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**A subordination results for a class of analytic
 functions defined by q -differential operator**

Abstract. In this paper, we derive several subordination results and integral means result for certain class of analytic functions defined by means of q -differential operator. Some interesting corollaries and consequences of our results are also considered.

1. Introduction and definitions

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. Also denote by \mathcal{T} a subclass of \mathcal{A} consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad z \in \Delta$$

introduced and studied by Silverman [22]. For two functions f and g given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

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their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n c_n z^n. \quad (2)$$

We briefly recall here the notion of q -operators, i.e. q -difference operator that plays vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of q -calculus was initiated by Jackson [7] and Kanas and Răducanu [12] have used the fractional q -calculus operators in investigations of certain classes of functions which are analytic in Δ . For details on q -calculus one can refer [2, 3, 7, 12, 16, 11, 26] and also the reference cited therein. For the convenience, we provide some basic definitions and concept details of q -calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$.

For $0 < q < 1$ the Jackson's q -derivative of a function $f \in \mathcal{A}$ is, by definition, given as follows [7]

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases} \quad (3)$$

and

$$D_q^2 f(z) = D_q(D_q f(z)).$$

From (3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1 - q^n}{1 - q},$$

is sometimes called *the basic number* n . Observe that if $q \rightarrow 1^-$, then $[n]_q \rightarrow n$.

For a function $h(z) = z^n$, we obtain $D_q h(z) = D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1}$, and as $q \rightarrow 1^-$ we note

$$D_q h(z) = q \rightarrow 1^- \quad ([n]_q z^{n-1}) = n z^{n-1} = h'(z),$$

where h' is the ordinary derivative. Recently, for $f \in \mathcal{A}$, Govindaraj and Sivasubramanian [11] defined and discussed the Sălăgean q -differential operator as follows

$$\mathcal{D}_q^0 f(z) = f(z),$$

$$\mathcal{D}_q^1 f(z) = z \mathcal{D}_q f(z),$$

$$\mathcal{D}_q^m f(z) = z \mathcal{D}_q^m (\mathcal{D}_q^{m-1} f(z)),$$

$$\mathcal{D}_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad z \in \Delta.$$

We note that if $q \rightarrow 1^-$,

$$D^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n \quad m \in \mathbb{N}_0, z \in \Delta$$

is the familiar Sălăgean derivative [21].

Now let

$$\mathbb{D}^0 f(z) = \mathcal{D}_q^m f(z),$$

$$\begin{aligned} \mathbb{D}_{\lambda,q}^{1,m} f(z) &= (1 - \lambda) \mathcal{D}_q^m f(z) + \lambda z (\mathcal{D}_q^m f(z))' \\ &= z + \sum_{n=2}^{\infty} [n]_q^m [1 + (n - 1)\lambda] a_n z^n, \end{aligned}$$

$$\begin{aligned} \mathbb{D}_{\lambda,q}^{2,m} f(z) &= (1 - \lambda) \mathbb{D}_{\lambda,q}^{1,m} f(z) + \lambda z (\mathbb{D}_{\lambda,q}^{1,m} f(z))' \\ &= z + \sum_{n=2}^{\infty} [n]_q^m [1 + (n - 1)\lambda]^2 a_n z^n. \end{aligned}$$

In general, we have

$$\begin{aligned} \mathbb{D}_{\lambda,q}^{\zeta,m} f(z) &= (1 - \lambda) \mathcal{D}_{\lambda,q}^{\zeta-1,m} f(z) + \lambda z (\mathcal{D}_{\lambda,q}^{\zeta-1,m} f(z))' \\ &= z + \sum_{n=2}^{\infty} [n]_q^m [1 + (n - 1)\lambda]^{\zeta} a_n z^n, \quad \lambda > 0, \zeta \in \mathbb{N}_0. \end{aligned}$$

We note that when $q \rightarrow 1^-$, we get the differential operator

$$\mathbb{D}_{\lambda}^{\zeta,m} f(z) = z + \sum_{n=2}^{\infty} n^m [1 + (n - 1)\lambda]^{\zeta} a_n z^n \quad \lambda > 0, m, \zeta \in \mathbb{N}_0.$$

We observe that for $m = 0$, we get the differential operator D^{ζ} defined by Al-Oboudi [5], and if $\zeta = 0$, we get Sălăgean differential operator D^m , see [21].

With the help of the differential operator $\mathbb{D}_{\lambda,q}^{\zeta,m}$, we say that a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha, \beta)$ if it satisfies

$$\Re \left(\frac{z (\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - \alpha \right) > \beta \left| \frac{z (\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - 1 \right|, \quad z \in \Delta,$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda > 0$, $m, \zeta \in \mathbb{N}_0$.

The family $\mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha, \beta)$ contains many well-known as well as many new classes of analytic univalent functions. For $\beta = 0$, $\zeta = 0$ and $m = 0$ we obtain the family of starlike functions of order α ($0 \leq \alpha < 1$) denoted by $\mathcal{S}^*(\alpha)$ and for $\beta = 0$, $\zeta = 0$ and $m = 1$ we have the family of convex functions of order α ($0 \leq \alpha < 1$) denoted by $\mathcal{K}(\alpha)$. For $\zeta = 0$ and $m = 0$ we obtain the class $\beta - \mathcal{UST}(\alpha)$ and for $\zeta = 0$ and $m = 1$ we get the class $\beta - \mathcal{UKV}(\alpha)$. The classes $\beta - \mathcal{UST}(\alpha)$ and $\beta - \mathcal{UKV}(\alpha)$

were introduced by Rønning [19], [20]. We observe that $\beta - \mathcal{UST}(0) \equiv \beta - \mathcal{UST}$ the class of uniformly β -starlike functions and $\beta - \mathcal{UKV}(0) \equiv \beta - \mathcal{UKV}$ the class of uniformly β -convex functions introduced by Kanas and Wiśniowska [13], [14], see also the work of Kanas and Srivastava [15], Goodman [9], [10], Ma and Minda [18] and Gangadharan et al. [8].

Before we state and prove our main result we need the following definitions and lemmas.

DEFINITION 1.1 (Subordination Principle)

Let g be analytic and univalent in Δ . If f is analytic in Δ , $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$, then the function f is subordinate to g in Δ and we write $f \prec g$.

DEFINITION 1.2 (Subordinating Factor Sequence)

A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever f is analytic, univalent and convex in Δ , we have the subordination given by

$$\sum_{n=2}^{\infty} b_n a_n z^n \prec f(z), \quad z \in \Delta, \quad a_1 = 1.$$

LEMMA 1.3 ([28])

The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\Re\left(1 + 2 \sum_{n=1}^{\infty} b_n z^n\right) > 0, \quad z \in \Delta.$$

LEMMA 1.4

Assume that

$$\sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^\zeta [n(\beta+1) - (\alpha+\beta)] |a_n| \leq 1 - \alpha, \quad (4)$$

then $f \in \mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha, \beta)$, where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda > 0$ and $m, \zeta \in \mathbb{N}_0$. The result is sharp for the function

$$f_n(z) = z - \frac{1 - \alpha}{[n]_q^m [1 + (n-1)\lambda]^\zeta [n(\beta+1) - (\alpha+\beta)]} z^n.$$

Proof. It suffices to show that

$$\beta \left| \frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - 1 \right| - \Re\left(\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - 1\right) \leq 1 - \alpha.$$

We have

$$\begin{aligned}
 \beta \left| \frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - 1 \right| &= \Re \left(\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - 1 \right) \\
 &\leq (1 + \beta) \left| \frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - 1 \right| \\
 &\leq \frac{(1 + \beta) \sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} (n-1) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} |a_n| |z|^{n-1}} \\
 &\leq \frac{(1 + \beta) \sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} (n-1) |a_n|}{1 - \sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} |a_n|}.
 \end{aligned}$$

The last expression is bounded from above by $1 - \alpha$ if

$$\sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} [n(\beta + 1) - (\alpha + \beta)] |a_n|$$

holds. It is obvious that the function f_n satisfies the inequality (4), and thus $1 - \alpha$ cannot be replaced by a larger number. Therefore we need only to prove that $f \in \mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha, \beta)$. Since

$$\begin{aligned}
 \Re \left(\frac{1 - \sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} n a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} a_n z^{n-1}} - \alpha \right) \\
 > \beta \left| \frac{\sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} (n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q^m [1 + (n-1)\lambda]^{\zeta} a_n z^{n-1}} \right|.
 \end{aligned}$$

Letting $z \rightarrow 1$ along the real axis, we obtain the desired inequality given in (4). and the proof is complete.

Let $\mathcal{S}_{\lambda,q}^{*,\zeta,m}(\alpha, \beta)$ denote the class of functions $f \in \mathcal{A}$ whose coefficients satisfy the condition (4). We note that $\mathcal{S}_{\lambda,q}^{*,\zeta,m}(\alpha, \beta) \subseteq \mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha, \beta)$.

2. Main Theorem

Employing the techniques used earlier by Srivastava and Attiya [27], Attiya [4] and Frasin [6], Singh [25] and others, we state and prove the following theorem.

THEOREM 2.1

Let the function f be defined by (1) be in the class $\mathcal{S}_{\lambda, q}^{*, \zeta, m}(\alpha, \beta)$, where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda > 0$, $\zeta \in \mathbb{N}_0$. Also let \mathcal{K} denote the familiar class of functions $f \in \mathcal{A}$ which are also univalent and convex in Δ . Then

$$\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}(f * g)(z) \prec g(z), \quad z \in \Delta, \quad g \in \mathcal{K}, \quad (5)$$

and

$$\Re(f(z)) > -\frac{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}, \quad z \in \Delta. \quad (6)$$

The constant $\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}$ is the best estimate.

Proof. Let $f \in \mathcal{S}_{\lambda, q}^{*, \zeta, m}(\alpha, \beta)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\begin{aligned} & \frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}(f * g)(z) \\ &= \frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]} \left(z + \sum_{n=2}^{\infty} a_n c_n z^n \right). \end{aligned}$$

Thus, by Definition 1.2, the assertion of our theorem will hold if the sequence

$$\left(\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]} a_n \right)_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.3, this will be the case if and only if

$$\Re \left(1 + 2 \sum_{n=1}^{\infty} \frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]} a_n z^n \right) > 0, \quad z \in \Delta. \quad (7)$$

Now

$$\begin{aligned} & \Re \left(1 + \frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)} \sum_{n=1}^{\infty} a_n z^n \right) \\ &= \Re \left(1 + \frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)} z \right. \\ & \quad \left. + \frac{1}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)} \right. \\ & \quad \left. \cdot \sum_{n=2}^{\infty} (1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha) a_n z^n \right) \\ & \geq 1 - \left(\frac{[2]_q^m(1+\lambda)^\zeta(\beta+2-\alpha)}{1-\alpha+[2]_q^m(1+\lambda)^\zeta(\beta+2-\alpha)} r \right) \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{1 - \alpha + [2]_q^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)} \\
 & \cdot \sum_{n=2}^{\infty} [2]_q^m [1 + (n - 1)\lambda] [n(\beta + 1) - (\alpha + \beta)] a_n r^n \\
 & > 1 - \frac{(1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)}{1 - \alpha + (1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)} r \\
 & - \frac{1 - \alpha}{1 - \alpha + (1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)} r > 0, \quad |z| = r.
 \end{aligned}$$

Notice that the last but one inequality follows from the fact that $[2]_q^m \sum_{n=2}^{\infty} [1 + (n - 1)\lambda] [n(\beta + 1) - (\alpha + \beta)]$ is an increasing function of $n (n \geq 2)$. Thus (7) holds true in Δ . This proves the inequality (5). The inequality (6) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$ in (5).

To prove the sharpness of the constant $\frac{(1+q)^m (1+\lambda)^\zeta (\beta+2-\alpha)}{2[1-\alpha+(1+q)^m (1+\lambda)^\zeta (\beta+2-\alpha)]}$, we consider the function $f_2 \in \mathcal{S}_{\lambda, q}^{*, \zeta, m}(\alpha, \beta)$ given by

$$f_2(z) = z - \frac{1 - \alpha}{(1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)} z^2,$$

where $-1 \leq \alpha < 1, \beta \geq 0, \lambda > 0, m, \zeta \in \mathbb{N}_0$. Thus from (5) we have

$$\frac{(1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)}{2[1 - \alpha + (1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)]} f_2(z) \prec \frac{z}{1 - z}.$$

It can be easily verified that

$$\min \left\{ \Re \left(\frac{(1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)}{2[1 - \alpha + (1 + q)^m (1 + \lambda)^\zeta (\beta + 2 - \alpha)]} f_2(z) \right) \right\} = -\frac{1}{2}, \quad z \in \Delta.$$

This shows that the constant $\frac{(1+q)^m (1+\lambda)^\zeta (\beta+2-\alpha)}{2[1-\alpha+(1+q)^m (1+\lambda)^\zeta (\beta+2-\alpha)]}$ is the best possible.

Putting $m = 0$ in Theorem 2.1 yields the following result obtained by Aouf et al. [1].

COROLLARY 2.2

Let f , defined by (1), be in the class $\mathcal{M}_\lambda^*(\zeta, \alpha, \beta)$, where $-1 \leq \alpha < 1, \beta \geq 0, \lambda > 0, \zeta \in \mathbb{N}_0$. Then

$$\frac{(1 + \lambda)^\zeta (\beta + 2 - \alpha)}{2[1 - \alpha + (1 + \lambda)^\zeta (\beta + 2 - \alpha)]} (f * g)(z) \prec g(z) \quad z \in \Delta, \quad g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{1 - \alpha + (1 + \lambda)^\zeta (\beta + 2 - \alpha)}{(1 + \lambda)^\zeta (\beta + 2 - \alpha)}, \quad z \in \Delta.$$

The constant $\frac{(1+\lambda)^\zeta (\beta+2-\alpha)}{2[1-\alpha+(1+\lambda)^\zeta (\beta+2-\alpha)]}$ is the best estimate.

If we put $m = 0$ and $\zeta = 0$ in Theorem 2.1, we obtain the next two results obtained by Frasin [6].

COROLLARY 2.3

Let f , defined by (1), be in the class $\beta - \mathcal{UST}(\alpha)$. Then

$$\frac{\beta + 2 - \alpha}{2(\beta + 3 - 2\alpha)}(f * g)(z) \prec g(z), \quad -1 \leq \alpha < 1, \beta \geq 0, z \in \Delta, g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{\beta + 3 - 2\alpha}{\beta + 2 - \alpha}, \quad z \in \Delta.$$

The constant $\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}$ is the best estimate.

COROLLARY 2.4

Let f , defined by (1), be in the class $\beta - \mathcal{UKV}(\alpha)$. Then

$$\frac{\beta + 2 - \alpha}{2\beta + 5 - 3\alpha}(f * g)(z) \prec g(z), \quad -1 \leq \alpha < 1, \beta \geq 0, z \in \Delta, g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{2\beta + 5 - 3\alpha}{2(\beta + 2 - \alpha)}, \quad z \in \Delta.$$

The constant $\frac{\beta+2-\alpha}{2\beta+5-3\alpha}$ is the best estimate.

Putting $m = 0$, $\zeta = 0$ and $\beta = 0$ in Theorem 2.1, we obtain the next two results obtained by Frasin [6].

COROLLARY 2.5

Let f , defined by (1), be in the class $\mathcal{S}^*(\alpha)$. Then

$$\frac{2 - \alpha}{6 - 4\alpha}(f * g)(z) \prec g(z), \quad z \in \Delta, g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{3 - 2\alpha}{2 - \alpha}, \quad z \in \Delta.$$

The constant $\frac{2-\alpha}{6-4\alpha}$ is the best estimate.

COROLLARY 2.6

Let f , defined by (1), be in the class $\mathcal{K}(\alpha)$. Then

$$\frac{2 - \alpha}{5 - 3\alpha}(f * g)(z) \prec g(z), \quad z \in \Delta, g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{5 - 3\alpha}{2(2 - \alpha)}, \quad z \in \Delta.$$

The constant $\frac{2-\alpha}{5-3\alpha}$ is the best estimate.

3. Integral Means Inequalities

LEMMA 3.1 ([17])

If the functions f and g are analytic in Δ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta.$$

In [22], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family \mathcal{T} and applied this function to resolve his integral means inequality, conjectured in [23] and settled in [24], that

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [24], Silverman also proved his conjecture for the subclasses $\mathcal{T}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of \mathcal{T} .

Applying Lemma 3.1 and Lemma 1.4, we prove the following result.

THEOREM 3.2

Suppose $f \in \mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha, \beta)$, $\eta > 0$, and f_2 is defined by

$$f_2(z) = z - \frac{1 - \alpha}{(1 + q)^m [1 + \lambda]^\zeta [\beta + 2 - \alpha]} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$ we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{8}$$

Proof. Observe that for $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ inequality (8) is equivalent to

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1 - \alpha}{[2]_q^m [1 + \lambda]^\zeta [\beta + 2 - \alpha]} z \right|^\eta d\theta.$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1 - \alpha}{[2]_q^m [1 + \lambda]^\zeta [\beta + 2 - \alpha]} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1 - \alpha}{[2]_q^m [1 + \lambda]^\zeta [\beta + 2 - \alpha]} w(z),$$

and using (4), we obtain that $w(z)$ is analytic in Δ , $w(0) = 0$ and

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{[2]_q^m [1 + \lambda]^\zeta [\beta + 2 - \alpha]}{1 - \alpha} |a_n| z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[n]_q^m [1 + (n-1)\lambda]^\zeta [n(\beta + 1) - (\alpha + \beta)]}{1 - \alpha} |a_n| \leq |z|. \end{aligned}$$

This completes the proof of Theorem 3.2.

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References

- [1] Aouf, Mohamed Kamal et al. " β -uniformly convex and starlike functions." *Proc. Pakistan Acad. Sci.* 46, no. 2 (2009): 97-101. Cited on 59.
- [2] Araci, Serkan et al. "A certain (p, q) -derivative operator and associated divided differences." *J. Inequal. Appl.* 2019 (2019): Paper No. 301. Cited on 54.
- [3] Aral, Ali, Vijay Gupta, and Ravi P. Agarwal. *Applications of q -calculus in operator theory*. New York: Springer, 2013. Cited on 54.
- [4] Attiya, Adel A. "On some applications of a subordination theorem." *J. Math. Anal. Appl.* 311, no. 2 (2005): 489-494. Cited on 57.
- [5] Al-Oboudi, Fatima M. "On univalent functions defined by a generalized Sălăgean operator." *Int. J. Math. Math. Sci.* no. 25-28 (2004): 1429-1436. Cited on 55.
- [6] Frasin, Basem Aref. "Subordination results for a class of analytic functions defined by a linear operator." *JIPAM. J. Inequal. Pure Appl. Math.* 7 (2006): Article 4. Cited on 57 and 60.
- [7] Jackson, Frederick H., "On q -functions and a certain difference operator." *Trans. Royal Soc. Edinb.* 46 (1908): 253-281. Cited on 54.
- [8] Gangadharan, A., Tirunelveli Nellaippan Shanmugam, and Hari Mohan Srivastava. "Generalized hypergeometric functions associated with k -uniformly convex functions." *Comput. Math. Appl.* 44, no. 12 (2002): 1515-1526. Cited on 56.
- [9] Goodman, Adolph W. "On uniformly convex functions." *Ann. Polon. Math.* 56, no. 1 (1991): 87-92. Cited on 56.
- [10] Goodman, Adolph W. "On uniformly starlike functions." *J. Math. Anal. Appl.* 155, no. 2 (1991): 364-370. Cited on 56.
- [11] Govindaraj, M., and Srikanth Sivasubramanian. "On a class of analytic functions related to conic domains involving q -calculus." *Anal. Math.* 43, no. 3 (2017): 475-487. Cited on 54.
- [12] Kanas, Stanisława R. and Dorina Răducanu. "Some class of analytic functions related to conic domains." *Math. Slovaca* 64, no. 5 (2014): 1183-1196. Cited on 54.
- [13] Kanas, Stanisława R., and Agnieszka Wiśniowska. "Conic regions and k -uniform convexity." *J. Comput. Appl. Math.* 105, no. 1-2 (1999): 327-336. Cited on 56.

- [14] Kanas, Stanisława R., and Agnieszka Wiśniowska. "Conic domains and starlike functions." *Rev. Roumaine Math. Pures Appl.* 45, no. 4 (2000): 647-657. Cited on 56.
- [15] Kanas, Stanisława R., and Hari Mohan Srivastava. "Linear operators associated with k -uniformly convex functions." *Integral Transform. Spec. Funct.* 9, no. 2 (2000): 121-132. Cited on 56.
- [16] Karahuseyin, Zeliha, Şahsene Altinkaya, and Sibel Yalçın. "On $H_3(1)$ Hankel determinant for univalent functions defined by using q -derivative operator." *TJMM* 9, no. 1 (2017): 25-33. Cited on 54.
- [17] Littlewood, John Edensor. "On Inequalities in the Theory of Functions." *Proc. London Math. Soc.* 23, no. 7 (2): 481-519. Cited on 61.
- [18] Ma, Wan Cang, and David Minda. "Uniformly convex functions." *Ann. Polon. Math.* 57, no. 2 (1992): 165-175. Cited on 56.
- [19] Rønning, Frode. "Uniformly convex functions and a corresponding class of starlike functions." *Proc. Amer. Math. Soc.* 118, no. 1 (1993): 189-196. Cited on 56.
- [20] Rønning, Frode. "On starlike functions associated with parabolic regions." *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 45, no. 1991 (1992): 117-122. Cited on 56.
- [21] Sălăgean, Grigore Stefan. "Subclasses of univalent functions." *Complex Analysis: Fifth Romanian- Finnish Seminar, Part I(Bucharest, 1981)* Vol. 1013 of *Lecture Notes in Mathematics*, 362-372. Berlin, New York: Springer-Verlag, 1983. Cited on 55.
- [22] Silverman, Herb. "Univalent functions with negative coefficients." *Proc. Amer. Math. Soc.* 51 (1975): 109-116. Cited on 53 and 61.
- [23] Silverman, Herb. "A survey with open problems on univalent functions whose coefficients are negative." *Rocky Mountain J. Math.* 21, no. 3 (1991): 1099-1125. Cited on 61.
- [24] Silverman, Herb. "Integral means for univalent functions with negative coefficients." *Houston J. Math.* 23, no. 1 (1997): 169-174. Cited on 61.
- [25] Singh, Sukhjit. "A subordination theorem for spirallike functions." *Int. J. Math. Math. Sci.* 24, no. 7 (2000): 433-435. Cited on 57.
- [26] Srivastava, Hari Mohan. "Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials." *Appl. Math. Inf. Sci.* 5, no. 3 (2011): 390-444. Cited on 54.
- [27] Srivastava, Hari Mohan, and Adel A. Attiya. "Some subordination results associated with certain subclasses of analytic functions." *JIPAM. J. Inequal. Pure Appl. Math.* 5, no. 4 (2004): 6pp. Cited on 57.
- [28] Wilf, Herbert Saul. "Subordinating factor sequences for convex maps of the unit circle." *Proc. Amer. Math. Soc.* 12 (1961): 689-693. Cited on 56.

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[64]

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