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Application of Pascal distribution series to Rønning type starlike and convex functions

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Abstract

In this article we investigate the connections between the Pascal distribution series and the class of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and its coefficients are probabilities of the Pascal distribution.More precisely ,we determine such connection with parabolic starlike and uniformly convex functions in the open unit disk $\mathbb{U}$.

Keywords: Starlike functions Convex functions Uniformly Starlike functions Uniformly Convex functions Hadamard product Pascal distribution series.
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1. Introduction

Let $\mathbb{U}$ represent the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}$ represent the set of analytic functions in $\mathbb{U}$. We suppose $\mathcal{A}$ denote the subset of $\mathcal{H}$ comprising of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathbb{U}, \tag{1}$$

normalized by $f(0) = 0 = f'(0) - 1$ and univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ whose members are

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{2}$$

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For functions $f_1(z) = z + \sum_{n=2}^{\infty} a_{n,1}z^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} a_{n,2}z^n$, in $\mathcal{A}$ then the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n, \quad z \in \mathbb{U}.$$

For $0 \leq \alpha < 1$, we let the well known subclasses of $\mathcal{A}$ as below:

1. $\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\}$
2. $\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}$
   and
3. $\mathcal{R}(\alpha) = \{ f \in \mathcal{A} : \Re\left(f'(z)\right) > \alpha \}$

where $z \in \mathbb{U}$. Obviously $\mathcal{S}^*(0) =: \mathcal{S}^*$, Further, $\mathcal{K} = \mathcal{K}(0)$. Further, note that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

Due to Ali et al., [1] and Murugusundaramoorthy et al., [9] we state $\mathcal{M}_\mu(\vartheta, \nu)$ and $\mathcal{N}_\mu(\vartheta, \nu)$ the subclasses of $\mathcal{A}$ as below:

For some ϑ $(0 \leq \vartheta < 1)$, μ $(0 \leq \mu \leq 1)$, $\nu \geq 0$ and $f \in \mathcal{A}$ be given by (1), we let $f \in \mathcal{M}_\mu(\vartheta, \nu)$ if it satisfy the analytic criteria

$$\Re\left(\frac{zf'(z)}{(1-\mu)z + \mu f(z)} - \vartheta\right) > \nu \left| \frac{zf'(z)}{(1-\mu)z + \mu f(z)} - 1 \right|, \ z \in \mathbb{U}$$

and also let $f \in \mathcal{N}_\mu(\vartheta, \nu)$, if it satisfy the criteria

$$\Re\left(\frac{zf'(z) + z^2f''(z)}{(1-\mu)z + \mu zf'(z)} - \vartheta\right) > \nu \left| \frac{zf'(z) + z^2f''(z)}{(1-\mu)z + \mu zf'(z)} - 1 \right|, \ z \in \mathbb{U}.$$

Note that$\mathcal{M}_1(\vartheta, \nu) \equiv \mathcal{S}_{\mathcal{P}}(\vartheta, \nu)$ and $\mathcal{N}_1(\vartheta, \nu) \equiv \mathcal{UCV}(\vartheta, \nu)$[3]. Further, denote $\mathcal{M}^*_\mu(\vartheta, \nu) = \mathcal{M}_\mu(\vartheta, \nu) \cap \mathcal{T}$ and $\mathcal{N}^*_\mu(\vartheta, \nu) = \mathcal{N}_\mu(\vartheta, \nu) \cap \mathcal{T}$, the subclasses of $\mathcal{T}$ also specializing the parameters we note the following:

1. $\mathcal{M}^*_1(\vartheta, \nu) \equiv \mathcal{TS}_{\mathcal{P}}(\vartheta, \nu)$[3]
2. $\mathcal{M}^*_1(0, \nu) \equiv \mathcal{TS}_{\mathcal{P}}(\nu)$[14]
3. $\mathcal{N}^*_1(\vartheta, \nu) \equiv \mathcal{UCT}(\vartheta, \nu)$[3]
4. $\mathcal{N}^*_1(0, \nu) \equiv \mathcal{UCT}(0\nu)$[14]

Example 1.1. *For some $\vartheta(0 \leq \vartheta < 1)$, $\nu \geq 0$, and fixing $\mu = 0$ and $f \in \mathcal{A}$ be given by(1),we let (i)$\mathcal{M}_0(\vartheta, \nu) \equiv \mathcal{USD}(\vartheta, \nu)$ if*

$$\Re\left(f'(z) - \vartheta\right) > \nu \left|f'(z) - 1\right| \ z \in \mathbb{U}$$

(ii) $\mathcal{N}_0(\vartheta, \nu) \equiv \mathcal{UCD}(\vartheta, \nu)$ *if*

$$\Re\left((zf'(z))' - \vartheta\right) > \nu \left|(zf'(z))' - 1\right|, \ z \in \mathbb{U}.$$

Murugusundaramoorthy et al., [9] have studied $\mathcal{M}_\mu(\vartheta, \nu)$ and $\mathcal{N}_\mu(\vartheta, \nu)$ based on Hurwitz-zeta functions. To prove our main results we need the following results, proved and stated (a special cases given) in [9].

Lemma 1.1. *[9] Let $f \in \mathcal{A}$ be given by (1),then f belongs to the class*

1. $\mathcal{M}_\mu(\vartheta, \nu)$ *if*

$$\sum_{n=2}^{\infty} [n(1+\nu) - \mu(\vartheta + \nu)]|a_n| \leq 1 - \vartheta. \tag{3}$$

2. $\mathcal{N}_\mu(\vartheta,\nu)$ *if*

$$\sum_{n=2}^{\infty} n[n(1+\nu)-\mu(\vartheta+\nu)]|a_n| \leq 1-\vartheta. \tag{4}$$

3. $\mathcal{S}_{\mathcal{P}}(\vartheta,\nu)$ *if*

$$\sum_{n=2}^{\infty}[n(1+\nu)-(\vartheta+\nu)]|a_n| \leq 1-\vartheta.$$

4. $\mathcal{UCV}(\vartheta,\nu)$ *if*

$$\sum_{n=2}^{\infty} n[n(1+\nu)-(\vartheta+\nu)]|a_n| \leq 1-\vartheta.$$

Lemma 1.2. *[9] Let $f \in \mathcal{A}$ be given by (1),then f belongs to the class*

1. $\mathcal{USD}(\vartheta,\nu)$ *if*

$$\sum_{n=2}^{\infty} n(1+\nu)|a_n| \leq 1-\vartheta.$$

2. $\mathcal{UCD}(\vartheta,\nu)$ *if*

$$\sum_{n=2}^{\infty} n^2(1+\nu)|a_n| \leq 1-\vartheta.$$

Remark 1.1. *The conditions given in Lemma 1.1 and 1.2 are both necessary and sufficient if $f \in \mathcal{T}$ be given by (2).*

Special functions (series) play a vital role in geometric function theory, exclusively in the proof by de Branges of the famous Bieberbach conjecture. The astonishing use of special functions (hypergeometric functions) has provoked renewed attention in function theory in the last few decades (see[4, 6, 12, 16, 17]) and lately by probability distribution series [2, 5, 8, 10, 11].

A variable χ is said to be Pascal distribution if it takes the values $0,1,2,3,\dots$ with probabilities $(1-q)^\kappa, \frac{q\kappa(1-q)^\kappa}{1!}, \frac{q^2\kappa(\kappa+1)(1-q)^\kappa}{2!}, \frac{q^3\kappa(\kappa+1)(\kappa+2)(1-q)^\kappa}{3!},\dots$ respectively, where q and κ are called the parameter,and thus

$$P(\chi=\varrho)=\binom{\varrho+\kappa-1}{\kappa-1}q^\varrho(1-q)^\kappa, \varrho=0,1,2,3,\dots$$

Lately, for $\kappa \geq 1; 0 \leq q \leq 1$, El-Deeb et al.[5] gave a power series whose coefficients are probabilities of Pascal distribution

$$\Phi_q^m(z)=z+\sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^\kappa z^n, \qquad z \in \mathbb{U} \tag{5}$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity. More recently, Bulboacă and Murugusundaramoorthy [2] introduced a linear operator by the convolution (or Hadamard) product

$$\mathcal{I}_q^\kappa : \mathcal{A} \to \mathcal{A}$$

which is defined as follows:

$$\mathcal{I}_q^\kappa f(z)=\Phi_q^\kappa(z) * f(z)=z+\sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^\kappa a_n z^n, \qquad z \in \mathbb{U} \tag{6}$$

Motivated by the aforementioned works on hypergeometric functions [4, 6, 12, 16, 17], and distribution function [2, 5, 8, 10, 11] we give the connections between Pascal distribution series with the classes $\mathcal{M}_\mu^*(\vartheta,\nu)$ and $\mathcal{N}_\mu^*(\vartheta,\nu)$ by applying the convolution operator given by (6).

For convenience throughout in the sequel, let $m \geq 1; 0 \leq q \leq 1$ and following notations:

$$\sum_{n=0}^{\infty}\binom{n+\kappa-1}{\kappa-1}q^n = \frac{1}{(1-q)^{\kappa}} \tag{7}$$

$$\sum_{n=0}^{\infty}\binom{n+\kappa}{\kappa}q^n = \frac{1}{(1-q)^{\kappa+1}} \tag{8}$$

$$\sum_{n=0}^{\infty}\binom{n+\kappa+1}{\kappa+1}q^n = \frac{1}{(1-q)^{\kappa+2}} \tag{9}$$

Theorem 1.1. *If* $\kappa \geq 1$ *then* $\Phi_q^{\kappa}(z) \in \mathcal{M}_{\mu}(\vartheta, \nu)$ *if*

$$\frac{(1+\nu)q\kappa}{(1-q)} + [(1+\nu) - \mu(\vartheta+\nu)]\left(1-(1-q)^{\kappa}\right) \leq 1-\vartheta. \tag{10}$$

Proof. Since $\Phi_q^{\kappa}(z) = z + \sum\limits_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa}z^n \in \mathcal{M}_{\mu}(\vartheta,\nu)$ by virtue of Lemma 1.1 and (3) it suits to show that

$$\mathfrak{L}_1(\kappa,\mu,\vartheta,\nu) = \sum_{n=2}^{\infty}[n(1+\nu) - \mu(\vartheta+\nu)]\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa} \leq 1-\vartheta.$$

Now by writing $n = (n-1)+1$ we get

$$\begin{aligned}
\mathfrak{L}_1(\kappa,\mu,\vartheta,\nu) &= (1+\nu)\sum_{n=2}^{\infty} n\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa} \\
&- \mu(\vartheta+\nu)\sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa} \\
&= (1+\nu)(1-q)^{\kappa}\sum_{n=2}^{\infty}(n-1)\binom{n+\kappa-2}{\kappa-1}q^{n-1} \\
&+ (1-q)^{\kappa}[(1+\nu)-\mu(\vartheta+\nu)]\sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1} \\
&= (1+\nu)(1-q)^{\kappa}\sum_{n=2}^{\infty}q\kappa\binom{n+\kappa-2}{\kappa}q^{n-2} \\
&+ (1-q)^{\kappa}[(1+\nu)-\mu(\vartheta+\nu)]\sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}.q^{n-1}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{L}_1(\kappa,\mu,\vartheta,\nu) = &\ (1+\nu)(1-q)^{\kappa}\sum_{n=0}^{\infty}q\kappa\binom{n+\kappa}{\kappa}q^n \\
+ &\ (1-q)^{\kappa}[(1+\nu)-\mu(\vartheta+\nu)]\left(\sum_{n=0}^{\infty}\binom{n+\kappa-1}{\kappa-1}q^n - 1\right) \\
\leq &\ (1+\nu)(1-q)^{\kappa}q\kappa\frac{1}{(1-q)^{\kappa+1}} \\
+ &\ (1-q)^{\kappa}[(1+\nu)-\mu(\vartheta+\nu)]\left(\frac{1}{(1-q)^{\kappa}} - 1\right) \\
= &\ \frac{(1+\nu)q\kappa}{(1-q)} + [(1+\nu)-\mu(\vartheta+\nu)]\left(1-(1-q)^{\kappa}\right).
\end{aligned}$$

But $\mathfrak{L}_1(\kappa,\mu,\vartheta,\nu)$ is bounded above by $1-\vartheta$ if (10) holds, which completes the proof. □

Theorem 1.2. *If* $\kappa \geq 1$ *then* $\Phi_q^\kappa(z), \in \mathcal{N}_\mu(\vartheta,\nu)$ *if*

$$\begin{aligned} & \frac{(1+\nu)\kappa(\kappa+1)q^2}{(1-q)^2} + \frac{[3(1+\nu)-\mu(\vartheta+\nu)]q\kappa}{1-q} \\ + \quad & [(1+\nu)-\mu(\vartheta+\nu)]\left(1-(1-q)^\kappa\right) \leq 1-\vartheta. \end{aligned} \tag{11}$$

Proof. Since $\Phi_q^\kappa(z) = z + \sum\limits_{n=2}^{\infty} \binom{n+\kappa-2}{\kappa-1} q^{n-1}(1-q)^\kappa z^n \in \mathcal{N}_\mu(\vartheta,\nu)$
according to Lemma 1.1 and (4), it enough to show that

$$\sum_{n=2}^{\infty} n[n(1+\nu)-\mu(\vartheta+\nu)] \binom{n+\kappa-2}{\kappa-1} q^{n-1}(1-q)^\kappa \leq 1-\vartheta.$$

Let

$$\mathfrak{L}_2(\kappa,\mu,\vartheta,\nu) = \sum_{n=2}^{\infty} \left(n^2(1+\nu)-n\mu(\vartheta+\nu)\right) \binom{n+\kappa-2}{\kappa-1} q^{n-1}(1-q)^\kappa.$$

Taking $n = 1+(n-1)$ and $n^2 = 1+3(n-1)+(n-1)(n-2)$, we can rewrite the above term as

$$\begin{aligned} \mathfrak{L}_2(\kappa,\mu,\vartheta,\nu) \quad = \quad & (1+\nu)(1-q)^\kappa \sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+\kappa-2}{\kappa-1}q^{n-1} \\ + \quad & [3(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty}(n-1)\binom{n+\kappa-2}{\kappa-1}q^{n-1} \\ + \quad & [(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1}. \end{aligned}$$

That is,

$$\begin{aligned} \mathfrak{L}_2(\kappa,\mu,\vartheta,\nu) = & (1+\nu)q^2(1-q)^\kappa \sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+\kappa-2}{\kappa-1}q^{n-3} \\ & + [3(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty} q\kappa(n-1)\binom{n+\kappa-2}{\kappa}q^{n-2} \\ & + [(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1} \\ = & (1+\nu)q^2(1-q)^\kappa \sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+\kappa-2}{\kappa-1}q^{n-3} \\ & + [3(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty} q\kappa(n-1)\binom{n+\kappa-2}{\kappa}q^{n-2} \\ & + [(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1} \\ = & (1+\nu)q^2(1-q)^\kappa \sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+\kappa-2}{\kappa-1}q^{n-3} \\ & + [3(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty} q\kappa(n-1)\binom{n+\kappa-2}{\kappa}q^{n-2} \\ & + [(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa \sum_{n=2}^{\infty}\binom{n+\kappa-2}{\kappa-1}q^{n-1} \end{aligned}$$

$$
\begin{aligned}
= &\quad (1+\nu)\kappa(\kappa+1)q^2(1-q)^\kappa \sum_{n=0}^{\infty}\binom{n+\kappa+1}{\kappa+1}q^n \\
+ &\quad [3(1+\nu)-\mu(\vartheta+\nu)]q\kappa(1-q)^\kappa \sum_{n=0}^{\infty}\binom{n+\kappa}{\kappa}q^n \\
+ &\quad [(1+\nu)-\mu(\vartheta+\nu)](1-q)^\kappa\left(\frac{1}{(1-q)^\kappa}-1\right) \\
= &\quad \frac{(1+\nu)\kappa(\kappa+1)q^2}{(1-q)^2}+\frac{[3(1+\nu)-\mu(\vartheta+\nu)]q\kappa}{1-q} \\
+ &\quad [(1+\nu)-\mu(\vartheta+\nu)]\left(1-(1-q)^\kappa\right).
\end{aligned}
$$

But $\mathfrak{L}_2(\kappa,\mu,\vartheta,\nu)$ is bounded above by $1-\vartheta$ if (11) holds. Thus the proof is complete. □

Corollary 1.1. *If $\kappa \geq 1$ then*

1. $\Phi_q^\kappa(z) \in \mathcal{S}_{\mathcal{P}}(\vartheta,\nu)$ *if* $\frac{(1+\nu)q\kappa}{(1-q)^{\kappa+1}} \leq 1-\vartheta$
2. $\Phi_q^\kappa(z) \in \mathcal{UCV}(\vartheta,\nu)$ *if* $\frac{(1+\nu)\kappa(\kappa+1)q^2}{(1-q)^{\kappa+2}}+\frac{[3+2\nu-\vartheta)]q\kappa}{(1-q)^{\kappa+1}} \leq 1-\vartheta.$
3. $\Phi_q^\kappa(z) \in \mathcal{USD}(\vartheta,\nu)$ *if* $(1+\nu)\left[\frac{q\kappa}{(1-q)}+1-(1-q)^\kappa\right] \leq 1-\vartheta$
4. $\Phi_q^\kappa(z) \in \mathcal{UCD}(\vartheta,\nu)$ *if* $(1+\nu)\left[\frac{\kappa(\kappa+1)q^2}{(1-q)^2}+\frac{3q\kappa}{1-q}+1-(1-q)^\kappa\right] \leq 1-\vartheta.$

2. Inclusion Properties

The class $\mathcal{R}^\tau(\upsilon,\delta)$ was given by Swaminathan [17] (for special cases see the references cited there in) and for $f \in \mathcal{R}^\tau(\upsilon,\delta)$ he proved the result given below:

Let $f \in \mathcal{A}$ be in $\mathcal{R}^\tau(\upsilon,\delta)$, $(\tau \in \mathbb{C}\backslash\{0\},\ 0<\upsilon\leq 1; \delta<1)$, if it holds the inequality

$$\left|\frac{(1-\upsilon)\frac{f(z)}{z}+\upsilon f'(z)-1}{2\tau(1-\delta)+(1-\upsilon)\frac{f(z)}{z}+\upsilon f'(z)-1}\right|<1, \quad (z\in\mathbb{U}).$$

Lemma 2.1. *[17] If $f \in \mathcal{R}^\tau(\upsilon,\delta)$ is of form (1), then*

$$|a_n| \leq \frac{2\,|\tau|\,(1-\delta)}{1+\upsilon(n-1)}, \quad n\in\mathbb{N}\setminus\{1\}. \tag{12}$$

The bounds given in (12) is sharp.

Making use of the Lemma 2.1,in the following theorem we will establish the connection between Pascal distribution series with the class $\mathcal{N}_\mu(\alpha,\nu)$.

Theorem 2.1. *If $\kappa\geq 1$ and $f\in\mathcal{R}^\tau(\upsilon,\delta)$, if the inequality*

$$\left[\frac{(1+\nu)q\kappa}{(1-q)}+[(1+\nu)-\mu(\vartheta+\nu)]\left(1-(1-q)^\kappa\right)\right] \leq \frac{\upsilon(1-\vartheta)}{2\,|\tau|\,(1-\delta)} \tag{13}$$

is satisfied, then $\mathcal{I}_q^\kappa f(z)\in\mathcal{N}_\mu(\alpha,\nu)$.

Proof. Let f be given by (1) and a member of $\mathcal{R}^{\tau}(\upsilon,\delta)$. By asset of Lemma 1.1 and (4) it suits to show that

$$\sum_{n=2}^{\infty} n[n(1+\nu)-\mu(\vartheta+\nu)]\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa}|a_n| \leq 1-\vartheta.$$

Since $f \in \mathcal{R}^{\tau}(\mu,\delta)$ then by Lemma 12 we have

$$|a_n| \leq \frac{2\,|\tau|\,(1-\delta)}{1+\upsilon(n-1)}, \quad n \in \mathbb{N}\setminus\{1\}.$$

Let

$$\begin{aligned}\mathfrak{L}_3(\kappa,\mu,\vartheta,\nu) &= \sum_{n=2}^{\infty} n[n(1+\nu)-\mu(\vartheta+\nu)]\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa}|a_n| \\ &\leq \; 2\,|\tau|\,(1-\delta)\sum_{n=2}^{\infty} n\frac{[n(1+\nu)-\mu(\vartheta+\nu)]}{1+\mu(n-1)}\binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa}.\end{aligned}$$

Since $1+\upsilon(n-1) \geq n\upsilon$, we get

$$\begin{aligned}\mathfrak{L}_3(\kappa,\mu,\vartheta,\nu) &\leq \frac{2\,|\tau|\,(1-\delta)}{\upsilon}\sum_{n=2}^{\infty}[n(1+\nu)-\mu(\vartheta+\nu)] \\ &\times \binom{n+\kappa-2}{\kappa-1}q^{n-1}(1-q)^{\kappa}.\end{aligned}$$

Proceeding as in Theorem 1.1, we get

$$\begin{aligned}&\mathfrak{L}_3(\kappa,\mu,\vartheta,\nu) \\ \leq \; &\frac{2\,|\tau|\,(1-\delta)}{\upsilon}\left[\frac{(1+\nu)q\kappa}{(1-q)} + [(1+\nu)-\mu(\vartheta+\nu)]\,(1-(1-q)^{\kappa})\right].\end{aligned}$$

But the expression $\mathfrak{L}_3(\kappa,\mu,\vartheta,\nu)$ is bounded above by $1-\vartheta$ if (15) holds. Thus the proof is complete. □

Corollary 2.1. *If $\kappa \geq 1$ and $f \in \mathcal{R}^{\tau}(\upsilon,\delta)$, if the inequality*

$$\left[\frac{(1+\nu)\kappa(\kappa+1)q^2}{(1-q)^{\kappa+2}} + \frac{[3+2\nu-\vartheta)]q\kappa}{(1-q)^{\kappa+1}}\right] \leq \frac{\upsilon(1-\vartheta)}{2\,|\tau|\,(1-\delta)} \tag{14}$$

is satisfied, then $\mathcal{I}_q^{\kappa}f(z) \in \mathcal{UCV}(\vartheta,\nu)$.

Corollary 2.2. *If $\kappa \geq 1$ and $f \in \mathcal{R}^{\tau}(\upsilon,\delta)$, if the inequality*

$$(1+\nu)\left[\frac{\kappa(\kappa+1)q^2}{(1-q)^2} + \frac{3q\kappa}{1-q} + 1-(1-q)^{\kappa}\right] \leq \frac{\upsilon(1-\vartheta)}{2\,|\tau|\,(1-\delta)} \tag{15}$$

is satisfied, then $\mathcal{I}_q^{\kappa}f(z) \in \mathcal{UCD}(\vartheta,\nu)$.

Remark 2.1. *The above conditions are also necessary for functions $\Phi_q^{\kappa}(z)$ of the form(5).*

Theorem 2.2. *Let $\kappa \geq 1$, and $\mathcal{L}(\kappa,z) = \int_0^z \frac{\mathcal{I}_q^{\kappa}(t)}{t}dt$ then $\mathcal{L}(\kappa,z) \in \mathcal{N}_{\mu}(\vartheta,\nu)$ if and only if*

$$\frac{(1+\nu)q\kappa}{(1-q)} + [(1+\nu)-\mu(\vartheta+\nu)]\,(1-(1-q)^{\kappa}) \leq 1-\vartheta. \tag{16}$$

Proof. Since

$$\mathcal{L}(\kappa, z) = z + \sum_{n=2}^{\infty} \binom{n+\kappa-2}{\kappa-1} q^{n-1}(1-q)^{\kappa} \frac{z^n}{n}$$

then by Theorem 1.1 we requisite to confirm that

$$\sum_{n=2}^{\infty} n[n(1+\nu) - \mu(\vartheta+\nu)] \frac{1}{n} \binom{n+\kappa-2}{\kappa-1} q^{n-1}(1-q)^{\kappa} \leq 1-\vartheta.$$

That is,

$$\sum_{n=2}^{\infty} [n(1+\nu) - \mu(\vartheta+\nu)] \binom{n+\kappa-2}{\kappa-1} q^{n-1}(1-q)^{\kappa} \quad \leq \quad 1-\vartheta.$$

Now by expressing $n = (n-1)+1$ and following the lines of Theorem 1.1, we get

$$\sum_{n=2}^{\infty} [n(1+\nu) - \mu(\vartheta+\nu)] \binom{n+\kappa-2}{\kappa-1} q^{n-1}(1-q)^{\kappa}$$
$$= \left[\frac{(1+\nu)q\kappa}{(1-q)} + [(1+\nu) - \mu(\vartheta+\nu)] \left(1-(1-q)^{\kappa}\right) \right]$$

which is bounded above by $1-\vartheta$ if (16) holds. □

Corollary 2.3. *Let* $\kappa \geq 1$ *and* $\mathcal{L}(\kappa, z) = \int_0^z \frac{\mathcal{I}_q^{\kappa}(t)}{t} dt$*,then*

1. $\mathcal{L}(\kappa, z) \in \mathcal{UCT}(\vartheta, \nu) \quad \Leftrightarrow \quad \frac{(1+\nu)q\kappa}{(1-q)^{\kappa+1}} \leq 1-\vartheta,$
   and
2. $\mathcal{L}(\kappa, z) \in \mathcal{UCD}(\vartheta, \nu) \quad \Leftrightarrow \quad (1+\nu)\left(\frac{q\kappa}{1-q} + 1 - (1-q)^m\right) \leq 1-\vartheta.$

Concluding Remark: By specializing $\mu = 0$ or $\mu = 1$ and fixing $\vartheta = 0$ in Theorems proved in present paper ,one can deduce for the classes studied in [14] and similar manner by taking $\nu = 0$ we can easily deduce for the function classes studied in [13].The details involved may be port as an exercise for the attracted reader.

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