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Analytic Functions of Complex Order Defined by Fractional Integrals Involving Fox's H-functions

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Abstract

We introduced certain new subclasses of analytic functions of complex order defined by fractional integrals involving Fox's H-functions in the unit disc and investigate the various properties and characteristics of analytic functions belonging to the subclasses $S_n(\lambda,b,\gamma)$. We also defined an another subclass $R_n(\lambda,b,\gamma)$ involving Fox's H- functions. Apart from deriving a set of coefficient bounds for each of these function classes, we establish several inclusion relationships involving the (n,δ) — neighbourhoods of analytic functions with negative coefficients belonging to these subclasses.

Keywords: Convex, Generalized Hypergeometric Functions, Hadamard Product, Inclusion Relations, (n, δ)-Neighborhood, Starlike, Univalent

1. Introduction

Denote by $A\ (n)$ the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_k z^k,\ a_k\geq 0,\ n\in N:=\{1,2,3,...\},\quad (1.1)$$

which are analytic and univalent in the open disc $U=\{z: z\in C, |z|<1\}$,note that $A(1)=A$. Also denote $\mathcal{T}(n)$ the class of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k,\ a_k\geq 0,\ n\in N:=\{1,2,3,...\},\quad (1.2)$$

which are analytic and univalent in the open disc $U=\{z: z\in C, |z|<1\}$, and note that $T(1)=T$,[20] be the subclass of $A(n)$ of univalent functions in U. For functions $f_j (j=1,2)$ given by

$$f_j(z)=z+\sum_{k=n+1}^{\infty}a_{k,j}z^k\ (j=1,2),$$

Let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_1 and f_2 defined by

$$(f_1*f_2)(z)=z+\sum_{k=n+1}^{\infty}a_{k,1}a_{k,2}z^k,\ z\in U.\quad (1.3)$$

Now we briefly recall the definitions of the special functions and operators of fractional calculus used in this paper. By a Fox's H-function we mean the generalized hyper geometric function defined by the Mellin-Barnes type contour integral

$$H_{p,q}^{m,n}\left[\sigma\left|\begin{matrix}(a_k,A_k)_1^p\\(b_k,B_k)_1^q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_c\frac{\prod_{k=1}^{m}\Gamma(b_k-B_ks)\prod_{j=1}^{n}\Gamma(1-a_j+sA_j)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+sB_k)\prod_{j=n+1}^{p}\Gamma(a_j-sA_j)}\sigma^s ds.\ (1.4)$$

here $\mathcal{L}$ is a suitable contour in C and the orders $(m;n;p;q)$ are integers $0\leq m\leq q$, $0\leq n\leq p$. For the conditions on the parameters $a_j\in\mathbb{R}$, $A_j>0\ (j=1,\ldots,p)$, $b_k\in\mathbb{R}$, $B_k>0\ (k=1,\ldots,q)$ and the types of contours, for existence and analyticity of function (1.4) in disks $\subset$ C, one can see 14,16 [11, App.], 10 etc. For $A_1=\cdots=A_p=1$,

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$B_1 = \cdots = B_q = 1$, (1.4) turns into the more popular *Meijer's G-function* (see [14], [16], [11, App.], [10]). The G— and H — functions encompass almost all the elementary and special functions. Among them, the *Mittag-Leffler function,* and the Wright's generalized hypergeometric functions ${}_p\Psi_q$ with irrational $A_j, B_k > 0$, give rather general and typical examples of H—functions, not reducible to G—functions:

$$p\Psi q\left(\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}:\sigma\right)=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\dots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\dots\Gamma(b_q+kB_q)}\frac{\sigma^k}{k!}$$
$$=H^{1,p}_{p,q+1}\left[-\sigma\left|\begin{matrix}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)\end{matrix}\right.\right] \quad (1.5)$$

when $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$; they turn into

$$p\Psi q\left(\begin{matrix}(a_1,1),\dots,(a_p,1)\\(b_1,1),\dots,(b_q,q)\end{matrix}:\sigma\right)=\left[\frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{i=1}^{q}\Gamma(b_j)}\right]{}_pF_q\left(a_1,\dots,a_p:b_1,\dots,b_q:\sigma\right). \quad (1.6)$$

Recently, Kiryakova[11], discussed the *Generalized Fractional Calculus* (GFC) with various applications to the special functions and integral transforms, to the hyper-Bessel operators, ODEs, dual and Volterra integral equations, univalent functions, etc. All the *classical FC operators* (see e.g. in 15), and most of their generalizations by different authors, fall in the GFC as very special cases, by taking multiplicities $m = 1, 2$, and some specific parameters.

Let m ≥ 1 be an integer; $\delta_i > 0$, $\gamma_i \in \mathbb{R}$, $\beta_j > 0$, i = 1,... m. We consider $\delta = (\delta_1,\dots, \delta_m)$ as a multi-order of fractional integration; $\gamma = (\gamma_1, \dots, \gamma_m)$ as multi-weight; $\beta = (\beta_1,\dots, \beta_m)$ as additional paramter. The integral operators defined follows:

$$I\,f(z)=I^{(\alpha_i),(\delta_i)}_{(\beta_i),m}f(z)=\int_0^1 H^{m,0}_{m,m}\left[\sigma\left|\begin{matrix}\left(\alpha_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\alpha_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right]f(z\sigma)d\sigma \quad (1.7)$$

if $\sum_{i=1}^{m}\delta_i > 0$, and as $I\,f(z) = f(z)$ if $\delta_1 = \delta_2 = \dots = \delta_m = 0$, are said to be multiple (m-tuple) *Erdélyi-Kober* fractional integration operators ([11]). More generally, all the operators of the form

$$\tilde{I}f(z)=z^{\delta_0}I^{(\alpha_i),(\delta_i)}_{(\beta_i),m}f(z) \text{ with } \delta\ge 0,$$

are briefly called as generalized *(m-tuple)* fractional integrals. The corresponding generalized fractional derivatives are denoted by

$$D^{(\alpha_i),(\delta_i)}_{(\beta_i),m}$$

and defined by means of explicit differintegral expressions, similarly to the idea for the classical Riemann-Liouville derivative (see e.g.[15]).

We state following lemma due to Kiryakova et al. [12, 13] to represent the generalized fractional calculus operator for functions in A and state some mapping properties of operators (1.7) in classes of analytic functions in the unit disk $U = \{z : |z| < 1\}$ based on the H—function theory.

Lemma 1.1 *For* $\delta_i \ge 0$, $\alpha_i \in \mathbb{R}$, $\beta_i > 0$ $(i = 1, \dots, m)$, *and each* $p > \max_i\left[-\beta_i(\alpha_i+1)\right]$,

$$I^{(\alpha_i),(\delta_i)}_{(\beta_i),m}\left\{z^p\right\}=\Omega_p z^p,\ \Omega_p=\prod_{i=1}^{m}\frac{\Gamma(\alpha_i+1+p/\beta_i)}{\Gamma(\alpha_i+\delta_i+1+p/\beta_i)}>0. \quad (1.8)$$

Then the conditions

$$\delta_i \ge 0,\ \alpha_i \ge -1,\ \beta_i > 0,\ i = 1,\dots,m, \quad (1.9)$$

ensure that (1.8) holds for each p ≥ 0. That is including A and its subclasses.

In view of (1.8), when working for the classes of functions A, T, and $A(n)$, $T(n)$, it is suitable to consider a normalized form of operators (1.7), involving the multiplier constant:

$$c:=\left[\Omega_1\right]^{-1}=\Omega^{-1}=\prod_{i=1}^{m}\frac{\Gamma(\alpha_i+\delta_i+1+1/\beta_i)}{\Gamma(\alpha_i+1+1/\beta_i)},(p=1)$$

namely we will work next with the (normalized) GFC operators

$$\tilde{I}^{(\alpha_i),(\delta_i)}_{(\beta_i),m}f(z):=cI^{(\alpha_i),(\delta_i)}_{(\beta_i),m}f(z). \quad (1.10)$$

From Lemma 1.1 and the more general results in [11 Ch.5, Sec.5.5], we have the following.

Theorem 1.1 *Under the parameters conditions (1.10):*

$$\delta \ge 0,\ \alpha_i > -1,\ \beta_i > 0\ (i = 1,\dots,m)$$

the generalized fractional integral $\tilde{I}^{(\alpha_i),(\delta_i)}_{(\beta_i),m}$ *maps the class A(n) into itself, and the image of a power series (1.1) has the form*

$$\tilde{I}f(z)=\tilde{I}^{(\alpha_i),(\delta_i)}_{(\beta_i),m}\left\{z+\sum_{k=n+1}^{\infty}a_kz^k\right\}=z+\sum_{k=n+1}^{\infty}\theta(k)a_kz^k\in\mathcal{A}(n) \quad (1.11)$$

with multipliers' sequence (k=n+1, n+2, ...):

$$\theta(k)=\prod_{i=1}^{m}\frac{\Gamma(\alpha_i+1+k/\beta_i)\Gamma(\alpha_i+\delta_i+1+1/\beta_i)}{\Gamma(\alpha_i+\delta_i+1+k/\beta_i)(\alpha_i+1+1/\beta_i)}>0. \quad (1.12)$$

In the class A the generalized fractional integral (1.7) can be represented by the hadamard product $\tilde{I}^{(\alpha_i),(\delta_i)}_{(\beta_i),m}f(z)=h(z)*f(z)$ where h(z) $\in A$ is expressed by the Wright's hypergeometric function(1.5)

$$h(z)=z+\sum_{k=2}^{\infty}\theta(k)z^k=z\Omega^{-1}{}_{m+1}\Psi_m\left(\begin{matrix}(1,1),\left(\alpha_i+1+\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\alpha_i+\delta_i+1+\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}:z\right). \quad (1.13)$$

These functions were extensively studied in[6, 5].

The main object of this paper is to define new subclasses of analytic functions of complex order in the unit disc and to investigate the various characteristics properties of analytic functions belonging to the new subclasses.

A function f $\in$ A(n) is starlike of complex order b ($b\in\mathbb{C}\setminus\{0\}$), that is $ST_n(b)$, if it also satisfies the following inequality

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0 \ \left(z\in U;b\in\mathbb{C}\setminus\{0\}\right). \quad (1.14)$$

Furthermore, a function f$\in A(n)$ is convex of complex order b ($b\in\mathbb{C}\setminus\{0\}$), that is $CV_n(b)$, if it also satisfies the following inequality

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0 \ \left(z\in U;b\in\mathbb{C}\setminus\{0\}\right). \quad (1.15)$$

The classes $ST_n(b)$ and $CV_n(b)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr et.al.[17] and Wiatrowski[23], respectively. Further motivated essentially by earlier investigations of [2,8] and [9], and making use of Fox's H- functions (the generalized hypergeometric functions), we introduced certain new subclasses of analytic functions in the unit disc and investigate the various properties and characteristics of analytic functions belonging to the new subclasses.

For $0\le\lambda\le 1$, we let $S_n(\lambda, b, \gamma)$ be the subclass of $A(n)$ consisting of functions of the form (1.2) and satisfying the inequality

$$\left|\frac{1}{b}\left(\frac{z(\tilde{I}f(z))'}{(1-\lambda)\tilde{I}f(z)+\lambda z(\tilde{I}f(z))'}-1\right)\right|<\gamma, \quad (1.16)$$

where $z\in U$, $b\in\mathbb{C}\setminus\{0\}$, $0\le\lambda<1$ and If (z) is given by (1.11).

Also let $R_n(\lambda, b, \gamma)$ be the subclass of $A(n)$ consisting of functions of the form (1.2) and satisfying the inequality

$$\left|\frac{1}{b}\left((1-\lambda)\frac{\tilde{I}f(z)}{z}+\lambda(\tilde{I}f(z))'-1\right)\right|<\gamma \quad (1.17)$$

where $z\in U$, $b\in C\setminus\{0\}$, $0<\gamma\le 1$, $0\le\lambda<1$ and If (z) is given by (1.11).

We deem it proper to mention some of the function classes which emerge from the function class $S_n(\lambda, b, \gamma)$ defined above. It is enough to choose suitable particular parameters m, γ_k, δ_k, β_k. For $m = 1$, we have the examples

Example 1. We observe that if we specialize that $\alpha_i = -1$ and $\delta_i = 1$ with $m = 1$, $\beta_i = 1$, in (1.11) gives the Biernaki operator $\tilde{I}^{-1,1}_{1,1}=Bf(z)=-log(1-z)*f(z)$ the class $S_n(\lambda, b, \gamma)$ reduces to the class $S_n^{\lambda}(b,\gamma)$ subclass of $A(n)$ consisting of functions of the form (1.2) and satisfying the inequality

$$\left|\frac{1}{b}\left(\frac{z(\mathcal{B}f(z))'}{(1-\lambda)\mathcal{B}f(z)+\lambda z(\mathcal{B}f(z))'}-1\right)\right|<\gamma,$$

where $z\in U$, $b\in\mathbb{C}\setminus\{0\}$, $0<\gamma\le 1$ and $0\le\lambda<1$.

Further by taking $\alpha_i = 0$ and $\delta_i = 1$ with $m = 1$, $\beta_i = 1$, in (1.11) gives the libera operator

$$\mathcal{L}f(z)=2I^{0,1}_{1,1}f(z)=z_2F_1(1,2;3;z)*f(z)$$

and generalized Libera operator

$$B_cf(z)=(c+1)I^{c-1,1}_{1,1}f(z)=z^{c+1}{}_2F_1(1,c+1;c+2;z)*f(z)$$

Example 2. By taking $\alpha_i = a - 2$ and $\delta_i = c - a$ with $m = 1$, $\beta_i = 1$, in (1.11) the class $S_n(\lambda, b, \gamma)$ reduces to the class $S_c^a(\lambda, b, \gamma)$ the subclass of $A(n)$ consisting of functions of the form (1.2) and satisfying the inequality

$$\left|\frac{1}{b}\left(\frac{z\left(L(a,c)f(z)\right)'}{(1-\lambda)L(a,c)f(z)+\lambda z\left(L(a,c)f(z)\right)'}-1\right)\right|<\gamma,$$

where $z \in U$, $b \in \mathbb{C} \setminus \{0\}$, $0 < \gamma \le 1, 0 \le \lambda < 1$ and

$$L(a,c)f(z) := z_2F_1(1,a;c;z)*f(z) = \frac{\Gamma(c)}{\Gamma(c)} I_{1,1}^{a-2,c-a} f(z).$$

called Carlsons-Shaffer's operator[4]

Example 3. For $D^\eta f(z) := z_2F_1(1, \eta + 1; 1; z) * f(z)$, $(\eta > -1)$, with $m = 1$ and $\beta = 1$, in (1.11) the class $S_n(\lambda, b, \gamma)$, reduces to the class $S_n(\eta, \lambda, b, \gamma)$ the subclass of $A(n)$ consisting of functions of the form (1.2) and satisfying the inequality

$$\left|\frac{1}{b}\left(\frac{z\left(D^\eta f(z)\right)'}{(1-\lambda)D^\eta f(z)+\lambda z\left(D^\eta f(z)\right)'}\right)\right|<\gamma,$$

where $z \in U$, $b \in \mathbb{C} \setminus \{0\}$, $0 < \gamma \le 1, 0 \le \lambda < 1$ and

$$D^\eta f(z) := z_2F_1(1,\eta+1;1;z)*f(z) = z + \sum_{k=n+1}^{\infty}\binom{\eta+k-1}{\eta-1} a_k z^k$$

Ruscheweyh Derivative operator[18]

Our definitions of function classes $S_n(\lambda, b, \gamma)$ and $R_n(\lambda, b, \gamma)$ are motivated essentially by earlier investigations[2] and[9], in each of which further details and references to other closely related subclasses can be found.

The main object of the present paper is to investigate the various properties and characteristics of analytic functions belonging to the subclasses $S_n(\lambda, b, \gamma)$ and $R_n(\lambda, b, \gamma)$ introduced. Apart from deriving a set of coefficient bounds for each of these function classes, we establish several inclusion relationships involving the (n, δ) — negihborhoods of analytic functions with negative and missing coefficients belonging to these subclasses. Also the special cases of some of these inclusion relations are shown to yield known results.

2. A Set of Coefficient Inequalities

In this section we obtain the coefficient inequalities for functions in the subclasses $S_n(\lambda, b, \gamma)$ and $R_n(\lambda, b, \gamma)$.

Theorem 2.1 *Let the function $f \in A(n)$ be defined by (1.2), then $f(z)$ is in the class $S_n(\lambda, b, \gamma)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(\left[1+\lambda(k-1)\right]\left(\gamma|b|-1\right)+k\right)\theta(k)a_k \le \gamma|b| \qquad (2.1)$$

where $\theta(k)$ is as defined in (1.12).

Proof. Let a function $f(z)$ of the form (1.2) belong to the class $S_n(\lambda, b, \gamma)$. Then in view of (1.11) and (1.16), we obtain the following inequality,

$$\left|\frac{\sum_{k=n+1}^{\infty}\left(\left[\lambda(k-1)+1\right]-k\right)\theta(k)a_k z^k}{z-\sum_{k=n+1}^{\infty}\left[\lambda(k-1)+1\right]\theta(k)a_k z^k}\right| \le \gamma|b|,\ z \in U.$$

Thus putting $z = r(0 \le r < 0)$,we obtain

$$\frac{\sum_{k=n+1}^{\infty}\left(\left[\lambda(k-1)+1\right]-k\right)\theta(k)a_k r^{k-1}}{1-\sum_{k=n+1}^{\infty}\left[\lambda(k-1)+1\right]\theta(k)a_k r^{k-1}} \le \gamma|b|,\ z \in U \qquad (2.2)$$

Hence, we observe that the expression in the denominator on the left-hand side of (2.2) is positive for $r = 0$ and also for all $r(0 < r < 1)$. Thus, by letting $r \to 1^-$ through real values, (2.2) leads us to the desired assertion (2.1) of Theorem 2.1.

Conversely, by applying (2.1) and setting $|z| = 1$, we find from (1.16) that

$$\left|z\left(\tilde{I}f(z)\right)' - (1-\lambda)\tilde{I}f(z) - \lambda z\left(\tilde{I}f(z)\right)'\right| - \gamma|b|\left|(1-\lambda)\tilde{I}f(z)+\lambda z\left(\tilde{I}f(z)\right)'\right|$$

$$= \left|\sum_{k=n+1}^{\infty}\left(\left[\lambda(k-1)+1\right]-k\right)\theta(k)a_k z^k\right| - \gamma|b| \left|z-\sum_{k=n+1}^{\infty}\left[\lambda(k-1)+1\right]\theta(k)a_k z^k\right|$$

$$< \sum_{k=n+1}^{\infty}\left(\left[1+\lambda(k-1)\right]\left(\gamma|b|-1\right)+k\right)\theta(k)a_k - \gamma|b| \le 0.$$

Hence, by the maximum modulus principle, we infer that $f(z) \in S_n(\lambda, b, \gamma)$, which evidently completes the proof of Theorem 2.1.

Similarly, we can prove the following theorem.

Theorem 2.2 *Let the function $f \in A(n)$ be defined by (1.2), then f(z) is in the class $R_n(\lambda, b, \gamma)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left[1+\lambda(k-1)\right]\theta(k)a_k \le \gamma|b| \qquad (2.3)$$

Where θ(k) is as defined in (1.12).

3. Inclusion Relations Involving the (*n*, *δ*)-Neighbourhoods

In this section, we establish several inclusion relations for the normalized analytic function classes $S_n(\lambda, b, \gamma)$ and $R_n(\lambda, b, \gamma)$ involving the (n, δ)—neighborhood . Following Goodman[7], Ruscheweyh[19], Silverman[21] and others[1,2,3,9], we recall the (n, δ)—neighborhood of a function $f \in A(n)$ by

$$N_{n,\delta}(f) := \left\{ f \in \mathcal{A}(n) \; : \; g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \le \delta \right\}. \quad (3.1)$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_{n,\delta}(e) := \left\{ f \in \mathcal{A}(n) \; : \; g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \le \delta \right\}. \quad (3.2)$$

Theorem 3.1 *If*

$$\delta := \frac{\gamma|b|(1+n)}{\left((\gamma|b|-1)[1+n\lambda]+n+1\right)\theta(n+1)}, (\gamma|b| \ge 1) \qquad (3.3)$$

then

$$S_n(\lambda, b, \gamma) \subset N_{n,\delta}(e). \qquad (3.4)$$

Proof. Let $f(z) \in S_n(\lambda, b, \gamma)$. Then, in view of the assertion (2.1) of Theorem 2.1, we have

$$\theta(n+1)\left([1+n\lambda](\gamma|b|-1)+n+1\right)\sum_{k=n+1}^{\infty} a_k \le \gamma|b|$$

which readily yields

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\gamma|b|}{\left([1+n\lambda](\gamma|b|-1)+n+1\right)\theta(n+1)}. \qquad (3.5)$$

Making use of (2.1) again, in conjunction with (3.5), we get

$$\begin{aligned}\theta(n+1)\sum_{k=n+1}^{\infty} ka_k &\le \gamma|b| + (1+n\lambda)(1-\gamma|b|)\theta(n+1)\sum_{k=n+1}^{\infty} a_k \\ &\le \gamma|b| + [1+n\lambda](1-\gamma|b|)\theta(n+1) \\ &\quad \frac{\gamma|b|}{\left[(1+n\lambda)(\gamma|b|-1)+n+1\right]\theta(n+1)} \\ &\le \frac{\gamma|b|(1+n)}{(1+n\lambda)(\gamma|b|-1)+n+1}.\end{aligned}$$

Hence

$$\sum_{k=n+1}^{\infty} ka_k \le \frac{\gamma|b|(1+n)}{\left[(1+n\lambda)(\gamma|b|-1)+n+1\right]\theta(n+1)} =: \delta, \; (\gamma|b|>1) \quad (3.6)$$

which, by means of the definition (3.2), establishes the inclusion relation (3.4) asserted by Theorem 3.1.

In a similar manner, by applying the assertion (2.3) of Theorem 2.2 instead of the assertion (2.1) of Theorem 2.1 to functions in the classes $R_n(\lambda, b, \gamma)$, we can prove the following relationship.

Theorem 3.2 *If*

$$\delta := \frac{\gamma|b|(n+1)}{(1+n\lambda)\theta(n+1)}, (\lambda \ge 1) \qquad (3.7)$$

then

$$R_n(\lambda, b, \gamma) \subset N_{n,\delta}(e). | \qquad (3.8)$$

4. Neighborhoods for the Classes $S_n^{\alpha}(\lambda, b, \gamma)$ $R_n^{\alpha}(\lambda, b, \gamma)$

In this last section, we determine the neighborhood properties for each the following functions classes $S_n^{\alpha}(\lambda, b, \gamma)$ and $R_n^{\alpha}(\lambda, b, \gamma)$. Here the class $S_n^{\alpha}(\lambda, b, \gamma)$ consists of functions $f(z) \in A(n)$ for which there exists another function $g(z) \in S_n(\lambda, b, \gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha \quad (z \in U; 0 \le \alpha < 1). \qquad (4.1)$$

Analogously, the class $R_n^{\alpha}(\lambda, b, \gamma)$ consists of functions $f(z) \in A(n)$ for which there exists another function $g(z) \in R_n(\lambda, b, \gamma)$ satisfying the inequality (4.1).

Theorem 4.1 *If* $g \in S_n(\lambda, b, \gamma)$ and

$$\alpha = 1 - \frac{\delta\left[(1+n\lambda)(\gamma|b|-1)+n+1\right]\theta(n+1)}{(n+1)\left[(1+n\lambda)(\gamma|b|-1)+n+1\right]\theta(n+1)-\gamma|b|}, \ (\gamma|b| \geq 1) \quad (4.2)$$

then

$$N_{n,\delta}(g) \subset S_n^{\alpha}(\lambda, b, \gamma). \quad (4.3)$$

Proof. Suppose that $f \in N_{n\delta}(g)$. We then find from the definition (3.1) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta$$

Which readily implies the coefficient inequality,

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1} \ (n \in \mathbb{N}).$$

Next, since $g \in S_n(\lambda, b, \gamma)$, we have

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\gamma|b|}{\left[(1+n\lambda)(\gamma|b|-1)+n+1\right]\theta(n+1)} \quad (4.4)$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=n+1}^{\infty}|a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k}$$

$$\leq \frac{\delta}{(n+1)} \frac{([1+n\lambda](\gamma|b|-1)+n+1)\theta(n+1)}{\left[\left[(1+n\lambda)(\gamma|b|-1)+n+1\right]\theta(n+1)-\gamma|b|\right]}$$

$$= 1 - \alpha$$

provided that α is given (4.2). Thus, by definition, $f \in S_n^{\alpha}(\lambda, b, \gamma)$ for α given by (4.2). This evidently completes our proof of Theorem 4.1.

Our proof of Theorem 4.2 is much akin to that of Theorem4.1.

Theorem 4.2 *If* $g \in R_n(\lambda, b, \gamma)$ *and*

$$\alpha = 1 - \frac{\delta[1+n\lambda]\theta(n+1)}{(n+1)\left[(1+n\lambda)\theta(n+1)-\gamma|b|\right]}, \ (\gamma|b| \geq 1) \quad (4.5)$$

then

$$N_{n,\delta}(g) \subset R_n^{\alpha}(\lambda, b, \gamma). \quad (4.6)$$

Concluding Remarks: By suitably specializing the various parameters involved in the operator $\tilde{I}^{(a_i),(\delta_i)}_{(\beta_i),m}$ and choosing $\lambda = 0$ and $\lambda = 1$ as illustrated in Examples 1, 2 and 3, we define various subclasses of $S_n(\lambda, b, \gamma)$, $R_n(\lambda, b, \gamma)$. Further from, Theorem 2.1 to Theorem 4.2, we can state the corresponding results for the new subclasses defined in Example 1 to 3 and also for many relatively more familiar function classes.

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