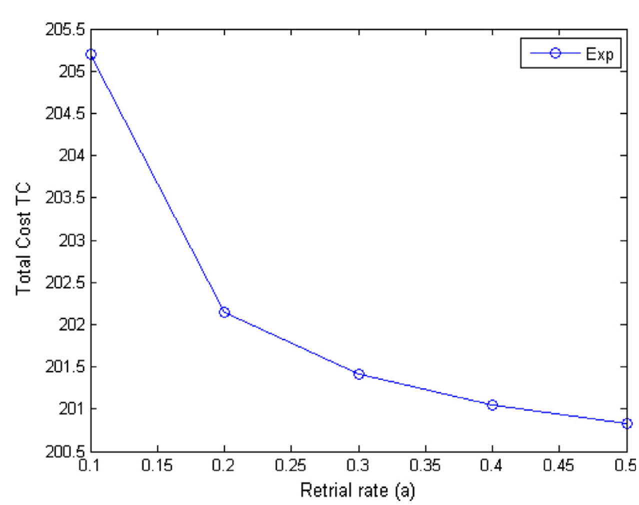

Figure 3 TC versus a.

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Table 4 The effect of feedback probability (p) on P_0, L_q and ψ.

Service distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
p	P_0	L_q	ψ	P_0	L_q	ψ	P_0	L_q	ψ
Feedback probability									
0.10	0.8664	0.1440	0.0199	0.7209	0.2764	0.0592	0.8453	0.1307	0.0275
0.20	0.8440	0.1461	0.0341	0.6751	0.2799	0.0896	0.8224	0.1311	0.0457
0.30	0.8154	0.1488	0.0523	0.6166	0.2848	0.1283	0.7930	0.1316	0.0690
0.40	0.7774	0.1525	0.0764	0.5394	0.2923	0.1794	0.7540	0.1325	0.1000
0.50	0.7245	0.1576	0.1099	0.4327	0.3056	0.2500	0.6995	0.1341	0.1432

Table 5 The effect of vacation rate (θ) on P_0, L_q and Π_v.

Vacation distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
θ	P_0	L_q	Π_v	P_0	L_q	Π_v	P_0	L_q	Π_v
Vacation rate									
1.00	0.6883	0.4325	0.1147	0.3601	0.8338	0.1100	0.6257	0.3802	0.1477
1.50	0.6926	0.3310	0.1066	0.3643	0.6081	0.0992	0.6371	0.2638	0.1357
2.00	0.6964	0.2918	0.0995	0.3681	0.4924	0.0901	0.6470	0.2050	0.1254
2.50	0.6997	0.2749	0.0933	0.3714	0.4132	0.0825	0.6555	0.1682	0.1165
3.00	0.7026	0.2674	0.0878	0.3744	0.3483	0.0761	0.6630	0.1419	0.1088

Table 6 The effect of lower speed service rate (μ_v) on P_0, L_q and Π_v.

Vacation distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
μ_v	P_0	L_q	Π_v	P_0	L_q	Π_v	P_0	L_q	Π_v
Lower service rate									
4.00	0.6646	0.0381	0.1329	0.3857	0.0429	0.1389	0.6621	0.1097	0.0834
5.00	0.6841	0.0340	0.1140	0.4021	0.0415	0.1229	0.6701	0.0898	0.0771
6.00	0.6987	0.0308	0.0998	0.4151	0.0399	0.1101	0.6770	0.0727	0.0716
7.00	0.7101	0.0281	0.0888	0.4257	0.0384	0.0998	0.6830	0.0579	0.0669
8.00	0.7192	0.0259	0.0799	0.4345	0.0370	0.0912	0.6883	0.0448	0.0627

Table 7 The effect of repair rate (ξ) on P_0, Lq and R.

Repair distribution	Exponential			Erlang-2 stage			Hyper-Exponential		
ξ	P_0	L_q	R	P_0	L_q	R	P_0	L_q	R
Repair rate									
1.00	0.6402	0.1554	0.0254	0.2535	0.3016	0.1259	0.6258	0.1327	0.0239
2.00	0.6509	0.1381	0.0127	0.2983	0.2584	0.0640	0.6363	0.1160	0.0114
3.00	0.6545	0.1321	0.0085	0.3135	0.2401	0.0429	0.6395	0.1105	0.0075
4.00	0.6563	0.1290	0.0064	0.3212	0.2304	0.0323	0.6411	0.1078	0.0056
5.00	0.6574	0.1271	0.0051	0.3259	0.2244	0.0258	0.6421	0.1062	0.0044

tables give the computed values of various characteristics of our model like, probability that the server is idle on working vacation P_0, the mean orbit size L_q, probability that server is idle during retrial rime (ψ), regular busy (Π_b), lower rate service (Π_v) and under repair (R) respectively. Note that the exponential distribution is $f(x) = \phi e^{-\phi x}, x > 0$, Erlang distribution is $f(x) = \frac{\phi^n}{(n-1)!} x^{n-1} e^{-\phi x}$, $x > 0$, where $n = 2$ and hyper-exponential distribution is $f(x) = c\phi e^{-\phi x} + (1-c)\phi^2 e^{-\phi^2 x}$, $x > 0$.

Algorithm to compute L_q:
Begin
input: λ, a, μ_b, p, b, θ, μ_v, α, and ξ.
Compute: ψ from Eq. (4.1).
Compute: Π_b from Eq. (4.2).
Compute: Π_v from Eq. (4.3).
Compute: R from Eq. (4.4).
Compute: L_q from Eq. (4.5).
Output: L_q.

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Table 4 shows that when feedback probability (p) increases, then the probability that server is idle on working vacation P_0 decreases the mean orbit size L_q increases and the probability that server is idle during retrial time ψ also increases for the values of $\lambda = 0.5$; $\mu_b = 8$; $a = 5$; $b = 0.8$; $\theta = 1$; $\mu_v = 4$; $\alpha = 0.2$ and $\xi = 2$. Table 5 shows that when vacation rate (θ) increases, the probability that server is idle on working vacation P_0 increases, then the mean orbit size L_q decreases and probability that server is on working vacation Π_v also decreases for the values of $\lambda = 1$; $\mu_b = 10$; $a = 5$; $b = 0.8$; $p = 0.3$; $\mu_v = 5$; $\alpha = 0.2$ and $\xi = 5$.

As expected from Table 5, increasing θ decreases the value of the L_q, Π_v and other performance measures. Based on the above, it is smaller for large values of μ_v and turns to zero when $\mu_v = \mu_b$. Another important case is $\mu_v = 0$, i.e., the server cannot provide service during a vacation period; the effect of the vacation rate θ has a noticeable effect on the system performance and cannot be ignored.

Table 6 shows that when lower speed service rate (μ_v) increases, the probability that server is idle on working vacation P_0 increases, then the mean orbit size L_q decreases and probability that server is on working vacation Π_v also decreases for the values of $\lambda = 1$; $\mu_b = 10$; $a = 3$; $b = 0.2$; $p = 0.3$; $\theta = 1$; $\mu_v = 5$ and $\alpha = 0.2$. Table 7 shows that when repair rate (ξ) increases, the probability that server is idle on working vacation P_0 increases, then the mean orbit size L_q decreases and probability that server is under repair R also decreases for the values of $\lambda = 1$; $\mu_b = 8$; $a = 5$; $b = 0.3$; $p = 0.3$; $\theta = 1$; $\mu_v = 5$ and $\alpha = 0.3$.

For the effect of the parameters λ, a, μ_b, μ_v, b, p, θ, α, ξ on the system performance measures, two dimensional graphs are drawn in Figs. 4–8. Figs. 4 and 5 show that the mean orbit size L_q decreases for the increasing the value of the retrial rate (a) and vacation rate (θ). Fig. 6 shows that the probability that the server is idle on working vacation P_0 increases for the increasing value of normal service rate (μ_b). In Fig. 7, we examine the behavior of the mean orbit size L_q decreases for increasing the value of lower speed service rate (μ_v). In Fig. 8, we depict the behavior of the server's availability (A_v) decreases for increasing the value of the breakdown rate (α).

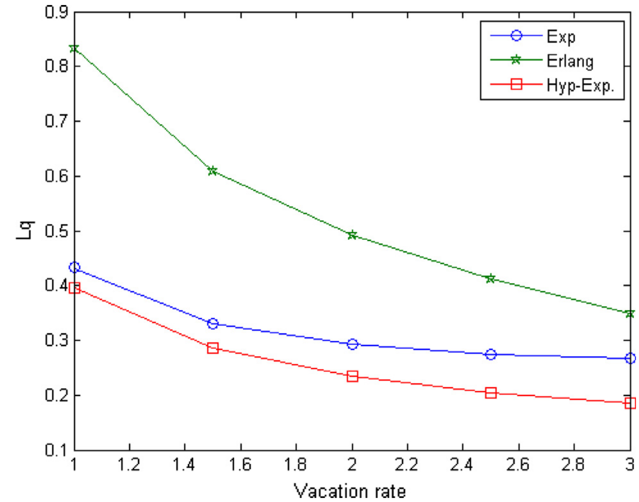

Figure 5 L_q versus θ.

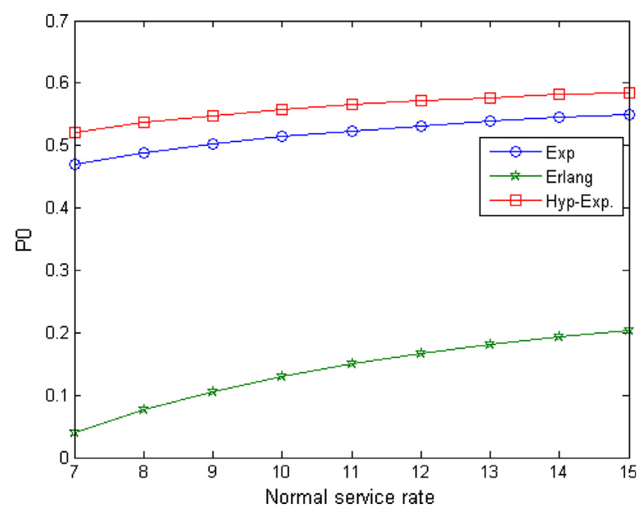

Figure 6 P_0 versus μ_b.

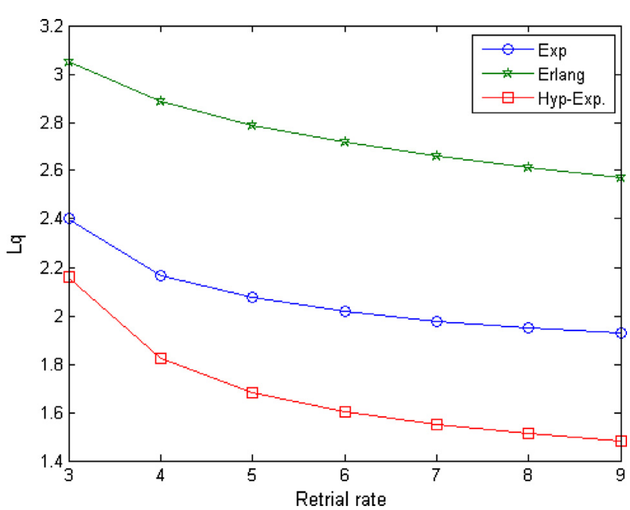

Figure 4 L_q versus a.

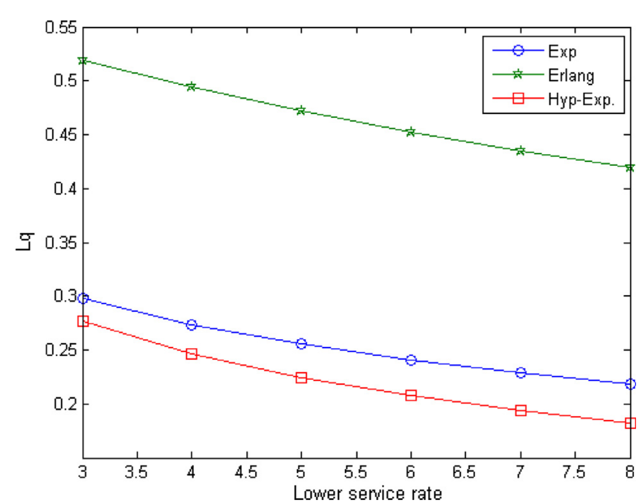

Figure 7 L_q versus μ_v.

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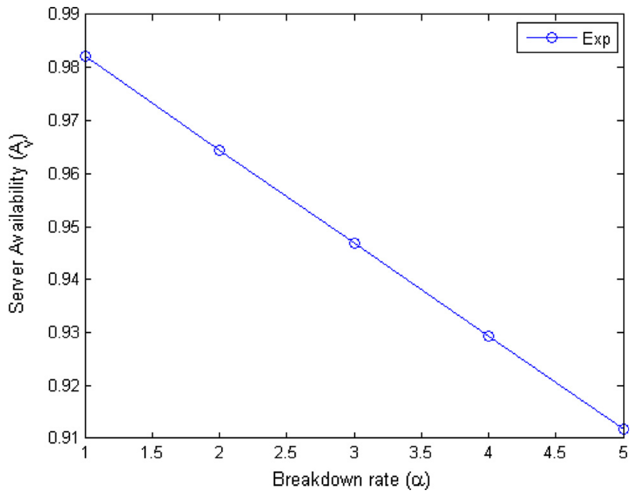

Figure 8 A_v versus α.

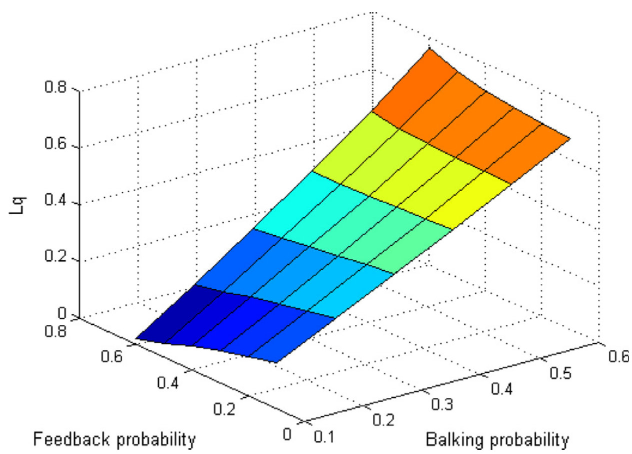

Figure 9 L_q versus b *and* p.

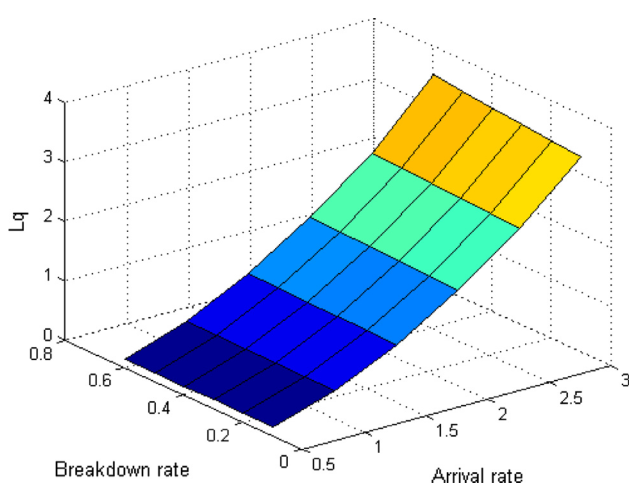

Figure 10 L_q versus λ and α.

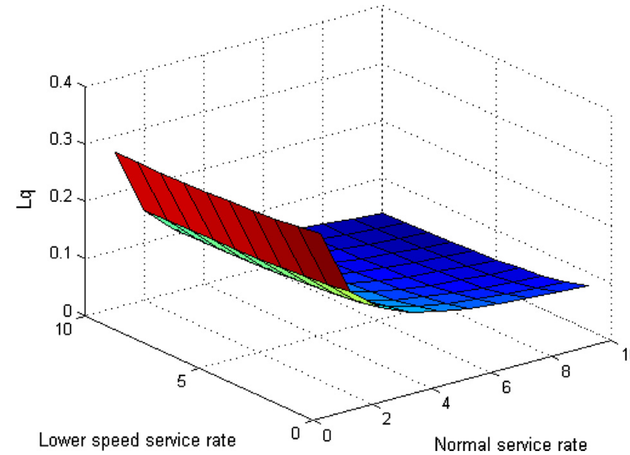

Figure 11 L_q versus μ_b and μ_v.

Three dimensional graphs are illustrated in Figs. 9–12. In Fig. 9, the surface displays an upward trend as expected for increasing the value of balking probability (b) and feedback probability (p) against the mean orbit size L_q. In Fig. 10, we examine the behavior of the mean orbit size L_q increases for increasing the value of arrival rate (λ) and the breakdown rate (α). In Fig. 11, the surface displays the downward trend as expected for increasing the value of the normal/lower speed service rates (μ_b, μ_v) against the mean orbit size L_q. In Fig. 12, we examine the behavior of the mean orbit size L_q decreases for increasing the value of vacation rate (θ) and repair rate (ξ).

From the above numerical examples, we observed that the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

9. Conclusion

In this investigation, we have studied a single server retrial queueing system with balking and feedback under multiple working vacation policy, where the busy server is subjected to breakdown and repair. The necessary and sufficient condition for the system to be stable is obtained. The analytical results that are validated with the help of numerical illustrations may be useful in various real-life situations to design the outputs. The probability generating functions for the numbers of customers in the system when it is free, busy, on working vacation and under repair are found by using the supplementary variable technique. Some important system performance measures are obtained. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, some numerical examples and cost optimization analysis are presented to study the impact of the system parameters and cost elements. The novelty of this investigation is the introduction of balking, server breakdown and repair in the presence of retrial queues with multiple working vacation policy. The motivation for this model comes from wide range applications in many real-time systems, for example in computer and communication network where messages are processed by a single server under working vacations and vacation interruption policy. Moreover, our model can be considered as a generalized version of many existing queueing

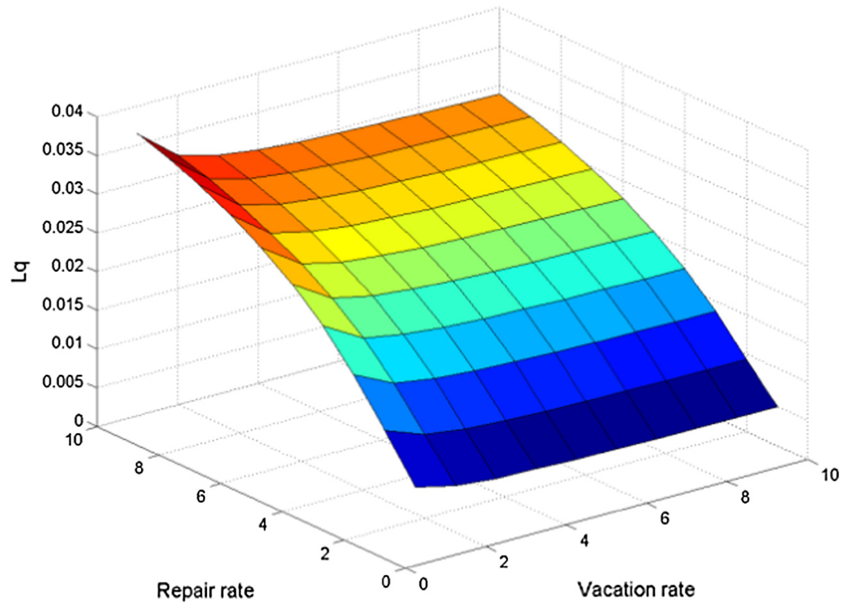

Figure 12 L_q versus θ *and* ξ.

models [25,20] equipped with many features and associated with many practical situations.

The present investigation includes features simultaneously such as

- Retrial queues
- Feedback
- Balking
- Multiple working vacations
- Vacation interruption
- Breakdowns and repairs

Our suggested model has potential practical real-life application in Simple Mail Transfer Protocol (SMTP) mail system uses to deliver the messages between mail servers. Other applications are in computer processing system and telephone consultation of medical service systems. This work can be further extended in many directions by incorporating the concepts of batch arrival, bulk service, working breakdowns, immediate Bernoulli feedbacks. Hopefully, this investigation will be of great help for the system managers to make decisions regarding the size of the system and other factors in a well-to-do manner.

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Appendix A

The embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if $\rho < R^*(\lambda)$ for our system to be stable, where $\rho = p + \lambda b\beta^{(1)}(1 + \alpha g^{(1)})$

Proof. To prove the sufficient condition of ergodicity, it is very convenient to use Foster's criterion (see Pakes [31]), which states that the chain $\{Z_n; n \in N\}$ is an irreducible and aperiodic Markov chain is ergodic if there exists a nonnegative function $f(j)$, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E[f(z_{n+1}) - f(z_n)/z_n = j]$ is finite for all $j \in N$ and $\psi_j \leqslant -\varepsilon$ for all $j \in N$, except perhaps for a finite number j's. In our case, we consider the function $f(j) = j$. Then we have

$$\psi_j = \begin{cases} p + \lambda b\beta^{(1)}(1 + \alpha g^{(1)}) - 1, & j = 0, \\ p + \lambda b\beta^{(1)}(1 + \alpha g^{(1)}) - R^*(\lambda), & j = 1, 2, \ldots \end{cases}$$

Clearly the inequality $p + \lambda b\beta^{(1)}(1 + \alpha g^{(1)}) < R^*(\lambda)$ is sufficient condition for Ergodicity. □

To prove the necessary condition, as noted in Sennott et al. [32], if the Markov chain $\{Z_n; n \geqslant 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \geqslant 0$ and there exits $j_0 \in \mathrm{N}$ such that $\psi_j \geqslant 0$ for $j \geqslant j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n; n \in N\}$. Then $p + \lambda b\beta^{(1)}(1 + \alpha g^{(1)}) \geqslant R^*(\lambda)$ implies the non-Ergodicity of the Markov chain.

References

[1] J.C. Ke, C.U. Wu, Z.G. Zhang, Recent developments in vacation models: a short survey, Int. J. Oper. Res. 7 (2010) 3–8.
[2] J. Artalejo, A. Gomez-Corral, Retrial Queueing Systems, Springer, Berlin, Germany, 2008.
[3] J.C. Ke, Operating characteristic analysis on the $M^{[X]}/G/1$ system with a variant vacation policy and balking, Appl. Math. Model. 31 (2007) 1321–1337.
[4] J. Wang, J. Li, A single server retrial queue with general retrial times and two phase service, J. Syst. Sci. Complex. 22 (2009) 291–302.

[5] S. Gao, J. Wang, Performance and reliability analysis of an M/G/1-G retrial queue with orbital search and non-persistent customers, Eur. J. Oper. Res. 236 (2014) 561–572.
[6] J.C. Ke, F.M. Chang, Modified vacation policy for M/G/1 retrial queue with balking and feedback, Comput. Ind. Eng. 57 (2009) 433–443.
[7] M. Baruah, K.C. Madan, T. Eldabi, Balking and re-service in a vacation queue with batch arrival and two types of heterogeneous service, J. Math. Res. 4 (2012) 114–124.
[8] B. Krishnakumar, S. Pavai Madheswari, S.R. Anantha Lakshmi, An M/G/1 Bernoulli feedback retrial queueing system with negative customers, Oper. Res. Int. J. 13 (2013) 187–210.
[9] P. Rajadurai, M.C. Saravanarajan, V.M. Chandrasekaran, Analysis of an M[X]/(G1, G2)/1 retrial queueing system with balking, optional re-service under modified vacation policy and service interruption, Ain Shams Eng. J. 5 (2014) 935–950.
[10] G. Choudhury, J.C. Ke, An unreliable retrial queue with delaying repair and general retrial times under Bernoulli vacation schedule, Appl. Math. Comput. 230 (2014) 436–450.
[11] G. Choudhury, K. Deka, A single server queueing system with two phases of service subject to server breakdown and Bernoulli vacation, Appl. Math. Model. 36 (2012) 6050–6060.
[12] S. Yang, J. Wu, Z. Liu, An M[X]/G/1 retrial G-queue with single vacation subject to the server breakdown and repair, Acta Math. Appl. Sin. 29 (2013) 579–596.
[13] P. Rajadurai, M. Varalakshmi, M.C. Saravanarajan, V.M. Chandrasekaran, Analysis of $M^{[X]}/G/1$ retrial queue with two phase service under Bernoulli vacation schedule and random breakdown, Int. J. Math. Oper. Res. 7 (2015) 19–41.
[14] P. Rajadurai, V.M. Chandrasekaran, M.C. Saravanarajan, Steady state analysis of batch arrival feedback retrial queue with two phases of service, negative customers, Bernoulli vacation and server breakdown, Int. J. Math. Oper. Res. 7 (2015) 519–546.
[15] P. Rajadurai, V.M. Chandrasekaran, M.C. Saravanarajan, Analysis of an $M^{[X]}/G/1$ unreliable retrial G-queue with orbital search and feedback under Bernoulli vacation schedule, Opsearch 53 (2016) 197–223.
[16] I. Dimitriou, A mixed priority retrial queue with negative arrivals, unreliable server and multiple vacations, Appl. Math. Model. 37 (2013) 1295–1309.
[17] I. Dimitriou, A preemptive resume priority retrial queue with state dependent arrivals, unreliable server and negative customers, Top 21 (2013) 542–571.
[18] L.D. Servi, S.G. Finn, M/M/1 queues with working vacations, Perform. Eval. 50 (2002) 41–52.
[19] D. Wu, H. Takagi, M/G/1 queue with multiple working vacations, Perform. Eval. 63 (2006) 654–681.
[20] D. Arivudainambi, P. Godhandaraman, P. Rajadurai, Performance analysis of a single server retrial queue with working vacation, Opsearch 51 (2014) 434–462.
[21] V.M. Chandrasekaran, K. Indhira, M.C. Saravanarajan, P. Rajadurai, A survey on working vacation queueing models, Int. J. Pure Appl. Math. 106 (2016) 33–41.
[22] Z. Liu, Y. Song, Geo/Geo/1 retrial queue with non-persistent customers and working vacations, J. Appl. Math. Comput. 42 (2013) 103–115.
[23] J. Li, N. Tian, The M/M/1 queue with working vacations and vacation interruption, J. Syst. Sci. Syst. Eng. 16 (2007) 121–127.
[24] S. Gao, Z. Liu, An M/G/1 queue with single working vacation and vacation interruption under Bernoulli schedule, Appl. Math. Model. 37 (2013) 1564–1579.
[25] S. Gao, J. Wang, W. Li, An M/G/1 retrial queue with general retrial times, working vacations and vacation interruption, Asia-Pac. J. Oper. Res. 31 (2014) 6–31.
[26] M. Zhang, Q. Liu, An M/G/1 G-queue with server breakdown, working vacations and vacation interruption, Opsearch 52 (2015) 256–270.
[27] M. Zhang, Z. Hou, M/G/1 queue with single working vacation, J. Appl. Math. Comput. 39 (2012) 221–234.
[28] A. Gomez-Corral, Stochastic analysis of a single server retrial queue with general retrial times, Naval Res. Logist. 46 (1999) 561–581.
[29] L. Tadj, G. Chodhury, Optimal design and control of queues, Top 13 (2005) 359–414.
[30] D.Y. Yang, C.H. Wu, Cost-minimization analysis of a working vacation queue with N-policy and server breakdowns, Comput. Ind. Eng. 82 (2015) 151–158.
[31] A.G. Pakes, Some conditions for ergodicity and recurrence of Markov chains, Oper. Res. 17 (1969) 1058–1061.
[32] L.I. Sennott, P.A. Humblet, R.L. Tweedi, Mean drifts and the non ergodicity of Markov chains, Oper. Res. 31 (1983) 783–789.

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