ARTICLE IN PRESS

Ain Shams Engineering Journal (2014) **xxx**, xxx–xxx

Ain Shams University

**Ain Shams Engineering Journal**

www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

# Analysis of an $M^{[X]}/(G_1, G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption

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Received 25 November 2013; revised 10 January 2014; accepted 16 February 2014

**KEYWORDS**

Two types of service;
Re-service;
Modified vacation;
Balking;
Service interruption

**Abstract** This paper deals with the steady state analysis of batch arrival retrial queueing system with two types of service under modified vacation policy, where each type consists of an optional re-service. An arriving batch may balk the system at some particular times. After the completion of each types of service the customers may re-service of the same type without joining the orbit or may leave the system. If the orbit is empty at the service completion of each types of service, the server takes at most $J$ vacations until at least one customer is received in the orbit when the server returns from a vacation. Busy server may breakdown at any instance and the service channel will fail for a short interval of time. The steady state probability generating function for system/orbit size is obtained by using the supplementary variable method. Some system performance measures and numerical illustrations are discussed.



## 1. Introduction

Recently there have been significant contributions to retrial queueing system (or queues with repeated attempts) in which arriving customer who finds the server busy upon arrival is obliged to leave the service area and repeat his demand after some time. Between trials, a blocked customer who remains in a retrial group is said to be in orbit. In queues, the customers are allowed to conduct retrials that have applications in telephone switching systems, telecommunication networks and computers are competing to gain service from a central processing unit. Moreover, retrial queues are also used as mathematical models of several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks, etc. There are plenty of literatures available on the retrial queues. We referred the works of Falin and Templeton [1], Aissani [2] and Artalejo [3], etc.

The concept of balking was first studied by Haight in 1957. In our model we assume that an arriving batch may join to the system for a service or leave the system. To the best of our knowledge very few works have been done with the concept of batch arrival retrial queueing system and balking. Such

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Peer review under responsibility of Ain Shams University.

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http://dx.doi.org/10.1016/j.asej.2014.02.003

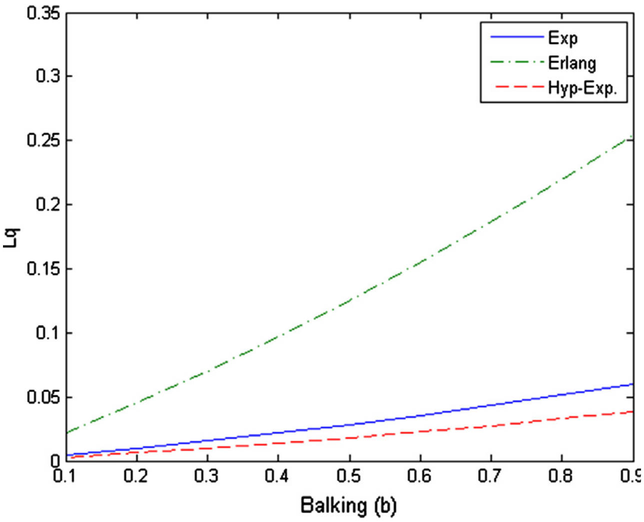


**Figure 1** $L_q$ versus $b$.

queueing models apply in many real world situations like web access, including call centers and computer systems, etc. Ke [4] studied the $M^{[x]}/G/1$ queue with variant vacations and balking. Some of the authors like Wang and Li [5,6] and Baruah et al. [7] discussed about the concept balking.

In our model, a single server provides two types of service, we assume that customers arrive in batch and they can choose any one type of service. In retrial queueing literature, the concept of feedback customers was discussed by many authors. Many queueing situations have the feature that the customers may be served repeatedly for a certain reason. After completion of service the customers have to join the queue and wait for their service once again. In this aspect, the concept of optional re-service can be considered as immediate feedback. The customer completes his service as first step and if he finds any defect in his service or wants service one more time, he will immediately get his service once again without joining the queue. Re-service have many real life application in situations like bank counters, working ATM machines, super markets, doctor clinics, etc. Authors like Madan and Baklizi [8], Madan et al. [9] and Baruah et al. [7] have discussed the concept of re-service.

In a vacation queueing system, the server may not be available for a period of time due to many reasons like, being checked for maintenance, working at other queues, scanning for new work (a typical aspect of many communication systems) or simply taking break. This period of time, when the server is unavailable for primary customers is referred as a vacation. Keilson and Servi [10] introduced the concept Bernoulli vacation, where the random decision whether to take a vacation or not are allowed only at instances when the system is not empty (and a service or vacation has just been completed). If the system is empty, the assumption for their model is that the server must take another vacation. Doshi [11] presented a survey on queueing systems with vacations. Krishnakumar and Arivudainambi [12] have investigated a single server retrial queue with Bernoulli schedule and general retrial times. Xu et al. [13] studied the concept of bulk input queue with working vacation. When the orbit becomes empty the server begins working vacation period during that period server gives service at lower rate or completely stops the service. Very recently Arivudainambi et al. [14] analyzed single server retrial queue with working vacation. Chang and Ke [15] examined a batch retrial model with $J$ vacations in which if orbit becomes empty, the server takes at most $J$ vacations repeatedly until at least one customer appears in the orbit upon returning from a vacation. By applying the supplementary variable technique, some important system characteristics are derived. This system has potential applications in packet-switched networks. Later, Ke and Chang [16] and Chen et al. [17] discussed a different $J$ vacation queueing models.

The service interruptions are unavoidable phenomenon in many real life situations. In most of the studies, it is assumed that the server is available in the service station on a permanent basis and service station never fails. However, these assumptions are practically unrealistic. In practice we often meet the case where service stations may fail and can be repaired. Applications of these models found in the area of computer communication networks and flexible manufacturing system, etc. Ke and Choudhury [18] discussed about the batch arrival retrial queueing system with two phases of service under the concept of breakdown and delaying repair. While the server is working with any phase of service, it may breakdown at any instance and the service channel will fail for a short interval of time. The repair process does not start immediately after a breakdown and there is a delay time for repair to start. Choudhury and Deka [19] considered a single server queue with two phases of service and the server is subject to breakdown while providing service to the customers. Further, Choudhury and Deka [20] developed the previous model with the concept of Bernoulli vacation. Chen et al. [17] studied a retrial queueing system with modified vacation and random breakdowns. Recently, Authors like Wang and Li [5,6], Choudhury et al. [21] and Rajadurai et al. [22] discussed about the retrial queueing systems with the concept of breakdown and repair.

In this paper, we have extended the study of Madan et al. [9] and Baruah et al. [7] by incorporating the concepts of retrial queues, almost $J$ vacations, breakdowns and delaying repair. The rest of this paper is organized as follows. In Section 2, the detailed description of the mathematical model and practical justification of the model are given. In Section 3, the system stability condition, the governing equations of our model and obtain the steady state solutions are obtained. Some performance measures are derived in Section 4. In Section 5 stochastic decomposition of the joint distribution of our model and important special cases are obtained. In Section 6 the effects of various parameters on the system performance are analyzed numerically. Summary of the work is presented in section 7.

## 2. Model descriptions

In this paper, we consider a batch arrival retrial queueing system with two types of service, balking under modified vacation policy and service interruption where the server provides each type consist of an optional re-service. The detailed description of the model is given as follows:

- ***Arrival process***: Customers arrive in batches according to a compound Poisson process with rate $\lambda$. Let $X_k$ denote the number of customers belonging to the $k$th

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arrival batch, where $X_k$, $k = 1, 2, 3, \ldots$ are with a common distribution $\Pr[X_k = n] = \chi_n$, $n = 1, 2, 3 \ldots$ and $X(z)$ denotes the probability generating function of $X$. We denote $X_{[k]}$ as the $k$th factorial moment of $X(z)$.

- ***Retrial process***: We assume that there is no waiting space and therefore if an arriving batch finds the server free, one of the customers from the batch begins his service and rest of them join into orbit. If an arriving batch of customers find the server being busy, vacation or breakdown, the arrivals either leave the service area with probability $1 - b$ or join pool of blocked customers called an orbit with probability $b$. Inter-retrial times have an arbitrary distribution $R(x)$ with corresponding Laplace-Stieltijes Transform (LST) $R^*(\theta)$.
- ***Service process***: There is a single server which provides two types of service and there is option for re-service. If an arriving customer finds the server free, then he may choose First Type of Service (FTS) with probability $p_1$ or may choose Second Type of Service (STS) with probability $p_2(p_1 + p_2 = 1)$. As soon as customer completes type $i$ service, he may repeat type $i$ service with probability $r_i$ $(i = 1, 2)$ or may leaves the system with probability $(1 - r_i)$. It is further assumed that either service may be repeated only once. The service times $S_i$ have general distribution function (d.f) $S_i(x)$ and Laplace- Stieltijes Transform $S_i^*(\theta)$ (for $i = 1, 2$) and finite $k$th moment $\beta_i^{(k)}$ $(i = 1, 2 \; and \; k = 1, 2)$.
- ***Vacation process***: Whenever the orbit is empty, the server leaves for a vacation of random length $V$. If no customer appears in the orbit when the server returns from a vacation, it leaves again for another vacation with the same length. Such pattern continues until it returns from a vacation to find at least one customer found in the orbit or it has already taken $J$ vacations. If the orbit is empty at the end of the $J$th vacation, the server remains idle for new arrivals in the system. At a vacation completion epoch the orbit is nonempty, the server waits for the customers in the orbit. The vacation time $V$ has distribution function $V(x)$ and LST $V^*(\theta)$ and finite $k$th moment $v^{(k)}(k = 1, 2)$.
- ***Breakdown process***: While the server is working with any types of service or re-service, it may breakdown at any time and the service channel will fail for a short interval of time i.e. server is down for a short interval of time. The breakdowns i.e. server's life times are generated by exogenous Poisson processes with rates $\alpha_1$ for FTS and $\alpha_2$ for STS, which we may call some sort of disaster during FTS and STS periods respectively.
- ***Repair process***: As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the arriving customer and waits for repair to start, which we may refer to as waiting period of the server. We define the waiting time as delay time. The delay time $D_i$ of the server for $i$th type of service follows with d.f. $D_i(y)$, LST $D_i^*(\theta)$ and finite $k$th moment $d_i^{(k)}$ $(i = 1, 2 \; and \; k = 1, 2)$. The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by $G_1$ for FTS and $G_2$ for STS) distributions of the server for both types of service are assumed to be arbitrarily distributed with d.f.$G_i(y)$, LST $G_i^*(\theta)$ and finite $k$th moment $g_i^{(k)}$ $(i = 1, 2 \; and \; k = 1, 2)$.
- Various stochastic processes involved in the system are assumed to be independent of each other.

### 2.1. Practical justifications of the suggested model

The proposed model can be applied in Wired Networks for selecting and maintaining routes in the routing table. Route selection is still one of the major challenging tasks in wired networks. In wired networks, all the routes will be maintained in a buffer alike, called Routing Table (orbit). The routing table can be either accessed for two reasons: route update (Type 1 service) and route usage (Type 2 service). In the first scenario, new routes would be added to the routing table, while the existing routes are used to proceed with the transmission. While updating the route, the re-service feature can check the routing table size to avoid the routing table overhead if required. When the size exceeds the limit, the server enters into the break down state. While using the existing route, the re-service feature can check whether the route is valid or not if required. If valid, it proceeds with the transmission. When not valid, the server enters into the break down state. When there are no routes in the routing table, the server moves to the sleep (vacation) state. The server system awakes up and checks the routing table status periodically. After a finishing these checking, the server moves itself from sleep to waiting state (idle) again to select a route for arriving routes.

The suggested model has also potential application in the transfer model of an email system. In Simple Mail Transfer Protocol (SMTP) mail system uses to deliver the messages between mail servers for relaying. When a mail transfer program contacts a server on a remote machine, it forms a Transmission Control Protocol (TCP) connection over which it communicates. Once the connection is in place, the two programs follow SMTP that allows the sender to identify it, specify a recipient and transfer an e-mail message. For receiving a group of messages, client applications usually use either the Post Office Protocol (POP) or the Internet Message Access Protocol (IMAP) to access their mail box accounts on a mail server. Typically, contacting a group of messages arrive at the mail server following the Poisson stream. When messages arrive at the mail server, it will be free then one of the messages from the group is selected to access (in POP or IMAP) and the rests will join to the buffer. In the buffer, each message waits and requires its service again after some time. The target server is the same as sender's mail server and the sending message will be possibly retransmitted to the server to request the receiving service immediately. The mail server may subject to electronic fails during service period and receive repair immediately. To keep the mail server functioning well, virus scan is an important maintenance activity for the mail server. It can be performed when the mail server is idle. This type of maintenance can be programmed to perform on a regular basis. However, these maintenance activities do not repeat continuously. When these activities are finished, mail server will enter the idle state again and wait for the contact messages to arrive. Because there is no mechanism to record how many contacting messages from various senders currently, it is appropriate for designing a program to collect information of contacting messages for the reason of efficiency. In this queueing scenario, the buffer in the sender mail server, the receiver mail server, the

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POP and IMAP, the retransmission policy and the maintenance activities correspond to the orbit, the server, the type1 and type2 service, the re-service discipline and the vacation policy respectively.

The results of this paper finds other applications in LAN, client-server communication, telephone systems, electronic mail services on internet, network and software designs of various computer communications systems, packet switched networks, production lines, In the operational model of WWW server for HTTP requests, call centers, inventory and production, maintenance and quality control in industrial organizations, etc.

## 3. System analysis

In this section, we develop the steady state difference-differential equations for the retrial system by treating the elapsed retrial times, the elapsed service times, the elapsed vacation times, the elapsed delay times and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states, the PGF for number of customers in the system and orbit.

In steady state, we assume that $R(0) = 0$, $R_i(\infty) = 1$, $S_i(0) = 0$, $S_i(\infty) = 1$, $V_j(0) = 0$, $V_j(\infty) = 1$ (for $i = 1, 2$ and $j = 1, 2 \ldots J$) are continuous at $x = 0$ and $D_i(0) = 0$, $D_i(\infty) = 1$, $R_i(0) = 0$, $R_i(\infty) = 1$ (for $i = 1, 2$) are continuous at $x = 0$ and $y = 0$. So that the function $\theta(x)$, $\mu_i(x)$, $\gamma(x)$, $\eta_i(y)$ and $\xi_i(y)$ are the conditional completion rates (hazard rates) for repeated attempts, service on both types, on vacation, under delaying repair on both types and repair on both types respectively (for $i = 1, 2$)

$$\theta(x)dx = \frac{dR(x)}{1 - R(x)}, \quad \mu_i(x)dx = \frac{dS_i(x)}{1 - S_i(x)},$$
$$\gamma(x)dx = \frac{dV(x)}{1 - V(x)}, \quad \eta_i(y)dy = \frac{dD_i(y)}{1 - D_i(y)},$$
$$\xi_i(y)dy = \frac{dG_i(y)}{1 - G_i(y)}.$$

In addition, let $R^0(t)$, $S_i^0(t)$, $V_j^0(t)$, $D_i^0(t)$, and $G_i^0(t)$ be the elapsed retrial times, service times, vacation times, delay times and repair times respectively at time $t$. Further, introduce the random variables,

$$C(t) = \begin{cases} 0, \text{ if the server is idle at time } t. \\ 1, \text{ if the server is busy on both types of service at time } t, \\ 2, \text{ if the server is on re-service of both types at time } t, \\ 3, \text{ if the server is on delaying repair of both types at time } t, \\ 4, \text{ if the server is on repair of both types at time } t, \\ 5, \text{ if the server is on vacation with the 1st vacation at time} t, \\ . \\ j+4, \text{ if the server is on vacation with the } j\text{th vacation at time } t, \\ . \\ J+4, \text{ if the server is on vacation with the } J\text{th vacation at time } t. \end{cases}$$

We also note that the state of the system at time $t$ can be described by the bivariate Markov process $\{C(t), N(t); t \geqslant 0\}$ where $C(t)$ denotes the server state $(0, 1, 2, 3, 4, 5 \ldots J + 4)$ depending on the server is idle, busy on both types of service, re-service on both types, delaying repair on both types, repair on both types, 1st vacation,... and $J$th vacation. $N(t)$ denotes the number of customers in the orbit.

If $C(t) = 0$ and $N(t) > 0$, then $R^0(t)$ represent the elapsed retrial time. If $C(t) = 1$ and $N(t) \geqslant 0$ then $S_1^0(t)$ and $S_2^0(t)$ corresponding to the elapsed time of the customer being served on both types. If $C(t) = 2$ and $N(t) \geqslant 0$ then $S_1^0(t)$ and $S_2^0(t)$ corresponding to the elapsed time of the customer being re-served on both types. If $C(t) = 3$ and $N(t) \geqslant 0$, then $D_1^0(t)$ and $D_2^0(t)$ corresponding to the elapsed time of the server being delayed repair on both types. If $C(t) = 4$ and $N(t) \geqslant 0$, then $G_1^0(t)$ and $G_2^0(t)$ corresponding to the elapsed time of the server being repaired on both types. If $C(t) = 5$ and $N(t) \geqslant 0$, then $V_1^0(t)$ corresponding to the elapsed $1^{st}$ vacation time. If $C(t) = j + 4$ and $N(t) \geqslant 0$, then $V_j^0(t)$ corresponding to the elapsed $j$th vacation time.

Let $\{t_n; n = 1, 2, \ldots\}$ be the sequence of epochs at which either a type 1 (or type 2) service period completion occurs, a vacation period ends or a repair period ends. The sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. It follows from Appendix A that $\{Z_n; n \in N\}$ is ergodic if and only if $\rho < 1$, then the system will be stable, where $\rho = X_{[1]}(1 - R^*(\lambda)) + \varpi$ and

$$\begin{aligned} \varpi = \lambda b X_{[1]} \Big\{ & p_1 \beta_1^{(1)} \Big[1 + \alpha_1 \Big(d_1^{(1)} + g_1^{(1)}\Big)\Big] \\ & + p_2 \beta_2^{(1)} \Big[1 + \alpha_2 \Big(d_2^{(1)} + g_2^{(1)}\Big)\Big] \\ & + r_1 p_1 \beta_1^{(1)} \Big[1 + \alpha_1 \Big(d_1^{(1)} + g_1^{(1)}\Big)\Big] \\ & + r_2 p_2 \beta_2^{(1)} \Big[1 + \alpha_2 \Big(d_2^{(1)} + g_2^{(1)}\Big)\Big] \Big\} \end{aligned}$$

For the process $\{N(t), t \geqslant 0\}$, we define the probabilities $P_0(t) = P\{C(t) = 0, N(t) = 0\}$ and the probability densities

$$\psi_n(x,t)dx = P\{C(t) = 0, N(t) = n, \ x \leqslant R^0(t) < x + dx\}, \quad \text{for } t \geqslant 0, \quad x \geqslant 0 \quad \text{and} \quad n \geqslant 1,$$

$$P_{i,n}(x,t)dx = P\{C(t) = 1, N(t) = n, \ x \leqslant S_i^0(t) < x + dx\}, \quad \text{for } t \geqslant 0, \quad x \geqslant 0, \quad n \geqslant 0 \quad \text{and} \quad (i = 1, 2),$$

$$\Pi_{i,n}(x,t)\mathrm{d}x = P\{C(t) = 2, \ N(t) = n, \ x \leqslant S_i^0(t) < x + \mathrm{d}x\}, \quad \text{for } t \geqslant 0, \quad x \geqslant 0, \quad n \geqslant 0 \quad \text{and} \quad (i = 1, 2),$$

$$Q_{i,n}(x,y,t)dy = P\{C(t) = 3, \ N(t) = n, \ y \leqslant D_i^0(t) < y + dy / S_i^0(t) = x\} \quad \text{for } t \geqslant 0, \quad (x,y) \geqslant 0, \quad n \geqslant 0 \quad \text{and} \quad (i = 1, 2),$$

$$R_{i,n}(x,y,t)dy = P\big\{C(t) = 4, \ N(t) = n, \ y \leqslant G_i^0(t) < y + dy / S_i^0(t) = x\big\}, \quad \text{for } t \geqslant 0, \quad (x,y) \geqslant 0, \quad n \geqslant 0 \quad \text{and} \quad (i = 1, 2),$$

$$\Omega_{j,n}(x,t)dx = P\Big\{C(t) = j + 4, \ N(t) = n, \ x \leqslant V_j^0(t) < x + dx\Big\}, \quad \text{for } t \geqslant 0, \ (x,y) \geqslant 0 \ \text{and} \ n \geqslant 0, \ (1 \leqslant j \leqslant J).$$

The following probabilities are used in sequent sections:

$P_0(t)$ is the probability that the system is empty at time $t$.

$\psi_n(x,t)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed retrial time of the test customer undergoing retrial is $x$.

$P_{i,n}(x,t)$, ($i = 1, 2$) is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed service time of the test customer undergoing service is $x$ in their respective types.

$\Pi_{i,n}(x,t)$, $(i = 1, 2)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed re-service time of the test customer undergoing re-service is $x$ in their respective types.

$\Omega_{j,n}(x,t)$, ($j = 1, 2, \ldots J$) is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed vacation time is $x$.

$Q_{i,n}(x,y,t)$ ($i = 1, 2$) is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed service time of the test customer undergoing service is $x$ and the elapsed delaying repair time of server is $y$ in their respective types.

$R_{i,n}(x,y,t)$ ($i = 1, 2$) is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed service time of the test customer undergoing service is $x$ and the elapsed repair time of server is $y$ in their respective types.

We assume that the stability condition is fulfilled in the sequel and so that we can set limiting probabilities for $x > 0$ and $n \geqslant 0$ and ($i = 1, 2$)

$$P_0 = \lim_{t\to\infty} P_0(t),\ \psi_n(x) = \lim_{t\to\infty}\psi_n(x,t) \quad \text{for} \quad n \geqslant 1, \quad P_{i,n}(x) = \lim_{t\to\infty} P_{i,n}(x,t), \quad \Pi_{i,n}(x) = \lim_{t\to\infty}\Pi_{i,n}(x,t),$$

$$Q_{i,n}(x,y) = \lim_{t\to\infty} Q_{i,n}(x,y,t),\ R_{i,n}(x,y) = \lim_{t\to\infty} R_{i,n}(x,y,t), \quad \Omega_{j,n}(x) = \lim_{t\to\infty}\Omega_{j,n}(x,t), \quad \text{for } (1 \leqslant j \leqslant J).$$

### 3.1. The steady state equations

By the method of supplementary variable technique [23], we obtain the following system of equations that govern the dynamics of the system behavior.

$$\lambda b P_0 = \int_0^\infty \Omega_{J,0}(x)\gamma(x)\,dx \tag{3.1}$$

$$\frac{d\psi_n(x)}{dx} + [\lambda + \theta(x)]\psi_n(x) = 0, \quad n \geqslant 1 \tag{3.2}$$

$$\frac{dP_{i,0}(x)}{dx} + [\lambda + \alpha_i + \mu_i(x)]P_{i,0}(x) = \lambda(1-b)P_{i,0}(x) + \int_0^\infty \xi_i(y)R_{i,0}(x,y)\,dy, \quad n = 0, \quad \text{for} \quad (i = 1, 2) \tag{3.3}$$

$$\frac{dP_{i,n}(x)}{dx} + [\lambda + \alpha_i + \mu_i(x)]P_{i,n}(x) = \lambda(1-b)P_{i,n}(x) + \lambda b\sum_{k=1}^{n}\chi_k P_{i,n-k}(x) + \int_0^\infty \xi_i(y)R_{i,n}(x,y)\,dy, \quad n \geqslant 1, \quad \text{for} \quad (i = 1, 2) \tag{3.4}$$

$$\frac{d\Pi_{i,0}(x)}{\mathrm{dx}} + [\lambda + \alpha_i + \mu_i(x)]\Pi_{i,0}(x) = \lambda(1-b)\Pi_{i,0}(x) + \int_0^\infty \xi_i(y)R_{i,0}(x,y)\,\mathrm{dy}, \quad n = 0, \quad \text{for} \quad (i = 1, 2) \tag{3.5}$$

$$\frac{d\Pi_{i,n}(x)}{\mathrm{dx}} + [\lambda + \alpha_i + \mu_i(x)]\Pi_{i,n}(x) = \lambda(1-b)\Pi_{i,n}(x) + \lambda b\sum_{k=1}^{n}\chi_k\Pi_{i,n-k}(x) + \int_0^\infty \xi_i(y)R_{i,n}(x,y)\,\mathrm{dy}, \quad n \geqslant 1, \quad \text{for} \quad (i = 1, 2) \tag{3.6}$$

$$\frac{d\Omega_{j,0}(x)}{dx} + [\lambda + \gamma(x)]\Omega_{j,0}(x) = \lambda(1-b)\Omega_{j,0}(x), \quad n = 0, \quad \text{for} \quad (j = 1, 2, \ldots J) \tag{3.7}$$

$$\frac{d\Omega_{j,n}(x)}{dx} + [\lambda + \gamma(x)]\Omega_{j,n}(x) = \lambda(1-b)\Omega_{j,n}(x) + \lambda b\sum_{k=1}^{n}\chi_k\Omega_{j,n-k}(x), \quad n \geqslant 1, \quad \text{for} \quad (j = 1, 2, \ldots J) \tag{3.8}$$

$$\frac{dQ_{i,0}(x,y)}{dy} + [\lambda + \eta_i(y)]Q_{i,0}(x,y) = \lambda(1-b)Q_{i,0}(x,y), \quad n = 0, \quad \text{for} \quad (i = 1, 2) \tag{3.9}$$

$$\frac{dQ_{i,n}(x,y)}{dy} + [\lambda + \eta_i(y)]Q_{i,n}(x,y) = \lambda(1-b)Q_{i,n}(x,y), + \lambda b\sum_{k=1}^{n}\chi_k Q_{i,n-k}(x,y), \quad n \geqslant 1, \quad \text{for} \quad (i = 1, 2) \tag{3.10}$$

$$\frac{dR_{i,0}(x,y)}{dy} + [\lambda + \xi_i(y)]R_{i,0}(x,y) = \lambda(1-b)R_{i,0}(x,y), \quad n = 0, \quad \text{for} \quad (i = 1, 2) \tag{3.11}$$

$$\frac{dR_{i,n}(x,y)}{dy} + [\lambda + \xi_i(y)]R_{i,n}(x,y) = \lambda(1-b)R_{i,n}(x,y), + \lambda b\sum_{k=1}^{n}\chi_k R_{i,n-k}(x,y), \quad n \geqslant 1, \quad \text{for} \quad (i = 1, 2) \tag{3.12}$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$\psi_n(0) = \sum_{j=1}^{J}\int_0^\infty \Omega_{j,n}(x)\gamma(x)\,dx + (1-r_1)\int_0^\infty P_{1,n}(x)\mu_1(x)\,dx + (1-r_2)\int_0^\infty P_{2,n}(x)\mu_2(x)\,dx + \int_0^\infty \Pi_{1,n}(x)\mu_1(x)\,dx + \int_0^\infty \Pi_{2,n}(x)\mu_2(x)\,dx, \quad n \geqslant 1 \tag{3.13}$$

$$P_{i,0}(0) = p_i\left(\int_0^\infty \psi_1(x)\theta(x)\,dx + \lambda b\chi_1 P_0\right), \quad n = 0, \quad \text{for} \quad (i = 1, 2) \tag{3.14}$$

$$P_{i,n}(0) = p_i\left[\int_0^\infty \psi_{n+1}(x)\theta(x)\,dx + \lambda\sum_{k=1}^{n}\chi_k\int_0^\infty \psi_{n-k+1}(x)\,dx + \lambda b\chi_{n+1}P_0\right], \quad n \geqslant 1, \quad \text{for} \quad (i = 1, 2) \tag{3.15}$$

$$\prod_{i,n}(0)=r_i\left[\int_0^\infty P_{i,n}(x)\mu_i(x)\,dx\right],\quad n\geqslant 1,\quad \text{for}\quad (i=1,2) \tag{3.16}$$

$$\Omega_{1,n}(0)=\begin{cases}(1-r_1)\int_0^\infty P_{1,0}(x)\mu_1(x)\,dx\\ +(1-r_2)\int_0^\infty P_{2,0}(x)\mu_2(x)\,dx\\ +\int_0^\infty \Pi_{1,0}(x)\mu_1(x)\,dx+\int_0^\infty \Pi_{2,0}(x)\mu_2(x)\,dx, & n=0\\ 0, & n\geqslant 1\end{cases} \tag{3.17}$$

$$\Omega_{j,n}(0)=\begin{cases}\int_0^\infty \Omega_{j-1,0}(x)\gamma(x)\,dx, & n=0,\quad j=2,3\ldots J\\ 0, & n\geqslant 1,\quad j=2,3\ldots.J\end{cases} \tag{3.18}$$

$$Q_{i,n}(x,0)=\alpha_i(P_{i,n}(x)+\Pi_{i,n}(x)),\quad n\geqslant 1,\quad \text{for}\quad (i=1,2) \tag{3.19}$$

$$R_{i,n}(x,0)=\int_0^\infty Q_{i,n}(x,y)\eta_i(y)\,dy,\quad n\geqslant 1,\quad \text{for}\quad (i=1,2) \tag{3.20}$$

The normalizing condition is

$$P_0+\sum_{n=1}^\infty\int_0^\infty \psi_n(x)\,dx+\sum_{n=0}^\infty\left[\sum_{i=1}^2\left(\int_0^\infty P_{i,n}(x)\,dx\right.\right.$$
$$+\int_0^\infty \Pi_{i,n}(x)\,dx+\int_0^\infty\int_0^\infty Q_{i,n}(x,y)\,dx\,dy$$
$$\left.\left.+\int_0^\infty\int_0^\infty R_{i,n}(x,y)\,dx\,dy\right)\right]+\sum_{j=1}^J\sum_{n=0}^\infty \Omega_{j,n}(x)\,dx=1 \tag{3.21}$$

### 3.2. The steady state solution

The probability generating function technique is used here to obtain the steady state solution of the retrial queueing model. To solve the above equations, we define the generating functions for $|z|\leqslant 1$, for ($i$ = 1, 2 and $j$ = 1, 2, … $J$) as follows:

$$\psi(x,z)=\sum_{n=1}^\infty \psi_n(x)z^n;\quad \psi(0,z)=\sum_{n=1}^\infty \psi_n(0)z^n;$$

$$P_i(x,z)=\sum_{n=0}^\infty P_{i,n}(x)z^n;\quad P_i(0,z)=\sum_{n=0}^\infty P_{i,n}(0)z^n;\quad \Pi_i(x,z)=\sum_{n=0}^\infty \Pi_{i,n}(x)z^n;$$

$$\Pi_i(0,z)=\sum_{n=0}^\infty \Pi_{i,n}(0)z^n;\quad \Omega_j(x,z)=\sum_{n=0}^\infty \Omega_{j,n}(x)z^n;\quad \Omega_j(0,z)$$
$$=\sum_{n=0}^\infty \Omega_{j,n}(0)z^n;\quad Q_i(x,y,z)=\sum_{n=0}^\infty Q_{i,n}(x,y)z^n;$$

$$Q_i(x,0,z)=\sum_{n=0}^\infty Q_{i,n}(x,0)z^n;\quad R_i(x,y,z)=\sum_{n=0}^\infty R_{i,n}(x,y)z^n;\quad R_i(x,0,z)$$
$$=\sum_{n=0}^\infty R_{i,n}(x,0)z^n\quad \text{and}\quad X(z)=\sum_{n=1}^\infty \chi_n z^n$$

Now multiplying the steady state equation and steady state boundary condition (3.1)–(3.20) by $z^n$ and summing over $n$, ($n$ = 0, 1, 2 …) for ($i$ = 1, 2 and $j$ = 1, 2 … $J$)

$$\frac{\partial\psi(x,z)}{\partial x}+[\lambda+\theta(x)]\psi(x,z)=0 \tag{3.22}$$

$$\frac{\partial P_i(x,z)}{\partial x}+[\lambda b(1-X(z))+\alpha_i+\mu_i(x)]P_i(x,z)$$
$$=\int_0^\infty \xi_i(y)R_i(x,y,z)\,dy,\quad \text{for}\quad (i=1,2) \tag{3.23}$$

$$\frac{\partial \Pi_i(x,z)}{\partial x}+[\lambda b(1-X(z))+\alpha_i+\mu_i(x)]\Pi_i(x,z)$$
$$=\int_0^\infty \xi_i(y)R_i(x,y,z)\,dy,\quad \text{for}\quad (i=1,2) \tag{3.24}$$

$$\frac{\partial \Omega_j(x,z)}{\partial x}+[\lambda b(1-X(z))+\gamma(x)]\Omega_j(x,z)=0 \tag{3.25}$$

$$\frac{\partial Q_i(x,y,z)}{\partial y}+[\lambda b(1-X(z))+\eta_i(y)]Q_i(x,y,z)=0 \tag{3.26}$$

$$\frac{\partial R_i(x,y,z)}{\partial y}+[\lambda b(1-X(z))+\xi_i(y)]R_i(x,y,z)=0 \tag{3.27}$$

$$\psi(0,z)=\sum_{j=1}^J\int_0^\infty \Omega_j(x,z)\gamma(x)\,dx+(1-r_1)\int_0^\infty P_1(x,z)\mu_1(x)\,dx$$
$$+(1-r_2)\int_0^\infty P_2(x,z)\mu_2(x)\,dx+\int_0^\infty \Pi_1(x,z)\mu_1(x)\,dx$$
$$+\int_0^\infty \Pi_2(x,z)\mu_2(x)\,dx-\sum_{j=1}^J\Omega_{j,0}(0)-\lambda bP_0 \tag{3.28}$$

$$P_i(0,z)=p_i\left[\frac{1}{z}\int_0^\infty \psi(x,z)\theta(x)\,dx+\frac{\lambda X(z)}{z}\int_0^\infty \psi(x,z)\,dx\right.$$
$$\left.+\frac{\lambda bX(z)}{z}P_0\right] \tag{3.29}$$

$$\prod_i(0,z)=r_i\int_0^\infty P_i(x,z)\mu_i(x)\,dx \tag{3.30}$$

$$Q_i(x,0,z)=\alpha_i(P_i(x,z)+\Pi_i(x,z)) \tag{3.31}$$

$$R_i(x,0,z)=\int_0^\infty Q_i(x,y,z)\eta_i(y)\,dy \tag{3.32}$$

Solving the partial differential Eqs. (3.1)–(3.12), it follows that

$$\psi(x,z)=\psi(0,z)[1-R(x)]e^{-\lambda x} \tag{3.33}$$
$$P_i(x,z)=P_i(0,z)[1-S_i(x)]e^{-A_i(z)x},\quad \text{for}\quad (i=1,2) \tag{3.34}$$
$$\Pi_i(x,z)=\Pi_i(0,z)[1-S_i(x)]e^{-A_i(z)x},\quad \text{for}\quad (i=1,2) \tag{3.35}$$
$$\Omega_j(x,z)=\Omega_j(0,z)[1-V(x)]e^{-b(z)x},\quad \text{for}\quad (1\leqslant j\leqslant J) \tag{3.36}$$
$$Q_i(x,y,z)=Q_i(x,0,z)[1-D_i(y)]e^{-b(z)y},\quad \text{for}\quad (i=1,2) \tag{3.37}$$
$$R_i(x,y,z)=R_i(x,0,z)[1-G_i(y)]e^{-b(z)y},\quad \text{for}\quad (i=1,2) \tag{3.38}$$

where $A_i(z)=b(z)+\alpha_i[1-D_i^*(b(z))G_i^*(b(z))]$ and $b(z)=\lambda b(1-X(z))$

From (3.7) we obtain,

$$\Omega_{j,0}(x)=\Omega_{j,0}(0)[1-V(x)]e^{-\lambda bx},\quad j=1,2,\ldots J \tag{3.39}$$

Multiplying with Eq. (3.39) by $\gamma(x)$ on both sides for $j$ = $J$ and integrating with respect to $x$ from 0 to $\infty$, then from (3.1) we have,

$$\Omega_{J,0}(0)=\frac{\lambda bP_0}{V^*(\lambda b)} \tag{3.40}$$

From Eq. (3.40) and solving (3.18) and (3.39) over the range $j$ = $J$ – 1,$J$ – 2, … 1, we get on simplification

$$\Omega_{j,0}(0)=\frac{\lambda bP_0}{[V^*(\lambda b)]^{J-j+1}},\quad j=1,2\ldots J-1 \tag{3.41}$$

From (3.18), (3.40) and (3.41), we get

$$\Omega_j(0,z) = \frac{\lambda b P_0}{[V^*(\lambda b)]^{J-j+1}}, \quad j = 1,2\ldots J \tag{3.42}$$

Integrating Eq. (3.39) from 0 to $\infty$ and using (3.40) and (3.41) again, we finally obtain

$$\Omega_{j,0}(0,z) = \frac{P_0(1-V^*(\lambda b))}{[V^*(\lambda b)]^{J-j+1}}, \quad j = 1,2\ldots J \tag{3.43}$$

Note that, $\Omega_{j,0}$ represents the steady-state probability that no customers appear while the server is on the $j$th vacation. Let us define $\Omega_0$ as the probability that no customers appear in the system while the server is on vacation. Then,

$$\Omega_0 = \frac{P_0(1-[V^*(\lambda)]^J)}{b[V^*(\lambda)]^J}, \tag{3.44}$$

Inserting Eqs. (3.34)–(3.36) and (3.42) in (3.43)

$$\begin{aligned}\psi(0,z) = {} & \lambda b P_0(N(z)-1) + (1-r_1)P_1(0,z)S_1^*(A_1(z)) \\ & + (1-r_2)P_2(0,z)S_2^*(A_2(z)) \\ & + \Pi_1(0,z)S_1^*(A_1(z)) + \Pi_2(0,z)S_2^*(A_2(z))\end{aligned} \tag{3.45}$$

where $N(z) = \frac{1-[V^*(\lambda b)]^J}{[V^*(\lambda b)]^J(1-V^*(\lambda b))}[V^*(\lambda b(1-X(z)))-1]$

Inserting Eqs. (3.33) and (3.34) in (3.29) and make some manipulation, finally we get,

$$P_1(0,z) = p_1\left\{\left(\frac{[R^*(\lambda)+X(z)(1-R^*(\lambda))]}{z}\right)\psi(0,z) + \frac{\lambda b X(z)}{z}P_0\right\}, \tag{3.46}$$

$$P_2(0,z) = p_2\left\{\left(\frac{[R^*(\lambda)+X(z)(1-R^*(\lambda))]}{z}\right)\psi(0,z) + \frac{\lambda b X(z)}{z}P_0\right\}, \tag{3.47}$$

Inserting Eqs. (3.46) and (3.35) in (3.30) and make some manipulation, finally we get,

$$\Pi_1(0,z) = r_1 P_1(0,z)S_1^*(A_1(z)) \tag{3.48}$$

Using (3.47), (3.35) and (3.30) we get

$$\Pi_2(0,z) = r_2 P_2(0,z)S_2^*(A_2(z)) \tag{3.49}$$

Using (3.46)–(3.49) in (3.45), we get

$$\psi(0,z) = \frac{Nr(z)}{Dr(z)} \tag{3.50}$$

$$\begin{aligned}Nr(z) = {} & \lambda b P_0\big[z(N(z)-1) + X(z)\big((1-r_1)p_1S_1^*(A_1(z)) \\ & +(1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\big[S_1^*(A_1(z))\big]^2 \\ & +r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\big]\end{aligned}$$

$$\begin{aligned}Dr(z) = {} & \{z - (R^*(\lambda)+X(z)(1-R^*(\lambda)))\big((1-r_1)p_1S_1^*(A_1(z)) \\ & +(1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\big[S_1^*(A_1(z))\big]^2 \\ & +r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\end{aligned}$$

Using Eqs. (3.50) and (3.46), we get,

$$P_1(0,z) = \lambda b P_0 p_1[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))+X(z)]/Dr(z), \tag{3.51}$$

Using Eqs. (3.50) and (3.47), we get,

$$P_2(0,z) = \lambda b P_0 p_2[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))+X(z)]/Dr(z), \tag{3.52}$$

Using Eqs. (3.48) and (3.51), we get,

$$\begin{aligned}\Pi_1(0,z) = {} & r_1\lambda b P_0 p_1[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda))) \\ & +X(z)]S_1^*(A_1(z))/Dr(z),\end{aligned} \tag{3.53}$$

Using Eqs. (3.49) and (3.52), we get,

$$\begin{aligned}\Pi_2(0,z) = {} & r_2\lambda b P_0 p_2[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda))) \\ & +X(z)]S_2^*(A_2(z))/Dr(z),\end{aligned} \tag{3.54}$$

Using Eqs. (3.51) and (3.53) in (3.31), we get,

$$Q_1(x,0,z) = \alpha_1\big(P_1(0,z)[1-S_1(x)]e^{-A_1(z)x} + \Pi_1(0,z)[1-S_1(x)]e^{-A_1(z)x}\big) \tag{3.55}$$

Using Eqs. (3.52) and (3.54) in (3.31), we get,

$$Q_2(x,0,z) = \alpha_2\big(P_2(0,z)[1-S_2(x)]e^{-A_2(z)x} + \Pi_2(0,z)[1-S_2(x)]e^{-A_2(z)x}\big), \tag{3.56}$$

Using Eqs. (3.37) and (3.32), we get,

$$R_1(x,0,z) = Q_1(x,0,z)D_1^*(b(z)) \tag{3.57}$$

$$R_2(x,0,z) = Q_2(x,0,z)D_2^*(b(z)) \tag{3.58}$$

Using (3.50)–(3.58) in (3.33)–(3.38), then the limiting probability generating functions $\psi(x,z)$, $P_1(x,z)$, $P_2(x,z)$, $\Pi_1(x,z)$, $\Pi_2(x,z)$, $\Omega_j(x,z)$, $Q_1(x,y,z)$, $Q_2(x,y,z)$, $R_1(x,y,z)$ and $R_2(x,y,z)$. We summarize the above results in the following Theorem 3.1.

**Theorem 3.1.** *Under the stability condition $\rho < 1$, the stationary distributions of the number of customers in the orbit and the server's state has the following PGF's (for $i = 1, 2$ and $1 \leqslant j \leqslant J$)*

$$\psi(x,z) = \frac{Nr(z)}{Dr(z)} \tag{3.59}$$

$$\begin{aligned}Nr(z) = {} & \lambda b P_0\big(z(N(z)-1) + X(z)\big((1-r_1)p_1S_1^*(A_1(z)) \\ & +(1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\big[S_1^*(A_1(z))\big]^2 \\ & +r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\big)(1-R(x))e^{-\lambda x}\end{aligned}$$

$$\begin{aligned}Dr(z) = {} & \{z - (R^*(\lambda)+X(z)(1-R^*(\lambda)))\big((1-r_1)p_1S_1^*(A_1(z)) \\ & +(1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\big[S_1^*(A_1(z))\big]^2 + r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\end{aligned}$$

$$\begin{aligned}P_i(x,z) = {} & \lambda b P_0 p_i[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda))) \\ & +X(z)](1-S_i(x))e^{-A_i(z)x}/Dr(z), \quad \text{for } (i=1,2)\end{aligned} \tag{3.60}$$

$$\begin{aligned}\Pi_i(x,z) = {} & \{r_i\lambda b P_0 p_i[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda))) \\ & +X(z)]\big(S_i^*(A_1(z))\big)(1-S_i(x))e^{-A_i(z)x}\}/Dr(z), \ \text{for } (i=1,2)\end{aligned} \tag{3.61}$$

$$\begin{aligned}Q_i(x,y,z) = {} & \{\lambda b P_0 p_i\alpha_i\{(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda))) \\ & +X(z)\}\big(1+r_i\big(S_i^*(A_1(z))\big)\big)(1-S_i(x))e^{-A_i(z)x} \\ & \times(1-D_i(y))e^{-b(z)y}\}/Dr(z), \quad \text{for } (i=1,2)\end{aligned} \tag{3.62}$$

$$\begin{aligned}R_i(x,y,z) = {} & \{\lambda b P_0 p_i\alpha_i\{(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda))) \\ & +X(z)\}\big(1+r_i\big(S_i^*(A_1(z))\big)\big) \ D_i^*(b(z))(1-S_i(x))e^{-A_i(z)x} \\ & \times(1-G_i(y))e^{-b(z)y}\}/Dr(z), \text{for} (i=1,2)\end{aligned} \tag{3.63}$$

$$\Omega_j(x,z) = \frac{\lambda b P_0(1-V(x))e^{-b(z)x}}{[V^*(\lambda b)]^{J-j+1}}, \quad j = 1,2,\ldots J \tag{3.64}$$

*where* $A_i(z) = b(z) + \alpha_i\big[1 - D_i^*(b(z))G_i^*(b(z))\big]$ and $b(z) = \lambda b[1-X(z)]$.

*Next we are interested in investigating the marginal orbit size distributions due to system state of the server.*

Please cite this article in press as: Rajadurai P et al., Analysis of an $M^{[X]}/(G_1,G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption, Ain Shams Eng J (2014), http://dx.doi.org/10.1016/j.asej.2014.02.003

**Theorem 3.2.** *Under the stability condition $\rho < 1$, the stationary distributions of the number of customers in the system when server being idle, busy on both types, re-service on both types, on vacation, under delaying repair on both types and under repair on both types are given by*

$$\psi(z) = \frac{Nr(z)}{Dr(z)} \tag{3.65}$$

$$\begin{aligned}Nr(z) = {} & bP_0(1-R^*(\lambda))\big(z(N(z)-1)+X(z)\big((1-r_1)p_1S_1^*(A_1(z))\\ & +(1-r_2)p_2S_2^*(A_2(z))+r_1p_1\big[S_1^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\big)\end{aligned}$$

$$\begin{aligned}Dr(z) = {} & \{z-(R^*(\lambda)+X(z)(1-R^*(\lambda)))\big((1-r_1)p_1S_1^*(A_1(z))\\ & +(1-r_2)p_2S_2^*(A_2(z))+r_1p_1\big[S_1^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\end{aligned}$$

$$\begin{aligned}P_i(z) = {} & \{\lambda bP_0p_i[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))\\ & +X(z)]\big(1-S_i^*[A_i(z)]\big)\}/A_i(z)Dr(z), \quad \text{for } (i=1,2)\end{aligned} \tag{3.66}$$

$$\begin{aligned}\Pi_i(z) = {} & \{r_i\lambda bP_0p_i[(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))\\ & +X(z)]\big(S_i^*(A_i(z))\big)\big(1-S_i^*(A_i(z))\big)\}/A_i(z)Dr(z), \text{ for } (i=1,2)\end{aligned} \tag{3.67}$$

$$\begin{aligned}Q_i(z) = {} & \{\lambda bP_0p_i\alpha_i\{(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))\\ & +X(z)\}\big(1+r_i\big(S_i^*(A_i(z))\big)\big)\big(1-S_i^*(A_i(z))\big)\\ & \big(1-D_i^*(b(z))\big)\}/A_i(z)b(z)Dr(z), \quad \text{for } (i=1,2)\end{aligned} \tag{3.68}$$

$$\begin{aligned}R_i(z) = {} & \{\lambda bP_0p_i\alpha_i\{(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))\\ & +X(z)\}D_i^*(b(z))\big(1+r_i\big(S_i^*(A_i(z))\big)\big)\big(1-S_i^*(A_i(z))\big)\\ & \big(1-G_i^*(b(z))\big)\}/A_i(z)b(z)Dr(z), \quad \text{for } (i=1,2)\end{aligned} \tag{3.69}$$

$$\Omega_j(z) = \frac{P_0([V^*(\lambda b(1-X(z)))]-1)}{(X(z)-1)[V^*(\lambda b)]^{J-j+1}}, \; j=1,2,\ldots J \tag{3.70}$$

where

$$P_0 = \left\{\frac{(1-X_{[1]}[1-R^*(\lambda)]-\varpi)}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1}\right\} \tag{3.71}$$

$$N'(1) = \frac{\{1-[V^*(\lambda b)]^J\}\lambda bX_{[1]}v^{(1)}}{[V^*(\lambda b)]^J(1-V^*(\lambda b))},$$

$A_i(z) = b(z)+\alpha_i\big[1-D_i^*(b(z))G_i^*(b(z))\big]$ and $b(z) = \lambda b[1-X(z)]$.

**Proof.** Integrating the above (3.59)–(3.61) and (3.64) equations with respect to $x$ and define the partial probability generating functions as, for ($i$ = 1, 2 and $1 \leqslant j \leqslant J$)

$$\psi(z) = \int_0^\infty \psi(x,z)\,dx, \;\; P_i(z) = \int_0^\infty P_i(x,z)\,dx, \;\; \Pi_i(z) = \int_0^\infty \Pi_i(x,z)\,dx, \;\; \Omega_j(z) = \int_0^\infty \Omega_j(x,z)\,dx.$$

Integrating the above (3.62) and (3.63) equations with respect to $x$ and $y$ define the partial probability generating functions as, for ($i$ = 1, 2)

$Q_i(x,z) = \int_0^\infty Q_i(x,y,z)\,dy, \;\; Q_i(z) = \int_0^\infty Q_i(x,z)\,dx, \;\; R_i(x,z) = \int_0^\infty R_i(x,y,z)\,dy, \;\; R_i(z) = \int_0^\infty R_i(x,z)\,dx$. Since, the only unknown is $P_0$ the probability that the server is idle when no customer in the orbit and it can be determined using the normalizing condition ($i$ = 1, 2 and $1 \leqslant j \leqslant J$). Thus, by setting $z$ = 1 in (3.65)–(3.69) and (3.70) and applying L-Hospitals rule whenever necessary and we get

$$P_0+\psi(1)+P_1(1)+P_2(1)+\Pi_1(1)+\Pi_2(1)+Q_1(1)+Q_2(1) + R_1(1)+R_2(1)+\sum_{j=1}^{J}\Omega_j(1) = 1. \quad \square$$

**Theorem 3.3.** *Under the stability condition $\rho < 1$, probability generating function of number of customers in the system and orbit size distribution at stationary point of time is*

$$K(z) = \frac{Nr_s(z)}{Dr(z)} \tag{3.72}$$

$$\begin{aligned}Nr(z) = {} & P_0\big(z\{1-\big((1-r_1)p_1S_1^*(A_1(z))+(1-r_2)p_2S_2^*(A_2(z))\\ & +r_1p_1\big[S_1^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\\ & \times\{(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))+X(z)\}\\ & -N(z)\{z-(R^*(\lambda)+X(z)(1-R^*(\lambda)))\\ & \big((1-r_2)p_1S_1^*(A_1(z))+(1-r_2)p_2S_2^*(A_2(z))\\ & +r_1p_1\big[S_2^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\\ & +[1-X(z)]\{z(b(1-R^*(\lambda))(N(z)-1)+1)\\ & -(R^*(\lambda)+X(z)(1-b)(1-R^*(\lambda)))\\ & \big((1-r_1)p_1S_1^*(A_1(z))+(1-r_2)p_2S_2^*(A_2(z))\\ & +r_1p_1\big[S_1^*(A_2(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\big)\end{aligned}$$

$$\begin{aligned}Dr(z) = {} & [1-X(z)]\big\{z-(R^*(\lambda)+X(z)(1-R^*(\lambda)))\big((1-r_1)p_1S_1^*(A_1(z))\\ & +(1-r_2)p_2S_2^*(A_2(z))+r_1p_1\big[S_1^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\big\}\end{aligned}$$

$$H(z) = \frac{Nr_o(z)}{Dr(z)} \tag{3.73}$$

$$\begin{aligned}Nr_o(z) = {} & P_0\big(\{1-\big((1-r_1)p_1S_1^*(A_1(z))+(1-r_2)p_2S_2^*(A_2(z))\\ & +r_1p_1\big[S_1^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\\ & \times\{(N(z)-1)(R^*(\lambda)+X(z)(1-R^*(\lambda)))+X(z)\}\\ & -N(z)\{z-(R^*(\lambda)+X(z)(1-R^*(\lambda)))\\ & \big((1-r_1)p_1S_1^*(A_1(z))+(1-r_2)p_2S_2^*(A_2(z))\\ & +r_1p_1\big[S_1^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\\ & +[1-X(z)]\{z(b(1-R^*(\lambda))(N(z)-1)+1)\\ & -(R^*(\lambda)+X(z)(1-b)(1-R^*(\lambda)))\\ & \big((1-r_1)p_1S_1^*(A_1(z))+(1-r_2)p_2S_2^*(A_2(z))\\ & +r_1p_1\big[S_1^*(A_1(z))\big]^2+r_2p_2\big[S_2^*(A_2(z))\big]^2\big)\}\big)\end{aligned}$$

*where $P_0$ is given in* Eq. 3.71.

**Proof.** The probability generating function of the number of customer in the system ($K(z)$) and the probability generating function of the number of customer in the orbit ($H(z)$) is obtained by using $K(z) = P_0+\psi(z)+\sum_{j=1}^{J}\Omega_j(z)+z(P_1(z)+P_2(z)+\Pi_1(z)+\Pi_2(z)+Q_1(z)+Q_2(z)+R_1(z)+R_2(z))H(z) = P_0+\psi(z) + \sum_{j=1}^{J}\Omega_j(z) + P_1(z) + P_2(z)+\Pi_1(z)+\Pi_2(z)+Q_1(z)+Q_2(z)+R_1(z)+R_2(z)$. Substituting Eqs. (3.65)–(3.71) in the above results (3.72) and (3.73) can be obtained by direct calculation. $\square$

## 4. Performance measures

In this section, we obtain some interesting probabilities when the system is in different states. We also derive system performance measures the mean number of customers in the

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orbit ($L_q$) using L-Hospitals rule and the reaming measures like the mean number of customers in the system ($L_s$), the average time a customer spends in the system ($W_s$) and the average time a customer spends in the queue ($W_q$) using the Little's formula. Since our results are numerically treatable; the performance measures give a hold for managerial implication. Note that (3.71) gives the steady state probability that the server is idle but available in the system. It follows from (3.65)–(3.69) and (3.70) that the probabilities of the server sate are as given in Theorem 4.1:

**Theorem 4.1.** *If the system satisfies the stability condition $\rho < 1$, then we get the following probabilities,*

(i) *Let $\psi$ be the steady state probability that the server is idle during the retrial time*

$$\psi = \left\{ \frac{b(1-R^*(\lambda))\{N'(1)+X_{[1]}-1+\varpi\}}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(ii) *Let $P_1$ be the steady-state probability that the server is busy on first type service,*

$$P_1 = \left\{ \frac{\lambda b p_1 \beta_1^{(1)}[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(iii) *Let $P_2$ be the steady-state probability that the server is busy on second type service,*

$$P_2 = \left\{ \frac{\lambda b p_2 \beta_2^{(1)}[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(iv) *Let $\Pi_1$ be the steady-state probability that the server is re-service on first type service,*

$$\Pi_1 = \left\{ \frac{\lambda b r_1 p_1 \beta_1^{(1)}[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(v) *Let $\Pi_2$ be the steady-state probability that the server is re-service on second type service,*

$$\Pi_2 = \left\{ \frac{\lambda b r_2 p_2 \beta_2^{(1)}[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(vi) *Let $\Omega$ be the steady state probability that the server is on vacation*

$$\Omega = \left\{ \frac{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}[1-R^*(\lambda)]-\varpi)}{\frac{N'(1)}{X_{[1]}}(bX_{[1]}(1-R^*(\lambda))+1)-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi))+1} \right\}$$

(vii) *Let $Q_1$ be the steady state probability that the server is under delaying repair time on first type service,*

$$Q_1 = \left\{ \frac{\alpha_1 \lambda b\left(p_1\beta_1^{(1)}d_1^{(1)}+r_1p_1\beta_1^{(1)}d_1^{(1)}\right)[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(viii) *Let $Q_2$ be the steady state probability that the server is under delaying repair time on second type service,*

$$Q_2 = \left\{ \frac{\alpha_2 \lambda b\left(p_2\beta_2^{(1)}d_2^{(1)}+r_2p_2\beta_2^{(1)}d_2^{(1)}\right)[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(ix) *Let $R_1$ be the steady state probability that the server is under repair time on first type service,*

$$R_1 = \left\{ \frac{\alpha_1 \lambda b\left(p_1\beta_1^{(1)}g_1^{(1)}+r_1p_1\beta_1^{(1)}g_1^{(1)}\right)[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

(x) *Let $R_2$ be the steady state probability that the server is under repair time on second type service,*

$$R_2 = \left\{ \frac{\alpha_2 \lambda b\left(p_2\beta_2^{(1)}g_2^{(1)}+r_2p_2\beta_2^{(1)}g_2^{(1)}\right)[N'(1)+X_{[1]}R^*(\lambda))]}{\frac{N'(1)}{X_{[1]}}(1-X_{[1]}(1-b)(1-R^*(\lambda)))-(1-R^*(\lambda))(X_{[1]}(1-b)+b(1-\varpi)+\varpi)+1} \right\}$$

**Proof.** Noting that

$$\psi = \lim_{z\to 1}\psi(z), \quad P_1 = \lim_{z\to 1}P_1(z), \quad P_2 = \lim_{z\to 1}P_2(z),$$
$$\Pi_1 = \lim_{z\to 1}\Pi_1(z), \quad \Pi_2 = \lim_{z\to 1}\Pi_2(z),$$
$$\Omega = \lim_{z\to 1}\sum_{j=1}^{J}\Omega_j(z), \quad Q_1 = \lim_{z\to 1}Q_1(z),$$
$$Q_2 = \lim_{z\to 1}Q_2(z), \quad R_1 = \lim_{z\to 1}R_1(z), \quad R_2 = \lim_{z\to 1}R_2(z).$$

The stated formula follows by direct calculation. □

**Theorem 4.2.** *Let $L_s$, $L_q$, $W_s$ and $W_q$ be the mean number of customers in the system, the mean number of customers in the orbit, average time a customer spends in the system and average time a customer spends in the orbit respectively, then under the stability condition, we have*

$$L_q = P_0\left[\frac{Nr'''(1)Dr''(1)-Dr'''(1)Nr''(1)}{3(Dr''(1))^2}\right]$$

$$Nr''(1) = 2N'(1)[X_{[1]}(1-b)(1-R^*(\lambda))-1]-2X_{[1]}(1-b[1-R^*(\lambda)] + (1-b)[1-R^*(\lambda)]\varpi)+2X_{[1]}^2(1-b)[1-R^*(\lambda)]$$

$$Nr'''(1) = 3N''(1)[X_{[1]}(1-b)(1-R^*(\lambda))-1]+3N'(1)[X_{[2]}(1 - b)[1-R^*(\lambda)]-3X_{[1]}b[1-R^*(\lambda)]]-3X_{[2]}[1-b[1 - R^*(\lambda)]-(R^*(\lambda)-b[1-R^*(\lambda)])\varpi]-3X_{[1]}(1 - R^*(\lambda))[b\tau+2X_{[1]}(1-b)\varpi]$$

$$Dr''(1) = -2X_{[1]}[1-X_{[1]}(1-R^*(\lambda))-\varpi]$$

$$Dr'''(1) = 3X_{[1]}[\tau+2X_{[1]}(1-R^*(\lambda))\varpi]+3X_{[2]}[1-2X_{[1]}(1-R^*(\lambda))-\varpi]$$

$$L_s = P_0\left[\frac{Nr'''(1)Dr''(1)-Dr'''(1)Nr''(1)}{3(Dr''(1))^2}\right]$$

$$Nr'''(1) = 3N''(1)[X_{[1]}(1-b)(1-R^*(\lambda))-1]+3N'(1)[X_{[2]}(1-b)[1 - R^*(\lambda)]-3X_{[1]}b[1-R^*(\lambda)]]-3X_{[2]}[1-b[1-R^*(\lambda)] - (R^*(\lambda)-b[1-R^*(\lambda)])\varpi]-6\varpi[N'(1)+X_{[1]}R^*(\lambda)]$$

$$-3X_{[1]}(1-R^*(\lambda))[b\tau+2X_{[1]}(1-b)\varpi] W_s = \frac{L_s}{\lambda E(X)} \text{ and } W_q = \frac{L_q}{\lambda E(X)}$$

*where*

$$N''(1) = \frac{\{1-[V^*(\lambda b)]^J\}(\lambda b)\left(\lambda b X_{[1]}^2 v^{(2)} + X_{[2]}v^{(1)}\right)}{[V^*(\lambda b)]^J(1-V^*(\lambda b))};$$

$$\tau = p_1(1-r_1)w_1 + p_2(1-r_2)w_2 + 2r_1p_1w_3 + 2r_2p_2w_4$$

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$$w_1 = \left[(\lambda b)^2 X^2_{[1]}\beta_1^{(2)}\left[1+\alpha_1\left(d_1^{(1)}+g_1^{(1)}\right)\right]^2 + \lambda b X_{[2]}\beta_1^{(1)}\left[1+\alpha_1\left(d_1^{(1)}+g_1^{(1)}\right)\right] + \alpha_1\beta_1^{(1)}(\lambda b)^2 X^2_{[1]}\left(d_1^{(2)}+2d_1^{(1)}g_1^{(1)}+g_1^{(2)}\right)\right]$$

$$w_2 = \left[(\lambda b)^2 X^2_{[1]}\beta_2^{(2)}\left[1+\alpha_2\left(d_2^{(1)}+g_2^{(1)}\right)\right]^2 + \lambda b X_{[2]}\beta_2^{(1)}\left[1+\alpha_2\left(d_2^{(1)}+g_2^{(1)}\right)\right] + \alpha_2\beta_2^{(1)}(\lambda b)^2 X^2_{[1]}\left(d_2^{(2)}+2d_2^{(1)}g_2^{(1)}+g_2^{(2)}\right)\right]$$

$$w_3 = \left[(\lambda b)^2 X^2_{[1]}\beta_1^{(2)}\left[1+\alpha_1\left(d_1^{(1)}+g_1^{(1)}\right)\right]^2 + \lambda b X_{[2]}\beta_1^{(1)}\left[1+\alpha_1\left(d_1^{(1)}+g_1^{(1)}\right)\right] + \alpha_1\beta_1^{(1)}(\lambda b)^2 X^2_{[1]}\left(d_1^{(2)}+2d_1^{(1)}g_1^{(1)}+g_1^{(2)}\right) + \lambda b X_{[1]}\beta_1^{(1)}\left[1+\alpha_1\left(d_1^{(1)}+g_1^{(1)}\right)\right]^2\right]$$

$$w_4 = \left[(\lambda b)^2 X^2_{[1]}\beta_2^{(2)}\left[1+\alpha_2\left(d_2^{(1)}+g_2^{(1)}\right)\right]^2 + \lambda b X_{[2]}\beta_2^{(1)}\left[1+\alpha_2\left(d_2^{(1)}+g_2^{(1)}\right)\right] + \alpha_2\beta_2^{(1)}(\lambda b)^2 X^2_{[1]}\left(d_2^{(2)}+2d_2^{(1)}g_2^{(1)}+g_2^{(2)}\right) + \lambda b X_{[1]}\beta_1^{(1)}\left[1+\alpha_1\left(d_1^{(1)}+g_1^{(1)}\right)\right]^2\right]$$

**Proof.** The mean number of customers in the orbit ($L_q$) under steady state condition is obtained by differentiating (3.73) with respect to $z$ and evaluating at $z = 1$

$$L_q = \frac{Nr(z)}{Dr(z)} = H'(1) = \lim_{z\to 1}\frac{d}{dz}H(z).$$

The mean number of customers in the system ($L_s$) under steady state condition is obtained by differentiating (3.72) with respect to $z$ and evaluating at $z = 1$

$$L_s = \frac{Nr(z)}{Dr(z)} = K'(1) = \lim_{z\to 1}\frac{d}{dz}K(z).$$

The average time a customer spends in the system ($W_s$) and average time a customer spends in the orbit ($W_q$) is obtained by using the Little's formula. The stated formulas followed by direct calculation. □

## 5. Stochastic decomposition and special cases

Stochastic decomposition has been widely observed among M/G/1 type queueing models with server vacations by Fuhrman and Cooper [24]. A key result in these analyses is that the number of customers in the system in steady-state at a random point in time is distributed as the sum of two independent random variables, one of which is the number of customers in the corresponding standard queueing system (in steady-state) at a random point in time, the other-random variable may have different probabilistic interpretations in specific cases depending on how the vacations are scheduled. Stochastic decomposition has also been observed to hold for some M/G/1 retrial queueing models by Krishnakumar and Arivudainambi [12].

$$\chi(z) = \frac{(1-\varpi)(1-z)((1-r_1)p_1S_1^*(A_1(z)) + (1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\left[S_1^*(A_1(z))\right]^2 + r_2p_2\left[S_2^*(A_2(z))\right]^2)}{\left\{\left((1-r_1)p_1S_1^*(A_1(z)) + (1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\left[S_1^*(A_1(z))\right]^2 + r_2p_2\left[S_2^*(A_2(z))\right]^2\right) - z\right\}}$$

Let $K(z)$ be the stationary system size distribution of $M^{[x]}/(G_1,G_2)/1$ retrial queueing system with optional re-service, balking, Modified vacation policy and server interruption is the convolution of two independent random variables $\chi(z)$ and $\varphi(z)$.

The mathematical version of the stochastic decomposition law is $K(z) = \chi(z).\ \varphi(z)$.

(i) The system size distribution of $M^{[x]}/(G_1, G_2)/1$ queueing system with optional re-service, balking and service interruption. (represented in first term of $K(z)$),
(ii) The conditional distribution of the number of customers in the vacation system at random point in time given the server is idle (represented in second term of $K(z)$).

The number of arrivals in the variant vacation system at a random point in time given that the server is on vacation or idle. In fact the second term can be also obtained through the vacation definition of our system, i.e., $\varphi(z) = \frac{N2(z)}{D2(z)} = \left(P_0 + P(z) + \sum_{j=1}^{J}\Omega_j(z)\right)\Big/\left(P_0 + P(1) + \sum_{j=1}^{J}\Omega_j(1)\right)$

$$\begin{aligned}
N2(z) = {} & P_0\Big(z\big\{1 - \big((1-r_1)p_1S_1^*(A_1(z)) + (1-r_2)p_2S_2^*(A_2(z)) \\
& + r_1p_1\left[S_1^*(A_1(z))\right]^2 + r_2p_2\left[S_2^*(A_2(z))\right]^2\big)\big\} \\
& \times\{(N(z)-1)(R^*(\lambda) + X(z)(1-R^*(\lambda))) + X(z)\} \\
& - N(z)\big\{z - (R^*(\lambda) + X(z)(1-R^*(\lambda)))\big((1-r_1)p_1S_1^*(A_1(z)) \\
& + (1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\left[S_1^*(A_1(z))\right]^2 + r_2p_2\left[S_2^*(A_2(z))\right]^2\big)\big\} \\
& + [1-X(z)]\{z(b(1-R^*(\lambda))(N(z)-1)+1) - (R^*(\lambda) \\
& + X(z)(1-b)(1-R^*(\lambda)))\big((1-r_1)p_1S_1^*(A_1(z)) \\
& + (1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\left[S_1^*(A_1(z))\right]^2 \\
& + r_2p_2[S_2^*(A_2(z))]^2\big)\}\Big) \times \big\{((1-r_1)p_1S_1^*(A_1(z)) \\
& + (1-r_2)p_2S_2^*(A_2(z)) + r_1p_1\left[S_1^*(A_1(z))\right]^2 \\
& + r_2p_2\left[S_2^*(A_2(z))\right]^2) - z\big\}D2(z) \\
= {} & (1-\varpi)(1-z)\big((1-r_1)p_1S_1^*(A_1(z)) + (1-r_2)p_2S_2^*(A_2(z)) \\
& + r_1p_1\left[S_1^*(A_1(z))\right]^2 + r_2p_2\left[S_2^*(A_2(z))\right]^2\big) \times Dr(z)
\end{aligned}$$

The first term can be obtained through the without vacation definition of our system.

From above stochastic decomposition law, we observe that $K(z) = \chi\ (z).\ \varphi(z)$ which conform that the decomposition result of Fuhrman and Cooper [24], also valid for this special vacation system.

### 5.1. Special cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

**Case (i):** *No Retrial, No balking, No vacation and No breakdown*Let $b = 0$; $\alpha_1 = \alpha_2 = 0$; $Pr[V = 0] = 1$ and $R^*(\lambda) \to \infty$. Our model can be reduced to batch arrival queueing system with two types of service with optional re-service. In this case, $K(z)$ can be simplified to the following expressions are coincided with the result in Madan et al. [9].

$$K(z) == \frac{P_0\Big((1-r_1)p_1S_1^*(\lambda-\lambda X(z)) + (1-r_2)p_2S_2^*(\lambda-\lambda X(z)) + r_1p_1\big[S_1^*(\lambda-\lambda X(z))\big]^2 + r_2p_2\big[S_1^*(\lambda-\lambda X(z))\big]^2\Big)}{z-\Big((1-r_1)p_1S_1^*(\lambda-\lambda X(z)) + (1-r_2)p_2S_2^*(\lambda-\lambda X(z)) + r_1p_1\big[S_1^*(\lambda-\lambda X(z))\big]^2 + r_2p_2\big[S_1^*(\lambda-\lambda X(z))\big]^2\Big)}$$

where $P_0 = 1-\lambda bX_{[1]}\Big\{(1-r_1)p_1\beta_1^{(1)} + (1-r_2)p_2\beta_2^{(1)} + 2r_1p_1\beta_1^{(1)} + 2r_2p_2\beta_2^{(1)}\Big\}$

**Case (ii):** *Single type, No re-service, No retrial and No breakdown*Let $p_2 = 0$, $Pr[S_2 = 0] = 1$, $r_1 = 1$; $b = 1$, $R^*(\lambda) \to 1$ and $\alpha_1 = \alpha_2 = 0$. Then we get a batch arrival queueing system with balking and modified vacations.

$$K(z) = \frac{(1-\rho)\big\{[z-1]S_1^*[\lambda b(1-X(z))]\big\}}{\big\{z-S_1^*[\lambda b(1-X(z))]\big\}} \times \frac{(1-[V^*(\lambda b)]^J)(V^*[\lambda b(1-X(z)]-1) + (X(z)-1)(1-[V^*(\lambda b)]^J)(V^*(\lambda b))^J}{\lambda b v^{(1)}(1-[V^*(\lambda b)]^J) + (V^*(\lambda b))^J(1-V^*(\lambda b))[X(z)-1]};$$

This results are coincide equivalent to the results by Ke [4].

**Case (iii):** *Single type, No re-service, No balking, single vacation and No breakdown*Let $p_2 = 0$, $Pr[S_2 = 0] = 1$, $r_1 = 0$; $b = 1$, $J = 1$ and $\alpha_1 = \alpha_2 = 0$. Our model can be reduced to an M/G/1 retrial queue with general retrial time under Bernoulli vacations.

$$K(z) = \frac{[R^*(\lambda)-\lambda\beta_1^{(1)}]\{(z+(1-z)R^*(\lambda))[1-V^*(\lambda-\lambda z)] + (1-z)V^*(\lambda)R^*(\lambda)\}S_1^*[\lambda-\lambda z]}{[\lambda v^{(1)} + V^*(\lambda)R^*(\lambda)]\big\{[R^*(\lambda)+z(1-R^*(\lambda))]S_1^*[\lambda-\lambda z]-z\big\}}$$

$$L_q = \lim_{z\to 1}H'(z) = \frac{2\lambda v^{(1)}(1-R^*(\lambda)) + \lambda^2 v^{(2)}}{2(\lambda v^{(1)} + V^*(\lambda)R^*(\lambda))} + \frac{2\lambda\beta_1^{(1)}(1-R^*(\lambda)) + \lambda^2\beta_1^{(2)}}{2(R^*(\lambda)-\lambda\beta_1^{(1)})}$$

In this case, the probability generating function of the number of customers in the system $K(z)$, the expected number of customers in the queue $Lq$ can be simplified the following expression and which is equivalent the results obtained by Krishnakumar and Arivudainambi [12].

**Case (iv):** *Single type, No re-service, No balking and No delaying repair*.Let $p_2 = 0$, $Pr[S_2 = 0] = 1$, $r_1 = 0$, $b = 1$ and $\eta_1 = \eta_2 = 0$. Our model can be reduced to an M/G/1 retrial queueing system with a modified vacations and server breakdowns.

$$K(z) = \left\{\frac{R^*(\lambda)-\lambda\beta_1^{(1)}}{N'(1)+R^*(\lambda)}\right\} \times \frac{(N(z)\{(R^*(\lambda)+z(1-R^*(\lambda))) + R^*(\lambda)(z-1)\}S_1^*[A_1(z)])}{\{z-S_1^*[A_1(z)](R^*(\lambda)+z(1-R^*(\lambda)))\}}$$

In this case, $K(z)$ can be simplified and the following expressions are coincided with the result in Chen et al. [17].

## 6. Numerical illustration

In this section, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. We consider retrial times, service times, vacation times, delay times and repair times are exponentially, Erlangianly and hyper-exponentially distributed. Further we assume that customers are arriving one by one, so $X_{[1]} = 1$ and $X_{[2]} = 0$. The arbitrary values to the parameters are so chosen such that they satisfy the stability condition. The following tables give the computed values of various characteristics of our model like, probability that the server is idle $P_0$, the mean orbit size $L_q$, probability that server is idle during retrial rime ($\psi$), busy on both types ($P_1, P_2$), re-service on both types ($\Pi_1, \Pi_2$), on vacation ($\Omega$), under delaying repair on both type ($Q_1, Q_2$) and repair on both types ($R_1, R_2$) respectively.

Where the exponential distribution is $f(x) = \upsilon e^{-\upsilon x}$, $x > 0$, Erlang-2 stage distribution is $f(x) = \upsilon^2 x e^{-\upsilon x}$, $x > 0$ and hyper-exponential distribution is $f(x) = c\upsilon e^{-\upsilon x} + (1-c)\upsilon^2 e^{-\upsilon^2 x}$, $x > 0$.

Table 1 shows that when balking probability $(1-b)$ increases, then the probability that server is idle $P_0$ increases, the mean orbit size $L_q$ decreases and probability that server is idle during retrial time $\psi$ also decreases for the values of $\lambda = 0.5$, $p_1 = 0.5$; $r_1 = 0.5$; $r_2 = 0.5$; $\theta = 2$; $\mu_1 = 8$; $\mu_2 = 10$; $\eta_1 = 6$; $\eta_2 = 8$; $\xi_1 = 6$; $\xi_2 = 8$; $\alpha_1 = 0.4$; $\alpha_2 = 0.6$; $\gamma = 5$; $J = 1$; $c = 0.7$. Table 2 shows that when first type probability ($p_1$) increases, then the mean orbit size $L_q$ increases and probability that server is busy on first type $P_1$ increasing and probability that server is busy on second type $P_2$ decreasing for the values of $\lambda = 0.5$, $r_1 = 0.5$; $b = 0.5$; $r_2 = 0.5$; $\theta = 2$; $\mu_1 = 8$; $\mu_2 = 10$; $\eta_1 = 6$; $\eta_2 = 8$; $\xi_1 = 6$; $\xi_2 = 8$; $\alpha_1 = 0.4$; $\alpha_2 = 0.6$; $\gamma = 5$; $J = 1$; $c = 0.7$. Table 3 shows that when re-service probability on first type ($r_1$) increases, then the probability that server is idle $P_0$ decreases, then the mean orbit size $L_q$ increases and probability that server is re-service on first

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**Table 1** The effect of balking probability $(1-b)$ on $P_0$ and $Lq$.

| Retrial distribution | Exponential | | | Erlang-2 stage | | | Hyper-exponential | | |
|---|---|---|---|---|---|---|---|---|---|
| $1-b$ | $P_0$ | $L_q$ | $\psi$ | $P_0$ | $L_q$ | $\psi$ | $P_0$ | $L_q$ | $\psi$ |
| *Balking probability* | | | | | | | | | |
| 0.10 | 0.7781 | 0.0599 | 0.0339 | 0.4945 | 0.2544 | 0.1237 | 0.8378 | 0.0384 | 0.0219 |
| 0.20 | 0.8032 | 0.0513 | 0.0271 | 0.5423 | 0.2196 | 0.1023 | 0.8565 | 0.0328 | 0.0175 |
| 0.30 | 0.8283 | 0.0433 | 0.0211 | 0.5931 | 0.1865 | 0.0819 | 0.8752 | 0.0276 | 0.0135 |
| 0.40 | 0.8535 | 0.0357 | 0.0157 | 0.6466 | 0.1551 | 0.0628 | 0.8937 | 0.0227 | 0.0100 |
| 0.50 | 0.8786 | 0.0286 | 0.0110 | 0.7025 | 0.1253 | 0.0454 | 0.9121 | 0.0181 | 0.0070 |

**Table 2** The effect of probability on first type $(p_1)$ on $P_0$ and $Lq$.

| Service distribution | Exponential | | | Erlang-2 stage | | | Hyper-exponential | | |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $L_q$ | $P_1$ | $P_2$ | $L_q$ | $P_1$ | $P_2$ | $L_q$ | $P_1$ | $P_2$ |
| *Type1 probability* | | | | | | | | | |
| 0.10 | 0.0265 | 0.0031 | 0.0223 | 0.1148 | 0.0060 | 0.0430 | 0.0167 | 0.0023 | 0.0163 |
| 0.20 | 0.0270 | 0.0062 | 0.0198 | 0.1174 | 0.0119 | 0.0382 | 0.0171 | 0.0046 | 0.0145 |
| 0.30 | 0.0275 | 0.0093 | 0.0173 | 0.1200 | 0.0179 | 0.0334 | 0.0174 | 0.0069 | 0.0127 |
| 0.40 | 0.0281 | 0.0124 | 0.0148 | 0.1227 | 0.0239 | 0.0287 | 0.0178 | 0.0092 | 0.0109 |
| 0.50 | 0.0286 | 0.0155 | 0.0124 | 0.1253 | 0.0298 | 0.0239 | 0.0181 | 0.0114 | 0.0091 |

**Table 3** The effect of re-service probability on first type $(r_1)$ on $P_0$ and $Lq$.

| Service distribution | Exponential | | | Erlang-2 stage | | | Hyper-exponential | | |
|---|---|---|---|---|---|---|---|---|---|
| $r_1$ | $P_0$ | $L_q$ | $\Pi_1$ | $P_0$ | $L_q$ | $\Pi_1$ | $P_0$ | $L_q$ | $\Pi_1$ |
| *reservice probability* | | | | | | | | | |
| 0.30 | 0.8826 | 0.0268 | 0.0046 | 0.7119 | 0.1168 | 0.0090 | 0.9149 | 0.0170 | 0.0034 |
| 0.40 | 0.8806 | 0.0277 | 0.0062 | 0.7072 | 0.1211 | 0.0119 | 0.9135 | 0.0175 | 0.0046 |
| 0.50 | 0.8786 | 0.0286 | 0.0077 | 0.7025 | 0.1253 | 0.0149 | 0.9121 | 0.0181 | 0.0057 |
| 0.60 | 0.8766 | 0.0295 | 0.0093 | 0.6979 | 0.1296 | 0.0179 | 0.9106 | 0.0187 | 0.0069 |
| 0.70 | 0.8746 | 0.0304 | 0.0108 | 0.6932 | 0.1340 | 0.0209 | 0.9092 | 0.0192 | 0.0080 |

**Table 4** The effect of number of vacations $(J)$ on $P_0$ and $Lq$.

| Vacation distribution | Exponential | | | Erlang-2 stage | | | Hyper-exponential | | |
|---|---|---|---|---|---|---|---|---|---|
| $J$ | $P_0$ | $L_q$ | $\Omega$ | $P_0$ | $L_q$ | $\Omega$ | $P_0$ | $L_q$ | $\Omega$ |
| *Number of vacations* | | | | | | | | | |
| 1.00 | 0.8888 | 0.0209 | 0.0467 | 0.7424 | 0.0875 | 0.0819 | 09195 | 0.0127 | 0.0363 |
| 2.00 | 0.8350 | 0.0268 | 0.0899 | 0.6411 | 0.1046 | 0.1486 | 0.8785 | 0.0167 | 0.0706 |
| 3.00 | 0.7851 | 0.0324 | 0.1299 | 0.5573 | 0.1187 | 0.2039 | 0.8397 | 0.0205 | 0.1031 |
| 4.00 | 0.7387 | 0.0375 | 0.1672 | 0.4871 | 0.1305 | 0.2501 | 0.8029 | 0.0240 | 0.1340 |
| 5.00 | 0.6956 | 0.0423 | 0.2018 | 0.4276 | 0.1405 | 0.2893 | 0.7679 | 0.0274 | 0.1632 |

**Table 5** The effect of vacation rate $(\gamma)$ on on $P_0$ and $Lq$.

| Vacation distribution | Exponential | | | Erlang-2 stage | | | Hyper-exponential | | |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $P_0$ | $L_q$ | $\Omega$ | $P_0$ | $L_q$ | $\Omega$ | $P_0$ | $L_q$ | $\Omega$ |
| *Vacation rate* | | | | | | | | | |
| 5.00 | 0.8888 | 0.0209 | 0.0467 | 0.7424 | 0.0875 | 0.0819 | 0.9195 | 0.0127 | 0.0363 |
| 6.00 | 0.8983 | 0.0195 | 0.0390 | 0.7621 | 0.0831 | 0.0689 | 0.9271 | 0.0117 | 0.0299 |
| 7.00 | 0.9052 | 0.0185 | 0.0335 | 0.7764 | 0.0801 | 0.0595 | 0.9325 | 0.0111 | 0.0254 |
| 8.00 | 0.9104 | 0.0179 | 0.0293 | 0.7872 | 0.0780 | 0.0523 | 0.9364 | 0.0106 | 0.0221 |
| 9.00 | 0.9144 | 0.0174 | 0.0261 | 0.7958 | 0.0765 | 0.0467 | 0.9395 | 0.0103 | 0.0195 |

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**Table 6** The effect of repair rate on first type FPS ($\xi_1$) on $P_0$ and $Lq$.

| Repair distribution | Exponential | | | Erlang-2 stage | | | Hyper-exponential | | |
|---|---|---|---|---|---|---|---|---|---|
| $\xi_1$ | $P_0$ | $L_q$ | $R_1$ | $P_0$ | $L_q$ | $R_1$ | $P_0$ | $L_q$ | $R_1$ |
| *Repair rate* | | | | | | | | | |
| 1.00 | 0.8968 | 0.0221 | 0.0093 | 0.7486 | 0.1092 | 0.0364 | 0.9263 | 0.0134 | 0.0069 |
| 2.00 | 0.8977 | 0.0202 | 0.0047 | 0.7567 | 0.0906 | 0.0182 | 0.9268 | 0.0121 | 0.0029 |
| 3.00 | 0.8980 | 0.0198 | 0.0031 | 0.7594 | 0.0864 | 0.0121 | 0.9270 | 0.0118 | 0.0018 |
| 4.00 | 0.8982 | 0.0196 | 0.0023 | 0.7607 | 0.0846 | 0.0091 | 0.9271 | 0.0118 | 0.0013 |
| 5.00 | 0.8983 | 0.0195 | 0.0019 | 0.7615 | 0.0837 | 0.0073 | 0.9271 | 0.0117 | 0.0010 |

type $\Pi_1$ also increase for the values of $\lambda = 0.5$, $p_1 = 0.5$; $b = 0.5$; $r_2 = 0.5$; $\theta = 2$; $\mu_1 = 8$; $\mu_2 = 10$; $\eta_1 = 6$; $\eta_2 = 8$; $\xi_1 = 6$; $\xi_2 = 8$; $\alpha_1 = 0.4$; $\alpha_2 = 0.6$; $\gamma = 5$; $J = 1$; $c = 0.7$.

Table 4 shows that when number of vacations ($J$) increases, then the probability that server is idle $P_0$ decreases, the mean orbit size $L_q$ increases and probability that server is on vacation $\Omega$ also increases for the values of $\lambda = 0.5$, $p_1 = 0.5$; $r_1 = 0.5$; $b = 0.5$; $r_2 = 0.5$; $\theta = 2$; $\mu_1 = 8$; $\mu_2 = 10$; $\eta_1 = 6$; $\eta_2 = 8$; $\xi_1 = 6$; $\xi_2 = 8$; $\alpha_1 = 0.4$; $\alpha_2 = 0.6$; $\gamma = 5$; $c = 0.7$. Table 5 shows that when vacation rate ($\gamma$) increases, then the probability that server is idle $P_0$ increases the mean orbit size $L_q$ decreases and probability that server is on vacation $\Omega$ also

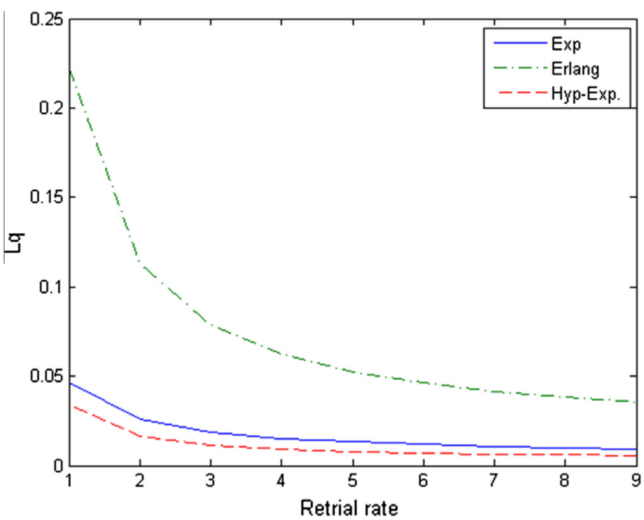


**Figure 2** $L_q$ versus $\theta$.

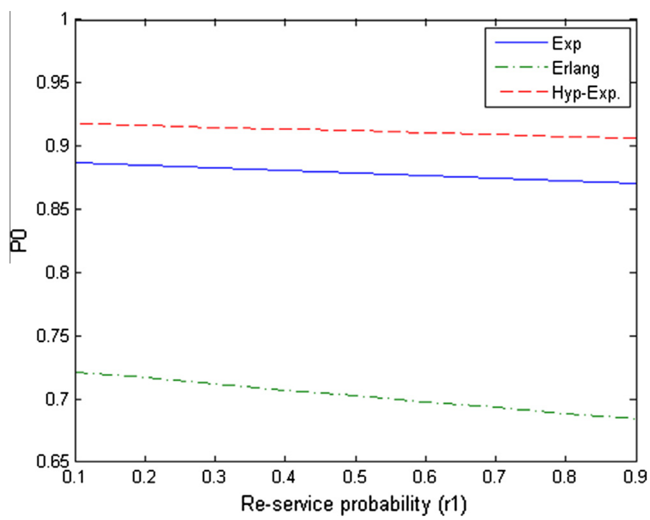


**Figure 4** $P_0$ versus $r_1$.

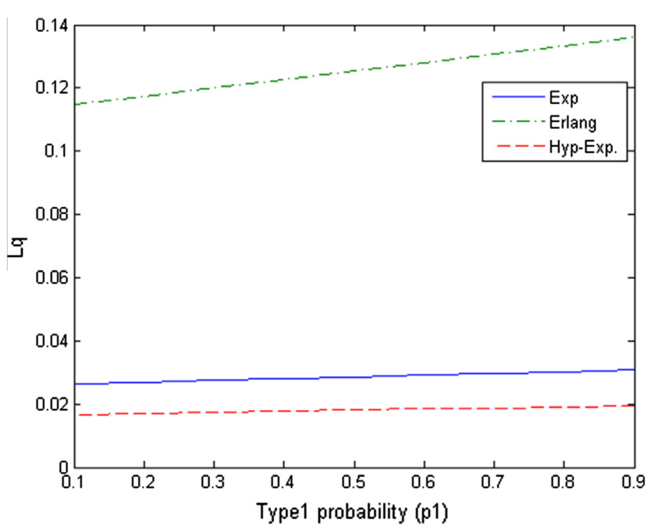


**Figure 3** $Lq$ versus $p_1$.

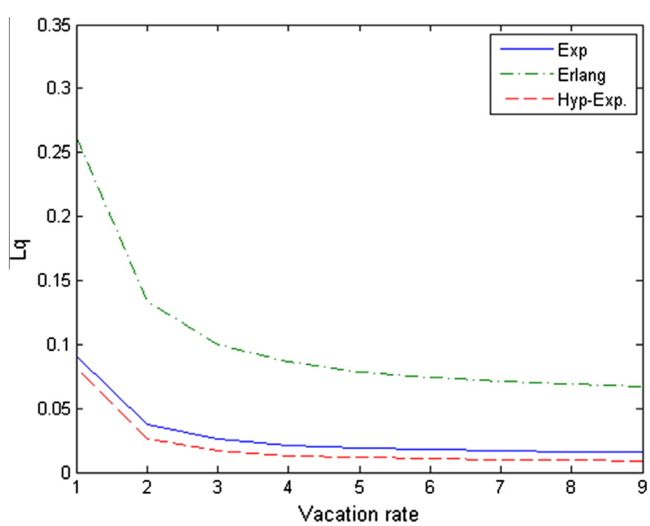


**Figure 5** $L_q$ versus $\gamma$.

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decreases for the values of $\lambda = 0.5$, $p_1 = 0.5$; $r_1 = 0.5$; $b = 0.5$; $r_2 = 0.5$; $\theta = 2$; $\mu_1 = 8$; $\mu_2 = 10$; $\eta_1 = 6$; $\eta_2 = 8$; $\xi_1 = 6$; $\xi_2 = 8$; $\alpha_1 = 0.4$; $\alpha_2 = 0.6$; $J = 1$; $c = 0.7$;. Table 6 shows that when repair rate on first type ($\xi_1$) increases, then the probability that server is idle $P_0$ increases, the mean orbit size $L_q$ decreases and probability that server is under repair on First type ($R_1$) also decrease for the values of $\lambda = 0.5$, $p_1 = 0.5$; $r_1 = 0.5$; $\gamma = 5$ $b = 0.5$; $r_2 = 0.5$; $\theta = 2$; $\mu_1 = 8$; $\mu_2 = 10$; $\eta_1 = 6$; $\eta_2 = 8$; $\xi_2 = 8$; $\alpha_1 = 0.4$; $\alpha_2 = 0.6$; $J = 1$; $c = 0.7$.

For the effect of the parameters $\theta$, $b$, $r_1$, $p_1$, $\gamma$, $J$, $\eta_1$ and $\xi_1$ on the system performance measures, two dimensional graphs are drawn in Figs. 1–8. Fig. 1 shows that the mean orbit size $L_q$ increases for increasing value of the non-balking probability ($b$). Fig. 2 shows that the mean orbit size $L_q$ decreases for increasing value of the retrial rate ($\theta$). Fig. 3 shows that the mean orbit size $L_q$ increases for increasing value of the first type probability ($p_1$). Fig. 4 shows that the idle probability $P_0$ decreasing for increasing value of the re-service probability on first type ($r_1$). Fig. 5 shows that the mean orbit size $L_q$ decreases for increasing value of the vacation rate ($\gamma$). Fig. 6 shows the mean orbit size $L_q$ increases for increasing value of the number of vacations ($J$). Fig. 7 shows the idle probability $P_0$ increases for increasing value of the delay rate on first type ($\eta_1$). Fig. 8 shows mean orbit size $L_q$ decreases for increasing value of the repair rate on first type ($\xi_1$).

Three dimensional graphs are illustrated in Figs. 9–12. In Fig. 9, the surface displays an upward trend as expected for increasing the value of the non-balking probability $b$ and first type probability $p_1$ against the mean orbit size $L_q$. Fig. 10 shows that the surface displays sharp fall trend as expected for increasing value of retrial rate $\theta$ and repair rate on first type $\xi_1$ against the mean orbit size $L_q$. The mean orbit size $L_q$ increases for increasing value of the number of vacations ($J$) and re-service probability on first type ($r_1$) is shown in Fig. 11. In Fig. 12, the surface displays downward trend as expected for increasing value of vacation rate $\gamma$ and delaying rate on first type $\eta_1$ against the server idle probability $P_0$.

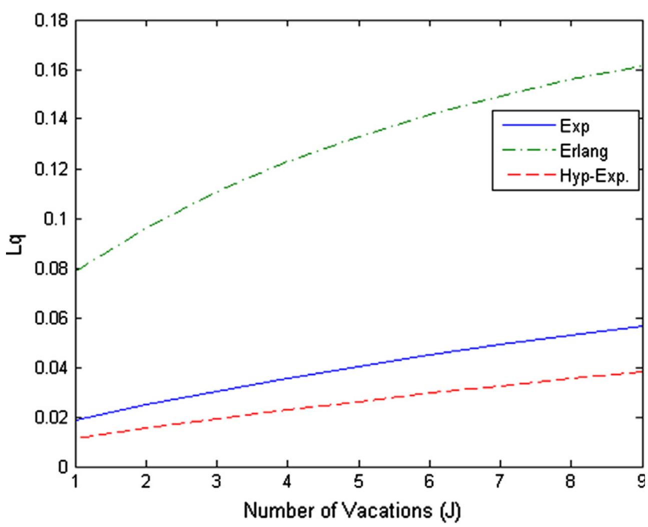


**Figure 6** $L_q$ versus $J$.

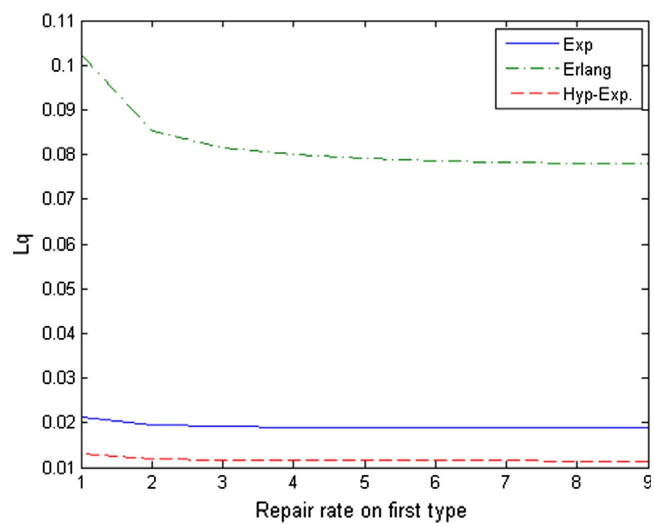


**Figure 8** $L_q$ versus $\xi_1$.

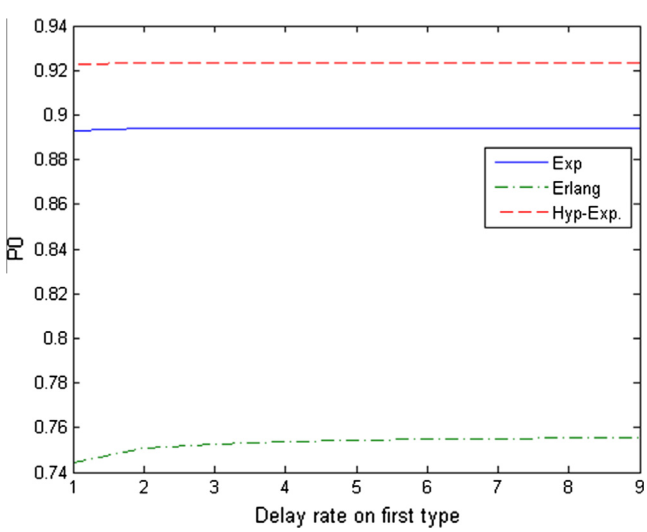


**Figure 7** $P_0$ versus $\eta_1$.

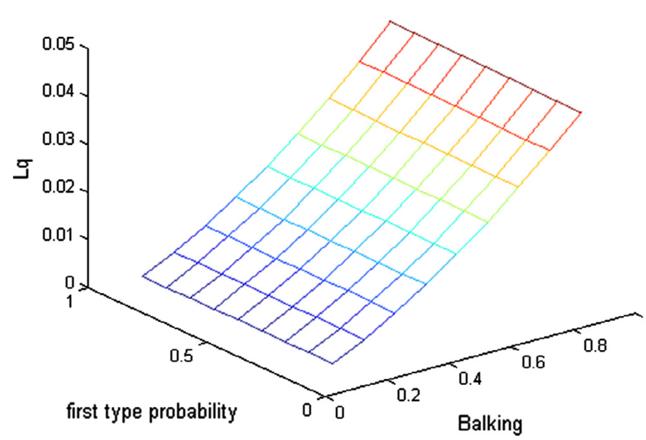


**Figure 9** $L_q$ versus $b$ and $p_1$.

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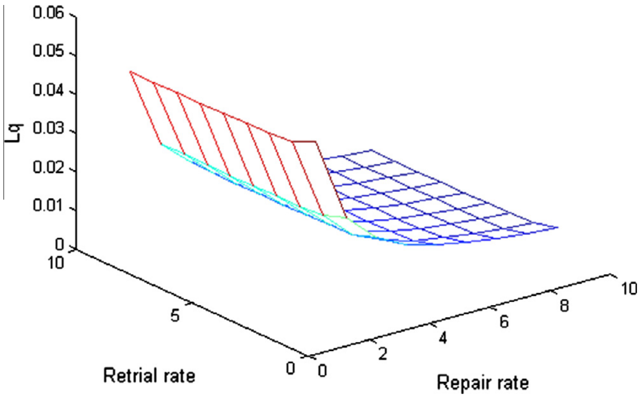


**Figure 10** $L_q$ versus $\theta$ and $\xi_1$.

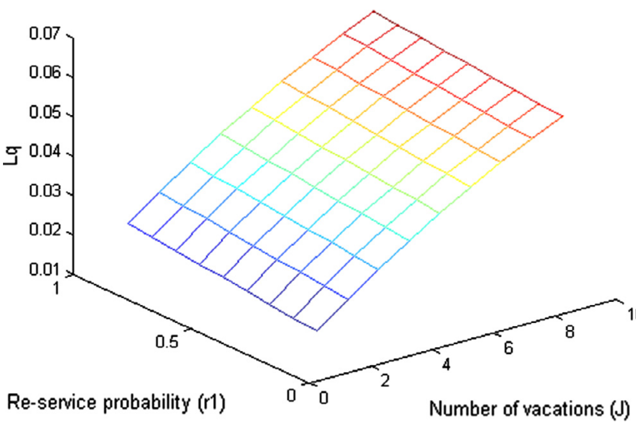


**Figure 11** $L_q$ versus $J$ and $r_1$.

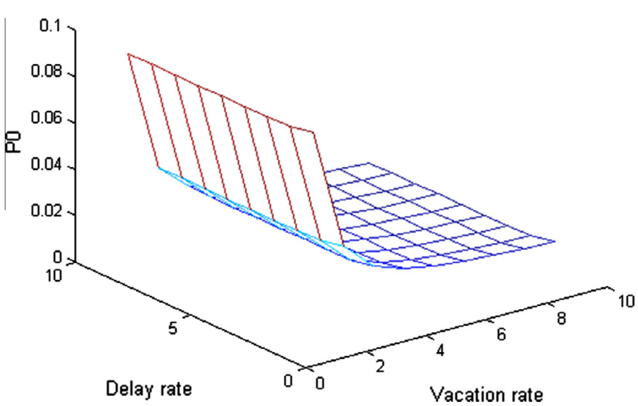


**Figure 12** $P_0$ versus $\gamma$ and $\eta_1$.

## 7. Conclusion

In this paper, we have studied a batch arrival retrial queueing system with balking, modified vacations subject to server breakdowns and delaying repair. Where the server provides two types of service and each type consist an optional re-service. The probability generating functions of the number of customers in the system and orbit are found by using the supplementary variable technique. The performance measures like, the mean number of customers in the system/orbit, the average waiting time of customer in the system/orbit and some system probabilities are obtained. Finally, the general decomposition law is shown to hold good for this model. The analytical results are validated with the help of numerical illustrations. This model finds potential application in Simple Mail Transfer Protocol (SMTP) mail system to deliver the messages between mail servers and Wired Networks for selecting routes from the routing table.


## Acknowledgement

The authors are thankful to the anonymous referees for their valuable comments and suggestions for the improvement of the paper.


**Appendix A.** The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if $\rho < 1$, where $\rho = X_{[1]}(1 - R^*(\lambda)) + \varpi$,

$$\varpi = \lambda b X_{[1]}\Big\{p_1\beta_1^{(1)}\Big[1+\alpha_1\Big(d_1^{(1)}+g_1^{(1)}\Big)\Big] +p_2\beta_2^{(1)}\Big[1+\alpha_2\Big(d_2^{(1)}+g_2^{(1)}\Big)\Big] +r_1p_1\beta_1^{(1)}\Big[1+\alpha_1\Big(d_1^{(1)}+g_1^{(1)}\Big)\Big]+r_2p_2\beta_2^{(1)}\Big[1+\alpha_2\Big(d_2^{(1)}+g_2^{(1)}\Big)\Big]\Big\}$$

**Proof** From Gomez-Corral [25], it is not difficult to see that $\{Z_n;\ n \in N\}$ is an irreducible and aperiodic Markov chain. To prove Ergodicity, we shall use the following Foster's criterion: an irreducible and aperiodic Markov chain is ergodic if there exists a nonnegative function $f(j)$, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E[f(z_{n+1}) - f(z_n)/z_n = j]$ is finite for all $j \in N$ and $\psi_j \leqslant -\varepsilon$ for all $j \in N$, except perhaps for a finite number $j$'s. In our case, we consider the function $f(j) = j$. then we have

$$\psi_j = \begin{cases} \varpi - 1, & j = 0, \\ X_{[1]}(1 - R^*(\lambda)) + \varpi - 1, & j = 1, 2 \ldots \end{cases}$$

Clearly the inequality $X_{[1]}(1 - R^*(\lambda)) + \varpi < 1$ is sufficient condition for Ergodicity.

The same inequality is also necessary for Ergodicity. As noted in Sennott et al. [26], we can guarantee non-Ergodicity, if the Markov chain $\{Z_n;\ n \geqslant 1\}$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \geqslant 0$ and there exits $j_0 \in N$ such that $\psi_j \geqslant 0$ for $j \geqslant j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a $k$ such that $m_{ij} = 0$ for $j < i - k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $\{Z_n;\ n \in N\}$. Then $X_{[1]}(1 - R^*(\lambda)) + \varpi \geqslant 1$ implies the non-Ergodicity of the Markov chain.

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Please cite this article in press as: Rajadurai P et al., Analysis of an $M^{[X]}/(G_1, G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption, Ain Shams Eng J (2014), http://dx.doi.org/10.1016/j.asej.2014.02.003