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An exponentially fitted finite difference scheme for a class of singularly perturbed delay differential equations with large delays

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Abstract This paper deals with singularly perturbed boundary value problem for a linear second order delay differential equation. It is known that the classical numerical methods are not satisfactory when applied to solve singularly perturbed problems in delay differential equations. In this paper we present an exponentially fitted finite difference scheme to overcome the drawbacks of the corresponding classical counter parts. The stability of the scheme is investigated. The proposed scheme is analyzed for convergence. Several linear singularly perturbed delay differential equations have been solved and the numerical results are presented to support the theory.

1. Introduction

Singular perturbation problems arise very frequently in fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. An overview of some existence and uniqueness results and applications of singularly perturbed equations may be obtained from [1–4]. Various approaches to the design and analysis of approximate numerical methods for singularly perturbed differential equations can be found in [5–8] and the references cited in them.

Delay differential equations arise widely in various application fields and are also described in technical devices like control circuits. Nowadays the delay differential equations are ubiquitous in various branches of bioscience and control theory: ecology, chemostat systems, epidemiology, immunology, compartmental studies, neutral network and the navigational control of ships and aircraft (with respectively large and short lags) and in more general control problems [9–21]. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then to react to it. Delay differential equations are of the retarded type if the delay argument does not occur in the highest order term. If we restrict this class in which the highest order term is multiplied by a small parameter, then we get singularly perturbed delay differential equations of

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retarded type. The numerical study of second order singularly perturbed differential-difference equation with small shift or delay has been given in [22–38] and references therein.

Amiraliyev and Cimen [39] have given an exponentially fitted difference scheme on a uniform mesh for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. Subburayan and Ramanujam [40] presented an initial value technique to solve singularly perturbed boundary value problem for the second order ordinary differential equations of convection–diffusion type with a delay. They also developed an asymptotic initial value technique to solve singularly perturbed boundary value problem for the second order ODE with the discontinuous convection–diffusion coefficient term [41].

In this paper we present an exponentially fitted finite difference scheme to solve singularly perturbed delay differential equation of second order with a large delay. For many singular perturbation problems a reduced problem is well defined and known a priori. We used Runge–Kutta method to solve the reduced problem. A condition at $x = 1$ is obtained by approximating the solution at $x = 1$ by the solution of the reduced problem at the same point. A fitting factor is introduced in a finite difference scheme and is obtained from the theory of singular perturbations. The stability of the scheme is investigated. The proposed scheme is analyzed for convergence. Several linear singularly perturbed delay differential equations have been solved and the numerical results are presented to support the theory.

2. Exponentially fitted finite difference method

To describe the method, we consider a singularly perturbed boundary value problem for the delay differential equation of the form:

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-1) = f(x), x \in [0,2] \tag{1}$$

subject to the interval and boundary conditions,

$$\begin{aligned} &y(x) = \phi(x), x \in [-1,0]; \\ &y(2) = \beta, \end{aligned} \tag{2}$$

where $0 < \varepsilon << 1$ and $a(x) \geqslant \alpha > 0, \theta \leqslant b(x) < 0, a(x), b(x), f(x)$ are given sufficiently smooth functions on $[0,2]$, $\phi(x)$ is smooth function on $[-1,0]$ and β is a given constant which is independent of ε. For small values of ε, the boundary value problem (1) along with (2) exhibits a strong boundary layer at $x = 2$ (c.f. [12]).

The linear ordinary differential equation (1) cannot, in general, be solved analytically because of the dependence of $a(x), b(x)$ and $f(x)$ on the spatial coordinate x. We divide the interval $[0,2]$ into $2N$ equal parts with constant mesh length h. Let $0 = x_0, x_1, \ldots, x_N = 1, x_{N+1}, x_{N+2}, \ldots, x_{2N} = 2$ be the mesh points. Then we have $x_i = ih; i = 0,1,2,\ldots,2N$. If we consider, the interval $[x_{i-1}, x_{i+1}]$ and the coefficients of Eq. (1) are evaluated at the midpoint of each interval, then we will obtain the differential equation

$$-\varepsilon y''(x) + a_i y'(x) = f_i - b_i y(x_i - 1), \quad x_{i-1} < x < x_{i+1}. \tag{3}$$

The analytical solution of Eq. (3) is of the form

$$y(x) = A_i + B_i e^{\frac{a_i(x-x_i)}{\varepsilon}} + \frac{x - x_i}{a_i}(f_i - b_i y(x_i - 1)), \quad x_{i-1} < x < x_{i+1}, \tag{4}$$

where A_i, B_i are arbitrary constants. We obtain the arbitrary constants A_i and B_i using the conditions $y(x_{i-1}) = y_{i-1}, y(x_i) = y_i, y(x_{i+1}) = y_{i+1}$.

From (4) we have,

$$y_{i-1} = A_i + B_i e^{\frac{-a_i h}{\varepsilon}} - \frac{h}{a_i}[f_i - b_i y(x_i - 1)], \tag{5}$$

$$y_{i+1} = A_i + B_i e^{\frac{a_i h}{\varepsilon}} + \frac{h}{a_i}[f_i - b_i y(x_i - 1)] \text{ and} \tag{6}$$

$$y_i = A_i + B_i. \tag{7}$$

Now, we have

$$y_{i+1} - 2y_i + y_{i-1} = B_i\left(e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}\right).$$

Therefore,

$$B_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}}. \tag{8}$$

Since, $A_i = y_i - B_i$, we have

$$A_i = \frac{\left(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}}\right)y_i - (y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}}. \tag{9}$$

Substituting (8) and (9) in (4), we get

$$y(x) = \frac{\left(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}} - 2e^{\frac{a_i(x-x_i)}{\varepsilon}}\right)y_i + \left(e^{\frac{a_i(x-x_i)}{\varepsilon}} - 1\right)(y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}} + \frac{(x - x_i)}{a_i}[f_i - b_i y(x_i - 1)]. \tag{10}$$

From (10), we have

$$y(x_{i-1}) = \frac{\left(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}} - 2e^{\frac{-a_i h}{\varepsilon}}\right)y_i + \left(e^{\frac{-a_i h}{\varepsilon}} - 1\right)(y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}} - \frac{h}{a_i}[f_i - b_i y(x_i - 1)],$$

$$y(x_{i+1}) = \frac{(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}} - 2e^{\frac{a_i h}{\varepsilon}})y_i + [e^{\frac{a_i h}{\varepsilon}} - 1](y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}} + \frac{h}{a_i}[f_i - b_i y(x_i - 1)].$$

Therefore, we have

$$y(x_{i+1}) - y(x_{i-1}) = \frac{y_{i+1} - 2y_i + y_{i-1}}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}}\left(e^{\frac{a_i h}{\varepsilon}} - e^{\frac{-a_i h}{\varepsilon}}\right) + \frac{2h}{a_i}[f_i - b_i y(x_i - 1)]. \tag{11}$$

Now, to find the solution in $[0,2]$, we consider the second order finite difference scheme

$$-\varepsilon\sigma(\rho)\left(\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}\right) + a(x_i)\left(\frac{y(x_{i+1}) - y(x_{i-1})}{2h}\right) = f(x_i) - b(x_i)y(x_i - 1) + O(h^2), 1 \leqslant i \leqslant 2N - 1 \tag{12}$$

where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (12) converges uniformly to the solution of (1), (2) and $\rho = \frac{h}{\varepsilon}$.

We consider $h = O(\varepsilon)$ i.e., $\frac{h}{\varepsilon}$ is finite.

Substituting (11) in (12), we get

$$-\frac{\varepsilon\sigma(\rho)}{h^2}(y_{i+1}-2y_i+y_{i-1})+\frac{a_i}{2h}\left(\frac{y_{i+1}-2y_i+y_{i-1}}{e^{\frac{a_ih}{\varepsilon}}-2+e^{\frac{-a_ih}{\varepsilon}}}\left(e^{\frac{a_ih}{\varepsilon}}-e^{\frac{-a_ih}{\varepsilon}}\right)+\frac{2h}{a_i}[f_i-b_iy(x_i-1)]\right)=f_i-b_iy(x_i-1)$$

$$\text{i.e.,} -\frac{\varepsilon\sigma(\rho)}{h^2}(y_{i+1}-2y_i+y_{i-1})+\frac{a_i}{2h}\left(\frac{y_{i+1}-2y_i+y_{i-1}}{e^{\frac{a_ih}{\varepsilon}}-2+e^{\frac{-a_ih}{\varepsilon}}}\right)\left(e^{\frac{a_ih}{\varepsilon}}-e^{\frac{-a_ih}{\varepsilon}}\right)=0$$

which implies

$$\frac{\varepsilon\sigma(\rho)}{h^2}=\frac{a_i}{2h}\left(\frac{e^{\frac{a_ih}{\varepsilon}}-e^{\frac{-a_ih}{\varepsilon}}}{e^{\frac{a_ih}{\varepsilon}}-2+e^{\frac{-a_ih}{\varepsilon}}}\right).$$

$$\therefore\ \sigma(\rho)=\frac{a_i\rho}{2}Coth\left(\frac{a_i\rho}{2}\right)\ \text{where}\ \rho=\frac{h}{\varepsilon} \tag{13}$$

which is the required fitting factor.

2.1. Numerical algorithm

Step 1. We obtain the reduced problem by setting $\varepsilon = 0$ in Eq. (1) with the appropriate interval condition. Let $y_0(x)$ be the solution of the reduced problem of (1), (2), i.e.;

$$a(x)y_0'(x)+b(x)y_0(x-1)=f(x) \tag{14}$$

$$\text{with } y_0(x)=\phi(x), -1\leqslant x\leqslant 0. \tag{15}$$

We solve (14), (15) by using Runge–Kutta method in $0 \leqslant x \leqslant 1$.

We consider $y_0(1) = \gamma$.

Step 2. To obtain the solution in $0 < x < 1$, we consider the numerical scheme from (12) which is of the form

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i, 1<i<N-1, \tag{16}$$

with the boundary conditions

$y_0 = \phi(0)$ and $y_N = \gamma$,

where

$$E_i=\frac{\varepsilon\sigma}{h^2}+\frac{a_i}{2h},$$

$$F_i=\frac{2\varepsilon\sigma}{h^2},$$

$$G_i=\frac{\varepsilon\sigma}{h^2}-\frac{a_i}{2h},\ \text{and}$$

$$H_i=-f_i+b_i\phi(x_{i-N}).$$

Similarly, to obtain the solution in $1 < x < 2$, we consider the numerical scheme from (12) which is of the form

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i, N+1<i<2N-1, \tag{17}$$

with the boundary conditions

$y_N = \gamma$ and $y_{2N} = \beta$,

where

$$E_i=\frac{\varepsilon\sigma}{h^2}+\frac{a_i}{2h},$$

$$F_i=\frac{2\varepsilon\sigma}{h^2},$$

$$G_i=\frac{\varepsilon\sigma}{h^2}-\frac{a_i}{2h},\ \text{and}$$

$$H_i=-f_i+b_iy(x_{i-N}).$$

This gives us the tridiagonal systems which can be solved easily by Thomas algorithm.

3. Thomas algorithm

We briefly discuss the Thomas algorithm to solve the tridiagonal system:

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i;\ i=1,2,3,\ldots\ldots,N-1 \tag{18}$$

subject to the boundary conditions

$$y_0=\phi(0); \tag{19a}$$

$$y_N=y(x_N)=\gamma. \tag{19b}$$

$$\text{We set } y_i=W_iy_{i+1}+T_i \text{ for } i=N-1,N-2,\ldots,2,1. \tag{20}$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined.

From (20), we have

$$y_{i-1}=W_{i-1}y_i+T_{i-1}. \tag{21}$$

Substituting (21) in (18), we have

$$E_i(W_{i-1}y_i+T_{i-1})-F_iy_i+G_iy_{i+1}=H_i.$$

$$\therefore\ y_i=\left(\frac{G_i}{F_i-E_iW_{i-1}}\right)y_{i+1}+\left(\frac{E_iT_{i-1}-H_i}{F_i-E_iW_{i-1}}\right) \tag{22}$$

By comparing (22) and (20), we get the recurrence relations

$$W_i=\left(\frac{G_i}{F_i-E_iW_{i-1}}\right) \tag{23a}$$

$$T_i=\left(\frac{E_iT_{i-1}-H_i}{F_i-E_iW_{i-1}}\right). \tag{23b}$$

To solve these recurrence relations for $i = 0, 1, \ldots, N-1$, we need the initial conditions for W_0 and T_0. For this we have $y_0 = \phi(0) = W_0y_1 + T_0$. If we choose $W_0 = 0$, then we get $T_0 = \phi(0)$. With these initial values, we compute W_i and T_i for $i = 1, 2, \ldots, N-1$ from (23) in forward process, and then obtain y_i in the backward process from (20) and (19b).

4. Stability analysis

We will now show that the algorithm is computationally stable. To check the stability of the proposed scheme, we follow Smith [42] and Turgut Öziş and Utku Erdoğan [43]. By stability, we mean that the effect of an error made in one stage of the calculation is not propagated into larger errors at later stages of the calculations. Let us now examine the recurrence relation given by (23a). Suppose that a small error e_{i-1} has been made in the calculation of W_{i-1}; then, we have

$\overline{W}_{i-1} = W_{i-1} + e_{i-1}$ and we are actually calculating

$$\overline{W}_i=\left(\frac{G_i}{F_i-E_i\overline{W}_{i-1}}\right). \tag{24}$$

From (24) and (23a), we have

$$e_i = \left(\frac{G_i}{F_i - E_i(W_{i-1}+e_{i-1})}\right) - \left(\frac{G_i}{F_i - E_iW_{i-1}}\right)$$
$$= \left(\frac{G_iE_ie_{i-1}}{(F_i - E_i(W_{i-1}+e_{i-1}))(F_i - E_iW_{i-1})}\right)$$
$$= \left(\frac{W_i^2E_i}{G_i}\right)e_{i-1} \tag{25}$$

under the assumption that the error is small initially. From the assumptions made earlier that $a(x) > 0$, we have

$$|F_i| \geqslant |E_i + G_i|;\; i = 1,2,3,\ldots\ldots,2N-1$$

From (23a) we have.

$$W_1 = \frac{G_1}{F_1} < 1,\; \text{since } F_1 > G_1.$$

$$W_2 = \frac{G_2}{F_2 - E_2W_1} < \frac{G_2}{F_2 - E_2};\; \text{since } W_1 < 1,$$

$$< \frac{G_2}{E_2 + G_2 - E_2} = 1;\; \text{since } F_2 \geqslant E_2 + G_2.$$

successively, it follows that

$$|e_i| = |W_i|^2\left|\frac{E_i}{G_i}\right||e_{i-1}|$$

Making use of the condition $|W_i| < 1, i = 0,1,2,\ldots\ldots,2N-1$ it follows that

$$|e_i| < |e_{i-1}|.$$

Therefore the recurrence relation (23a) is stable. Similarly we can prove that the recurrence relation (23b) is also stable. Finally the convergence of the Thomas algorithm is ensured by the condition $|W_i| < 1$, $i = 1,2,3,\ldots,2N-1$.

5. Convergence analysis

Multiplying the Eq. (16) by $\frac{h^2}{\varepsilon}$ we get

$$(\sigma + u_i)y_{i-1} - (2\sigma + v_i)y_i + (\sigma + w_i)y_{i+1} + g_i + T_i = 0 \tag{26}$$

where

$$u_i = \frac{a_ih}{2\varepsilon},\quad v_i = 0,\quad w_i = -\frac{a_ih}{2\varepsilon},\quad g_i = \frac{h^2}{\varepsilon}(f_i - b_i\phi(x_{i-N})).$$

Incorporating the boundary conditions $y_0 = \phi(x_0) = \phi(0), y_N = \gamma$ we obtain the system of equations in the matrix form as

$$(D+P)y + M + T(h) = 0 \tag{27}$$

where

$$D = [\sigma, -2\sigma, \sigma] = \begin{bmatrix} -2\sigma & \sigma & 0 & \ldots & 0 \\ \sigma & -2\sigma & \sigma & \ldots & 0 \\ 0 & . & . & \ldots & . \\ . & . & . & \ldots & . \\ 0 & \ldots & 0 & \sigma & -2\sigma \end{bmatrix}$$

and $$P = [u_i, v_i, w_i] = \begin{bmatrix} v_1 & w_1 & 0 & \ldots & 0 \\ u_2 & v_2 & w_2 & \ldots & 0 \\ 0 & . & . & \ldots & . \\ . & . & . & \ldots & . \\ 0 & \ldots & 0 & u_{N-1} & v_{N-1} \end{bmatrix}$$

are tridiagonal matrices of order $N-1$, and $M = [g_1 + (\sigma + u_1)\phi(0), g_2, g_3, \ldots, g_{N-2}, g_{N-1} + (\sigma + w_{N-1})\gamma]^T$, $T(h) = O(h^2)$ and $y = [y_1, y_2, \ldots y_{N-1}]^T$, $T(h) = [T_1, T_2, \ldots T_{N-1}]^T$, $\mathbf{0} = [0, 0, \ldots, 0]^T$ are the associated vectors of Eq. (27).

Let $Y = [Y_1, Y_2, \ldots, Y_{N-1}]^T \cong y$ which satisfies the equation

$$(D+P)Y + M = 0 \tag{28}$$

Let $e_i = Y_i - y_i, i = 1,2,\ldots,N-1$ be the discretization error so that

$$E = [e_1, e_2, \ldots e_{N-1}]^T = Y - y.$$

Subtracting Eq. (27) from Eq. (28) we get

$$(D+P)E = T(h). \tag{29}$$

Let $|a_i| \leqslant K$

Let $p_{i,j}$ be the (i,j)th element of the matrix P, then

$$|p_{i,i+1}| = |w_i| \leqslant h\frac{K}{2\varepsilon};\quad i = 1,2,\ldots,N-2 \tag{30a}$$

$$|p_{i,i-1}| = |u_i| \leqslant h\frac{K}{2\varepsilon};\; i = 2,3,\ldots,N-1 \tag{30b}$$

Thus for sufficiently small h (i.e., as $h \to 0$), we have

$$\sigma + |p_{i,i+1}| \neq 0,\; i = 1,2,\ldots,N-2 \tag{31a}$$

$$\sigma + |p_{i,i-1}| \neq 0,\; i = 2,3,\ldots,N-1 \tag{31b}$$

Hence, the matrix $(D+P)$ is irreducible [44].

Let S_i be the sum of absolute values of the elements of the ith row of the matrix $(D+P)$, then.

$S_i = |-\sigma| + |w_i|$ for $i = 1$
$S_i = |u_i| + |w_i|$ for $i = 2,3,\ldots,N-2$
$S_i = |-\sigma| + |u_i|$ for $i = N-1$

Let $K_* = \min_{1\leqslant i\leqslant N-1}|a_i|, K^* = \max_{1\leqslant i\leqslant N-1}|a_i|$ then $0 < K_* \leqslant K \leqslant K^*$.

For sufficiently small h, $(D+P)$ is monotone [44,45]. Hence $(D+P)^{-1}$ exists and $(D+P)^{-1} \geqslant 0$.

From the error Eq. (29) we have $\|E\| \leqslant \|(D+P)^{-1}\| \cdot \|T\|$. For sufficiently small h, we have

$$S_i > h\left(\frac{K}{2\varepsilon}\coth\left(\frac{K\rho}{2}\right) + \frac{K}{2\varepsilon}\right) > hQ_1 \;\text{ for i} = 1 \tag{32a}$$

where $Q_1 = \frac{K}{2\varepsilon}\coth\left(\frac{K\rho}{2}\right) + \frac{K}{2\varepsilon}$

$$S_i > h\frac{K}{\varepsilon} > hQ_2 \;\text{ for i} = 2,3,\ldots,\text{N}-2 \tag{32b}$$

where $Q_2 = \frac{K}{\varepsilon}$

$$S_i > h\left(\frac{K}{2\varepsilon}\coth\left(\frac{K\rho}{2}\right) + \frac{K}{2\varepsilon}\right) > hQ_1 \;\text{ for i} = \text{N}-1 \tag{32c}$$

Table 1 Numerical results of Example 1 for different values of perturbation parameter ε.

ε ↓	N → 32	64	128	256	512	1024
2^{-5}	5.1552e−004	1.4652e−004	3.7888e−005	9.5547e−006	2.3939e−006	5.9883e−007
2^{-6}	7.5922e−004	2.6637e−004	7.5145e−005	1.9455e−005	4.9061e−006	1.2292e−006
2^{-7}	8.3676e−004	3.8575e−004	1.3534e−004	3.8180e−005	9.8729e−006	2.4898e−006
2^{-8}	8.4092e−004	4.2513e−004	1.9441e−004	6.8209e−005	1.9242e−005	4.9774e−006
2^{-9}	8.4093e−004	4.2724e−004	2.1425e−004	9.7588e−005	3.4253e−005	9.6682e−006
2^{-10}	8.4093e−004	4.2725e−004	2.1532e−004	1.0755e−004	4.8915e−005	1.7177e−005
2^{-11}	8.4093e−004	4.2725e−004	2.1532e−004	1.0808e−004	5.3879e−005	2.4506e−005
2^{-12}	8.4093e−004	4.2725e−004	2.1532e−004	1.0808e−004	5.4147e−005	2.6966e−005
2^{-13}	8.4093e−004	4.2725e−004	2.1532e−004	1.0808e−004	5.4148e−005	2.7100e−005
2^{-14}	8.4093e−004	4.2725e−004	2.1532e−004	1.0808e−004	5.4148e−005	2.7100e−005
E^N	8.4093e−004	4.2725e−004	2.1532e−004	1.0808e−004	5.4148e−005	2.7100e−005
R^N	9.7692e−001	9.8860e−001	9.9433e−001	9.9717e−001	9.9859e−001	9.9931e−001

Table 2 Numerical results of Example 2 for different values of perturbation parameter ε.

ε ↓	N → 32	64	128	256	512	1024
2^{-5}	2.0741e−002	5.4962e−003	1.3953e−003	3.5024e−004	8.7651e−005	2.1918e−005
2^{-6}	3.5344e−002	1.0725e−002	2.8528e−003	7.2423e−004	1.8177e−004	4.5489e−005
2^{-7}	4.5585e−002	1.8006e−002	5.4949e−003	1.4561e−003	3.6979e−004	9.2808e−005
2^{-8}	4.7271e−002	2.3160e−002	9.1484e−003	2.7805e−003	7.3682e−004	1.8705e−004
2^{-9}	4.7302e−002	2.4016e−002	1.1672e−002	4.6105e−003	1.3988e−003	3.7082e−004
2^{-10}	4.7302e−002	2.4033e−002	1.2104e−002	5.8590e−003	2.3143e−003	7.0214e−004
2^{-11}	4.7302e−002	2.4033e−002	1.2112e−002	6.0756e−003	2.9352e−003	1.1594e−003
2^{-12}	4.7302e−002	2.4033e−002	1.2112e−002	6.0797e−003	3.0438e−003	1.4691e−003
2^{-13}	4.7302e−002	2.4033e−002	1.2112e−002	6.0797e−003	3.0458e−003	1.5234e−003
2^{-14}	4.7302e−002	2.4033e−002	1.2112e−002	6.0797e−003	3.0458e−003	1.5244e−003
E^N	4.7302e−002	2.4033e−002	1.2112e−002	6.0797e−003	3.0458e−003	1.5244e−003
R^N	9.7692e−001	9.8860e−001	9.9433e−001	9.9717e−001	9.9859e−001	1.0003e+000

Let $(D+P)^{-1}_{i,k}$ be the (i,k)th element of $(D+P)^{-1}$ and we define

$\|(D+P)^{-1}\| = \max_{1\leqslant i\leqslant N-1} \sum_{k=1}^{N-1}(D+P)^{-1}_{i,k}$ and $\|T(h)\| = \max_{1\leqslant i\leqslant N-1}|T_i|$.

Since $(D+P)^{-1}_{i,k} \geqslant 0$ and $\sum_{k=1}^{N-1}(D+P)^{-1}_{i,k}\cdot S_k = 1$ for $i = 1,2,\ldots,N-1$, we have,

$$(D+P)^{-1}_{i,k} \leqslant \frac{1}{S_k} < \frac{1}{hQ_1} \text{ for } k = 1.$$

$(D+P)^{-1}_{i,k} \leqslant \frac{1}{S_k} < \frac{1}{hQ_1}$ for $k = N-1$.

Further $\sum_{k=2}^{N-2}(D+P)^{-1}_{i,k} \leqslant \frac{1}{\min_{2\leqslant k\leqslant N-2} S_k} \leqslant \frac{1}{hQ_2}$ for $i = 2,3\ldots,N-2$.

From the error Eq. (29), using Eq. (32) we get

$$\|E\| = \frac{1}{h}\left|\frac{1}{Q_1}+\frac{1}{Q_2}+\frac{1}{Q_1}\right| \times O(h^2) = O(h). \tag{33}$$

This establishes the first order convergence of the finite difference scheme (16). From Eq. (33) it can be observed that the proposed method is ε-uniformly convergent, since the error $\|E\| = C\cdot h$, where C is independent of the perturbation parameter ε.

Remark. A similar analysis for convergence may be carried out for finite difference scheme (17).

6. Numerical examples

To demonstrate the applicability of the method we consider four boundary value problems of singularly perturbed linear differential difference equations exhibiting boundary layer at the right end of the interval $[0,2]$. These problems were widely discussed in the literature. Since the exact solutions of the problems are not known, the maximum absolute errors for the examples are calculated using the following double mesh principle

$$E^N_\varepsilon = \max_{0\leqslant i\leqslant N}\left|y^N_i - y^{2N}_{2i}\right|.$$

For a value of N, the ε-uniform maximum absolute error is calculated by the formula

$$E^N = \max_\varepsilon E^N_\varepsilon.$$

The numerical rate of convergence for all the examples has been calculated by the formula

$$R^N = \frac{\log|E^N_\varepsilon/E^{2N}_\varepsilon|}{\log 2}.$$

Example 1 [40, p. 247]. Consider the following singularly perturbed delay differential equation

Table 3 Numerical results of Example 3 for different values of perturbation parameter ε.

$\varepsilon \downarrow$	$N \rightarrow$ 32	64	128	256	512	1024
2^{-5}	1.4272e−003	6.2958e−004	2.0947e−004	5.7659e−005	1.4802e−005	3.7252e−006
2^{-6}	1.4424e−003	7.2567e−004	3.1741e−004	1.0583e−004	2.9132e−005	7.4754e−006
2^{-7}	1.4425e−003	7.3344e−004	3.6586e−004	1.5976e−004	5.3246e−005	1.4646e−005
2^{-8}	1.4425e−003	7.3349e−004	3.6978e−004	1.8369e−004	8.0213e−005	2.6706e−005
2^{-9}	1.4425e−003	7.3349e−004	3.6981e−004	1.8566e−004	9.2033e−005	4.0190e−005
2^{-10}	1.4425e−003	7.3349e−004	3.6981e−004	1.8567e−004	9.3020e−005	4.6064e−005
2^{-11}	1.4425e−003	7.3349e−004	3.6981e−004	1.8567e−004	9.3026e−005	4.6558e−005
2^{-12}	1.4425e−003	7.3349e−004	3.6981e−004	1.8567e−004	9.3026e−005	4.6561e−005
2^{-13}	1.4425e−003	7.3349e−004	3.6981e−004	1.8567e−004	9.3026e−005	4.6561e−005
2^{-14}	1.4425e−003	7.3349e−004	3.6981e−004	1.8567e−004	9.3026e−005	4.6561e−005
E^N	1.4425e−003	7.3349e−004	3.6981e−004	1.8567e−004	9.3026e−005	4.6561e−005
R^N	9.7574e−001	9.8801e−001	9.9404e−001	9.9703e−001	9.9852e−001	9.9926e−001

Table 4 Numerical results of Example 4 for different values of perturbation parameter ε.

$\varepsilon \downarrow$	$N \rightarrow$ 32	64	128	256	512	1024
2^{-5}	1.9126e−004	6.2523e−005	1.7063e−005	4.3683e−006	1.0987e−006	2.7510e−007
2^{-6}	2.2400e−004	9.7185e−005	3.1769e−005	8.6781e−006	2.2223e−006	5.5895e−007
2^{-7}	2.2703e−004	1.1381e−004	4.8981e−005	1.6049e−005	4.3824e−006	1.1219e−006
2^{-8}	2.2705e−004	1.1535e−004	5.7355e−005	2.4657e−005	8.0719e−006	2.2029e−006
2^{-9}	2.2705e−004	1.1536e−004	5.8131e−005	2.8790e−005	1.2377e−005	4.0479e−006
2^{-10}	2.2705e−004	1.1536e−004	5.8136e−005	2.9180e−005	1.4423e−005	6.2006e−006
2^{-11}	2.2705e−004	1.1536e−004	5.8136e−005	2.9182e−005	1.4618e−005	7.2188e−006
2^{-12}	2.2705e−004	1.1536e−004	5.8136e−005	2.9182e−005	1.4620e−005	7.3164e−006
2^{-13}	2.2705e−004	1.1536e−004	5.8136e−005	2.9182e−005	1.4620e−005	7.3171e−006
2^{-14}	2.2705e−004	1.1536e−004	5.8136e−005	2.9182e−005	1.4620e−005	7.3171e−006
E^N	2.2705e−004	1.1536e−004	5.8136e−005	2.9182e−005	1.4620e−005	7.3171e−006
R^N	9.7692e−001	9.8860e−001	9.9433e−001	9.9717e−001	9.9859e−001	9.9930e−001

Table 5 Comparison of maximum error for Example 1.

Method	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
Present method	4.2725e−004	2.1532e−004	1.0808e−004	5.4148e−005	2.7100e−005
Subburayan & Ramanujam [40]	7.8585e−004	2.7331e−004	8.5983e−005	2.6631e−005	8.2968e−006

$$\varepsilon y''(x) - 3y'(x) + y(x-1) = 0,$$
$$y(x) = 1, -1 \leqslant x \leqslant 0, \quad y(2) = 2.$$

The numerical results are presented in Table 1 for different values of perturbation parameter ε.

The exact solution of this problem is given by

$$y(x)=\begin{cases} 1+c_1\left[\exp\left(\frac{3x}{\varepsilon}\right)-1\right]+\frac{x}{3};\, x\in[0,1],\\ c_2+\frac{x}{3}+\frac{(x-1)^2}{18}+\frac{\varepsilon x}{27}-\frac{c_1 x}{3}-\frac{c_1 x}{3}\exp\left(\frac{3(x-1)}{\varepsilon}\right)\\ +\exp\left(\frac{3(x-2)}{\varepsilon}\right)\left[\frac{23}{18}-\frac{2\varepsilon}{27}-c_2+\frac{2c_1}{3}+\frac{2c_1}{3}\exp\left(\frac{3}{\varepsilon}\right)\right],\, x\in[1,2], \end{cases}$$

where

$$c_1 = \exp\left(\frac{-6}{\varepsilon}\right)\left[\frac{\frac{4\varepsilon}{9}-\frac{\varepsilon^2}{27}-3}{3-4\exp\left(\frac{-6}{\varepsilon}\right)+\frac{2\varepsilon}{3}\left[\exp\left(\frac{-3}{\varepsilon}\right)-\exp\left(\frac{-6}{\varepsilon}\right)\right]}\right],$$

$$c_2 = \left[\frac{1-\frac{23}{18}\exp\left(\frac{-3}{\varepsilon}\right)+\frac{2\varepsilon}{27}\exp\left(\frac{-3}{\varepsilon}\right)-\frac{\varepsilon}{27}}{1-\exp\left(\frac{-3}{\varepsilon}\right)} + \frac{c_1\exp\left(\frac{3}{\varepsilon}\right)\left[1-\exp\left(\frac{-3}{\varepsilon}\right)-\frac{2}{3}\exp\left(\frac{-6}{\varepsilon}\right)\right]}{1-\exp\left(\frac{-3}{\varepsilon}\right)}\right].$$

For this Example, the numerical solution and the exact solution for $\varepsilon = 2^{-5}$ are plotted in graphs, shown in Fig. 1.

To show the layer behaviour of the solution, the numerical solution has been plotted for Example 1 for $\varepsilon = 2^{-5}, 2^{-10}, 2^{-15}$, shown in Fig. 2.

It has been observed that the thickness of the boundary layer reduces to zero as the perturbation parameter $\varepsilon \rightarrow 0$.

Example 2 [40, p. 247]. Consider the following singularly perturbed delay deferential equation

$$\varepsilon y''(x) - 2y'(x) + 5y(x-1) = 0,$$
$$y(x) = 1, -1 \leqslant x \leqslant 0, \quad y(2) = 2.$$

The numerical results are presented in Table 2 for different values of perturbation parameter ε.

Example 3 [40, p.247]. Consider the following singularly perturbed delay deferential equation

$$\varepsilon y''(x) - (x+10)y'(x) + y(x-1) = -x,$$
$$y(x) = x, -1 \leqslant x \leqslant 0, \quad y(2) = 2.$$

Table 6 Comparison of maximum error for Example 2.

Method	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
Present method	2.4033e−002	1.2112e−002	6.0797e−003	3.0458e−003	1.5244e−003
Subburayan & Ramanujam [40]	2.0157e−002	7.0103e−003	2.2055e−003	6.8308e−004	2.1281e−004

Table 7 Comparison of maximum error for Example 3.

Method	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
Present method	7.3349e−004	3.6981e−004	1.8567e−004	9.3026e−005	4.6561e−005
Subburayan & Ramanujam [40]	2.6473e−003	8.3944e−004	2.5834e−004	8.0254e−005	2.4315e−005

Table 8 Comparison of maximum error for Example 4.

Method	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
Present method	1.1536e−004	5.8136e−005	2.9182e−005	1.4620e−005	7.3171e−006
Subburayan & Ramanujam [40]	6.2066e−004	1.9525e−004	6.0439e−005	1.8797e−005	5.6409e−006

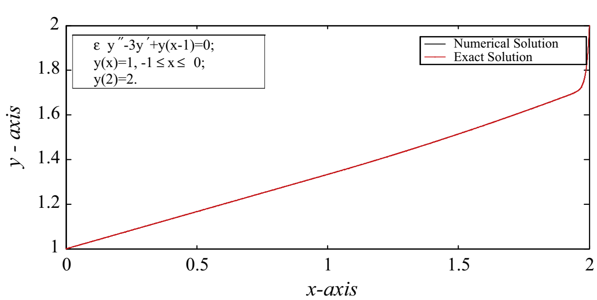

Fig. 1 Graph of exact solution and numerical solution of Example 1 for $\varepsilon = 2^{-5}$.

The numerical results are presented in Table 3 for different values of perturbation parameter ε.

Example 4 [40, p. 247]. Consider the following singularly perturbed delay deferential equation

$$\varepsilon y''(x) - 5y'(x) + \frac{1}{2}y(x-1) = \begin{cases} -1; 0 \leqslant x \leqslant 1 \\ 1; 1 \leqslant x \leqslant 2 \end{cases},$$

$y(x) = 1, -1 \leqslant x \leqslant 0, \quad y(2) = 2.$

The numerical results are presented in Table 4 for different values of perturbation parameter ε.

We compared our results with the results available in [40] for Examples 1–4 and are tabulated in Tables 5–8 respectively. The method that is proposed in [40] is an initial value technique. It is having slightly higher rate of convergence than the proposed method, but it is not a uniformly convergent method with respect to the perturbation parameter ε. Our aim is to propose a numerical scheme which converges uniformly with respect to the perturbation parameter ε and has been achieved.

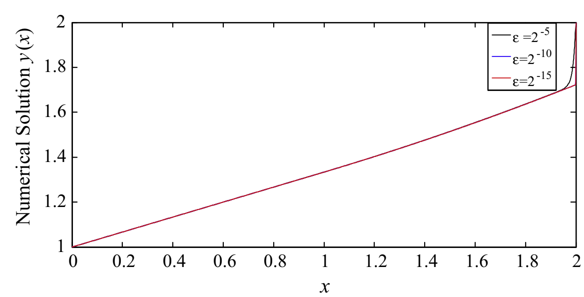

Fig. 2 Graph of numerical solution of Example 1 for $\varepsilon = 2^{-5}, 2^{-10}, 2^{-15}$.

7. Discussion and conclusions

In this paper we present an exponentially fitted finite difference scheme to solve singularly perturbed delay differential equation of second order with a large delay. The convergence analysis of the scheme has been derived and is found that the present method converges uniformly of order one with respect to the perturbation parameter ε. The numerical rate of convergence is also calculated and is found that the theoretical rate of convergence is matching with that of numerical rate of convergence. We have implemented the present method on four linear examples with right-end boundary layer by taking different values ofε. Numerical results are presented in tables. From the results, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. To show the layer behaviour of the solution, the numerical solution has been plotted for Example 1 for $\varepsilon = 2^{-5}, 2^{-10}, 2^{-15}$. It has been observed that the thickness of the boundary layer reduces to zero as the perturbation parameter ε tends to zero. On the

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basis of the numerical results of a variety of examples, it is concluded that the present method offers significant advantage for the linear singularly perturbed differential difference equations.

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