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Approximation of B-continuous and B-differentiable functions by GBS operators of q-Bernstein–Schurer–Stancu type

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Abstract: Bărbosu and Muraru (2015) introduced the bivariate generalization of the q-Bernstein–Schurer–Stancu operators and constructed a GBS operator of q-Bernstein–Schurer–Stancu type. The concern of this paper is to obtain the rate of convergence in terms of the partial and complete modulus of continuity and the degree of approximation by means of Lipschitz-type class for the bivariate operators. In the last section we estimate the degree of approximation by means of Lipschitz class function and the rate of convergence with the help of mixed modulus of smoothness for the GBS operator of q-Bernstein–Schurer–Stancu type. Furthermore, we show comparisons by some illustrative graphics in Maple for the convergence of the operators to some functions.

Key words: q-Bernstein–Schurer–Stancu operators, partial moduli of continuity, B-continuous, B-differentiable, GBS operators, modulus of smoothness, degree of approximation

1. Introduction

In 1987, q-based Bernstein operators were defined and studied by Lupas [21]. In 1997, another q-based Bernstein operator was proposed by Phillips [23]. Since then q-based operators have become an active research area. Muraru [22] introduced and investigated the q-Bernstein–Schurer operators. She obtained the Korovkin-type approximation theorem and the rate of convergence of the operators in terms of the first modulus of continuity. The term B-continuous and B-differentiable function was first introduced by Bögel in [12] and [13] wherein he studied some important results using these concepts. Further, in 1966, Dobrescu and Matei [15] gave some approximation properties for bivariate Bernstein polynomials using a generalized boolean sum. The Korovkin-type theorem for approximation of B-continuous functions using GBS operators is due to Badea et al. [5]. A very well known Shisha–Mond theorem [27] for B-continuous functions is given by Badea et al. [6]. GBS operators of Schurer–Stancu type were introduced by Bărbosu [7]. Agrawal et al. [1] defined bivariate q-Bernstein–Schurer–Kantorovich operators by using q-Riemann integral and studied the rate of convergence of these operators. Sidharth et al.[28] introduced GBS operators of Bernstein–Schurer–Kantorovich type and studied the degree of approximation by means of the mixed modulus of smoothness and the mixed Peetre's K-functional. For some important contributions in this direction we refer to [cf. [8, 10, 11, 17–19, 24–26] etc.].

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Very recently Bărbosu and Muraru [9] defined the q-Bernstein–Schurer–Stancu operators for the bivariate case as follows:

Let p_1, p_2 be nonnegative integers, $I = [0, 1 + p_1] \times [0, 1 + p_2]$ and $J = [0, 1] \times [0, 1]$.

Let $\{q_m\}$ and $\{q_n\}$ be sequences in $(0, 1)$ such that $q_m \to 1$, $q_m^m \to a \, (0 \leq a < 1)$, as $m \to \infty$ and $q_n \to 1$, $q_n^n \to b \, (0 \leq b < 1)$, as $n \to \infty$. Further, let $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$ and

$$S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C(I) \to C(J)$$

then for any $f \in C(I)$ we have

$$\begin{aligned} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f; q_m, q_n, x, y) &= \sum_{k_1=0}^{m+p_1} \sum_{k_2=0}^{n+p_2} \begin{bmatrix} m+p_1 \\ k_1 \end{bmatrix}_{q_m} \begin{bmatrix} n+p_2 \\ k_2 \end{bmatrix}_{q_n} \prod_{s=0}^{m+p_1-k_1-1} (1 - q_m^s x) \\ &\quad \times \prod_{r=0}^{n+p_2-k_2-1} (1 - q_n^r y) x^{k_1} y^{k_2} f_{k_1,k_2}, \end{aligned} \tag{1.1}$$

where $f_{k_1,k_2} = f\left(\frac{[k_1]_{q_m}+\alpha_1}{[m]_{q_m}+\beta_1}, \frac{[k_2]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right)$.

Let X and Y be compact subsets of $\mathbb{R}$. A function $f : X \times Y \longrightarrow$ R is called a B-continuous (Bögel continuous) function at $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x, y); (x_0, y_0)] = 0,$$

where $\Delta f[(x, y); (x_0, y_0)]$ denotes the mixed difference defined by

$$\Delta f[(x, y); (x_0, y_0)] = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0).$$

For any $(x, y) \in J$, the q-GBS operator of Bernstein–Schurer–Stancu type $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C(I) \to C(J)$, associated to $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ is defined as:

$$\begin{aligned} U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f(t,s); q_m, q_n, x, y) &= S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f(t,y) + f(x,s) - f(t,s); q_m, q_n, x, y) \\ &= \sum_{k_1=0}^{m+p_1} \sum_{k_2=0}^{n+p_2} \begin{bmatrix} m+p_1 \\ k_1 \end{bmatrix}_{q_m} \begin{bmatrix} n+p_2 \\ k_2 \end{bmatrix}_{q_n} \prod_{s=0}^{m+p_1-k_1-1} (1 - q_m^s x) \\ &\quad \times \prod_{r=0}^{n+p_2-k_2-1} (1 - q_n^r y) x^{k_1} y^{k_2} \{f_{k_1} + f_{k_2} - f_{k_1,k_2}\}, \end{aligned} \tag{1.2}$$

where

$$f_{k_1}(y) = f\left(\frac{[k_1]_{q_m}+\alpha_1}{[m]_{q_m}+\beta_1}, y\right), \; f_{k_2}(x) = f\left(x, \frac{[k_2]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right), \; f_{k_1,k_2} = f\left(\frac{[k_1]_{q_m}+\alpha_1}{[m]_{q_m}+\beta_1}, \frac{[k_2]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right).$$

In what follows, let $\|\cdot\|_{C(I)}$ denote the sup-norm on I.

2. Preliminaries

Lemma 1 *[9] Let $e_{i.j}: I \to I, e_{i,j}(x,y) = x^i y^j (0 \le i+j \le 2, i,j (integers))$ be the test functions. Then the following equalities hold for the operators given by (1.1):*

(i) $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_{0,0}; q_m, q_n, x, y) = e_{0,0}(x,y),$

(ii) $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_{1,0}; q_m, q_n, x, y) = \frac{[m+p_1]_{q_m} x + \alpha_1}{[m]_{q_m} + \beta_1},$

(iii) $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_{0,1}; q_m, q_n, x, y) = \frac{[n+p_2]_{q_n} y + \alpha_2}{[n]_{q_n} + \beta_2},$

(iv) $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_{2,0}; q_m, q_n, x, y) = \frac{1}{([m]_{q_m} + \beta_1)^2} \left([m+p_1]_{q_m}^2 x^2 + [m+p_1]_{q_m} x(1-x) + 2\alpha_1 [m+p_1]_{q_m} x + \alpha_1^2\right),$

(v) $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_{0,2}; q_m, q_n, x, y) = \frac{1}{([n]_{q_n} + \beta_2)^2} \left([n+p_2]_{q_n}^2 y^2 + [n+p_2]_{q_n} y(1-y) + 2\alpha_2 [n+p_2]_{q_n} y + \alpha_2^2\right).$

Lemma 2 *For $(x,y) \in J$, we have*

(i) $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x)^2; q_m, q_n, x, y) = \frac{1}{([m+\beta_1]_{q_m})^2}\{((q_m^m [p_1]_{q_m} - \beta_1)x + \alpha_1)^2 + [m+p_1]_{q_m} x(1-x)\},$

(ii) $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((s-y)^2; q_m, q_n, x, y) = \frac{1}{([n+\beta_2]_{q_n})^2}\{((q_n^n [p_2]_{q_n} - \beta_2)y + \alpha_2)^2 + [n+p_2]_{q_n} y(1-y)\}.$

Lemma 3 *For $(x,y) \in J$, we have*

(i) $\lim_{m\to\infty} [m]_{q_m} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x); q_m, q_n, x, y) = \alpha_1 - \beta_1 x,$

(ii) $\lim_{n\to\infty} [n]_{q_n} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((s-y); q_m, q_n, x, y) = \alpha_2 - \beta_2 y,$

(iii) $\lim_{m\to\infty} [m]_{q_m} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x)^2; q_m, q_n, x, y) = x(1-x),$

(iv) $\lim_{n\to\infty} [n]_{q_n} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((s-y)^2; q_m, q_n, x, y) = y(1-y).$

Similarly, it can be shown that

$$S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x)^4; q_m, q_n, x, y) = O(\frac{1}{[m]_{q_m}^2}), \text{ as } m \to \infty \text{ uniformly in } x \in [0,1], \tag{2.1}$$

and

$$S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((s-y)^4; q_m, q_n, x, y) = O(\frac{1}{[n]_{q_n}^2}), \text{ as } n \to \infty \text{ uniformly in } y \in [0,1]. \tag{2.2}$$

3. Main results

For $f \in C(I)$, the complete modulus of continuity for the bivariate case is defined as follows:

$$\bar{\omega}(f;\delta_1,\delta_2) = \sup\left\{|f(t,s)-f(x,y)| : (t,s),(x,y)\in I \text{ and } |t-x|\le\delta_1,\ \ |s-y|\le\delta_2\right\},$$

where $\bar{\omega}(f,\delta_1,\delta_2)$ satisfies the following properties:

(i) $\bar{\omega}(f,\delta_1,\delta_2)\to 0$, if $\delta_1\to 0$ and $\delta_2\to 0$;

(ii) $|f(t,s)-f(x,y)| \le \bar{\omega}(f,\delta_1,\delta_2)\left(1+\dfrac{|t-x|}{\delta_1}\right)\left(1+\dfrac{|s-y|}{\delta_2}\right)$.

Details of the complete modulus of continuity for the bivariate case can be found in [2].

Further, the partial moduli of continuity with respect to x and y are given by

$$\omega_1(f;\delta) = \sup\left\{|f(x_1,y)-f(x_2,y)| : y\in[0,1+p_2] \ \ and \ \ |x_1-x_2|\le\delta\right\},$$

and

$$\omega_2(f;\delta) = \sup\left\{|f(x,y_1)-f(x,y_2)| : x\in[0,1+p_1] \ \ and \ \ |y_1-y_2|\le\delta\right\}.$$

It is clear that they satisfy the properties of the usual modulus of continuity.

Let $C^2(I) := \left\{f\in C(I) : f_{xx}, f_{xy}, f_{yx}, f_{yy}\in C(I)\right\}$.

The norm on the space $C^2(I)$ is defined as

$$||f||_{C^2(I)} = ||f||_{C(I)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(I)} + \left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(I)}\right).$$

For $f\in C(I)$, let us consider the following K-functional:

$$K_2(f,\delta) = inf\{||f-g||_{C(I)} + \delta||g||_{C^2(I)} : g\in C^2(I)\}, \tag{3.1}$$

where $\delta>0$.

By [16], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta) \le C\bar{\omega}_2(f,\sqrt{\delta}), \tag{3.2}$$

where $\bar{\omega}_2(f,\sqrt{\delta})$ denotes the second order modulus of continuity for the bivariate case.

Let δ_m and δ_n be defined as

$$\begin{aligned} \delta_m &= max_{x\in[0,1]}\{S_{m,p_1}^{(\alpha_1,\beta_1)}((t-x)^2;q_m,x)\}^{1/2} \\ &= \frac{1}{[m]_{q_m}+\beta_1}\sqrt{4\max_{x\in[0,1]}\left(((q_m^m[p_1]_{q_m}-\beta)x+\alpha_1)^2+[m+p_1]_{q_m}\right)}, \end{aligned}$$

$$\text{and} \quad \begin{aligned} \delta_n &= max_{y\in[0,1]}\{S_{n,p_2}^{(\alpha_2,\beta_2)}((s-y)^2;q_n,y)\}^{1/2} \\ &= \frac{1}{[n]_{q_n}+\beta_2}\sqrt{4\max_{y\in[0,1]}\left(((q_n^n[p_2]_{q_n}-\beta_2)y+\alpha_2)^2+[n+p_2]_{q_n}\right)}. \end{aligned}$$

Theorem 1 *Let $f \in C(I)$. Then we have the inequality*

$$||S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,.,.) - f||_{C(J)} \leq 2(\omega_1(f;\delta_m) + \omega_2(f;\delta_n)). \tag{3.3}$$

Proof By the definition of partial moduli of continuity, Lemma 1, and using the Cauchy–Schwarz inequality we may write

$$\begin{aligned}
&|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y) - f(x,y)| \\
\leq\;& S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|f(t,s) - f(x,y)|;q_m,q_n,x,y) \\
\leq\;& S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|f(t,s) - f(t,y)|;q_m,q_n,x,y) + S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|f(t,y) - f(x,y)|;q_m,q_n,x,y) \\
\leq\;& S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(\omega_2(f;|s-y|);q_m,q_n,x,y) + S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(\omega_1(f;|t-x|);q_m,q_n,x,y) \\
\leq\;& \omega_2(f;\delta_n)\left[1 + \frac{1}{\delta_n} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|s-y|;q_m,q_n,x,y)\right] + \omega_1(f;\delta_m)\left[1 + \frac{1}{\delta_m} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|t-x|;q_m,q_n,x,y)\right] \\
\leq\;& \omega_2(f;\delta_n)\left[1 + \frac{1}{\delta_n}\left(S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((s-y)^2;q_m,q_n,x,y)\right)^{1/2}\right] \\
&+\omega_1(f;\delta_m)\left[1 + \frac{1}{\delta_m}\left(S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2;q_m,q_n,x,y)\right)^{1/2}\right] \\
\leq\;& \omega_2(f;\delta_n)\left(1 + \frac{1}{\delta_n}\frac{1}{[n]_{q_n} + \beta_2}\sqrt{4\max_{y\in[0,1]}\left(((q_n^n[p_2]_{q_n} - \beta_2)y + \alpha_2)^2 + [n+p_2]_{q_n}\right)}\right) \\
&+\omega_1(f;\delta_m)\left(1 + \frac{1}{\delta_m}\frac{1}{[m]_{q_m} + \beta_1}\sqrt{4\max_{x\in[0,1]}\left(((q_m^m[p_1]_{q_m} - \beta_1)x + \alpha_1)^2 + [m+p_1]_{q_m}\right)}\right).
\end{aligned}$$

Hence, we achieve the desired result. □

Theorem 2 *Let $f \in C(I)$ and $0 < q_m, q_n < 1$. Then for all $(x,y) \in J$, we have*

$$||S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,.,.) - f||_{C(J)} \leq 4\bar{\omega}(f,\delta_m,\delta_n).$$

Proof Using the linearity and positivity of the operator $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)$, we have

$$\begin{aligned}
&|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y) - f(x,y)| \\
\leq\;& S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|f(t,s) - f(x,y)|;q_m,q_n,x,y) \\
\leq\;& \omega(f;\delta_m,\delta_n)\left(S^{(\alpha_1,\beta_1)}_{m,p_1}(e_0;q_{m_1},x) + \frac{1}{\delta_m}S^{(\alpha_1,\beta_1)}_{m,p_1}(|t-x|;q_m,x)\right) \\
&\times\left(S^{(\alpha_2,\beta_2)}_{n,p_2}(e_0;q_n,y) + \frac{1}{\delta_n}S^{(\alpha_2,\beta_2)}_{n,p_2}(|s-y|;q_n,y)\right).
\end{aligned} \tag{3.4}$$

Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| &\leq \bar{\omega}(f;\delta_m,\delta_n)\left\{\left(1+\frac{1}{\delta_m}\sqrt{S_{m,p_1}^{(\alpha_1,\beta_1)}((t-x)^2;q_m,x)}\right)\right.\\ &\left.\times\left(1+\frac{1}{\delta_n}\sqrt{S_{n,p_2}^{(\alpha_2,\beta_2)}((s-y)^2;q_n,y)}\right)\right\}\\ &\leq 4\bar{\omega}(f;\delta_m,\delta_n).\end{aligned}$$

This completes the proof. □

Example 1 *For* $n,m=10,p_1,p_2=2,\alpha_1=3,\beta_1=4,\alpha_2=5,\beta_2=7,q_1,q_2=0.5,\ q_1,q_2=0.7$, *and* $q_1,q_2=0.9$ *the convergence of the operators* $S_{10,10,2,2}^{(3,4,5,7)}(f;.5,.5,x,y)$ *(yellow),* $S_{10,10,2,2}^{(3,4,5,7)}(f;.7,.7,x,y)$ *(pink),* $S_{10,10,2,2}^{(3,4,5,7)}(f;.9,.9,x,y)$ *(blue) to* $f(x,y)=x\left(x-\frac{1}{4}\right)\left(y-\frac{3}{7}\right)$ *(red) is illustrated by Figure 1.*

Example 2 *For* $m,n=10,\ \alpha_1,\alpha_2=1,\ \beta_1,\beta_2=2,\ p_1,p_2=1$, *the comparison of the convergence of* q*-Bernstein–Schurer–Stancu (blue) given by* $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)$ *and the operators bivariate* q*-Bernstein–Schurer (green),* q*-Bernstein–Stancu (red), to* $f(x,y)=2x\cos(\pi x)y^3$ *(yellow) with* $q_m=m/(m+1),q_n=1-1/\sqrt{n}$ *is illustrated in Figure 2.*

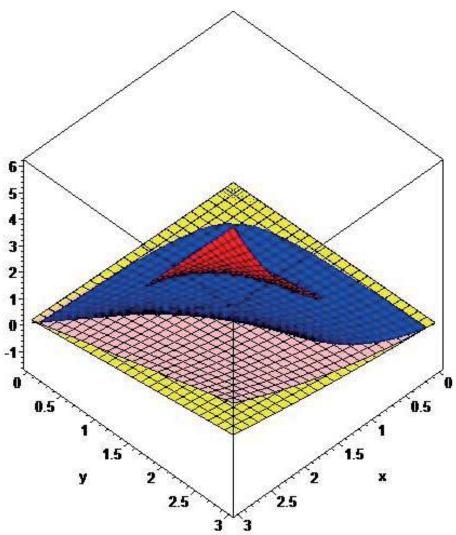

Figure 1

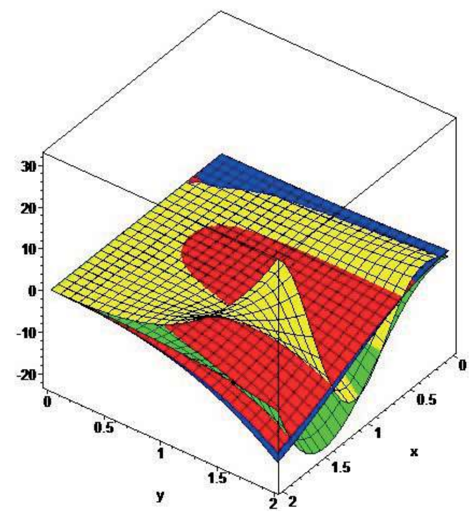

Figure 2

3.1. Degree of approximation

Now we estimate the degree of approximation for the bivariate operators (1.1) by means of the Lipschitz class.

For $0<\xi\leq 1$ and $0<\gamma\leq 1$, for $f\in C(I)$ we define the Lipschitz class $Lip_M(\xi,\gamma)$ for the bivariate case as follows:

$$|f(t,s)-f(x,y)|\leq M|t-x|^{\xi}|s-y|^{\gamma}.$$

Theorem 3 *Let $f \in Lip_M(\xi,\gamma)$. Then we have*

$$||S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,.,.)-f|| \leq M\delta_m^{\xi}\delta_n^{\gamma}.$$

Proof By our hypothesis, we may write

$$\begin{aligned}|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| &\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|f(t,s)-f(x,y)|;q_m,q_n,x,y)\\ &\leq MS_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|t-x|^{\xi}|s-y|^{\gamma};q_m,q_n,x,y)\\ &= MS_{m,p_1}^{(\alpha_1,\beta_1)}(|t-x|^{\xi};q_m,x)S_{n,p_2}^{(\alpha_2,\beta_2)}(|s-y|^{\gamma};q_n,y).\end{aligned}$$

Now, using Hölder's inequality with $u_1=\frac{2}{\xi}$, $v_1=\frac{2}{2-\xi}$ and $u_2=\frac{2}{\gamma}$ and $v_2=\frac{2}{2-\gamma}$, we have

$$\begin{aligned}S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| &\leq MS_{m,p_1}^{(\alpha_1,\beta_1)}((t-x)^2;q_m,x)^{\frac{\xi}{2}}S_{m,p_1}^{(\alpha_1,\beta_1)}(e_0;q_m,x)^{\frac{2-\xi}{2}}\\ &\quad\times S_{n,p_2}^{(\alpha_2,\beta_2)}((s-y)^2;q_n,y)^{\frac{\gamma}{2}}S_{n,p_2}^{(\alpha_2,\beta_2)}(e_0;q_n,y)^{\frac{2-\gamma}{2}}\\ &\leq M\delta_m^{\xi}\delta_n^{\gamma}.\end{aligned}$$

Hence, the proof is completed. □

Theorem 4 *Let $f \in C(I)$ and $(x,y)\in J$. Then we have*

$$|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| \leq \|f_x\|_{C(I)}\delta_m+\|f_y\|_{C(I)}\delta_n.$$

Proof Let $(x,y)\in J$ be a fixed point. Then we can write

$$f(t,s)-f(x,y)=\int_x^t f_u(u,s)d_qu+\int_y^s f_v(x,v)d_qv.$$

Now, applying $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(.;q_m,q_n,x,y)$ on both sides, we have

$$\begin{aligned}|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| &\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_x^t f_u(u,s)d_qu;q_m,q_n,x,y\right)\\ &\quad+S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_y^s f_v(x,v)d_qv;q_m,q_n,x,y\right).\end{aligned}$$

Since

$$\left|\int_x^t f_u(u,s)d_qu\right| \leq \|f_x\|_{C(I)}|t-x| \text{ and } \left|\int_y^s f_v(x,v)d_qv\right| \leq \|f_y\|_{C(I)}|s-y|,$$

we have

$$|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| \leq \|f_x\|_{C(I)}S_{m,p_1}^{(\alpha_1,\beta_1)}(|t-x|;q_m,x)+\|f_y\|_{C(I)}S_{n,p_2}^{(\alpha_2,\beta_2)}(|s-y|;q_n,y).$$

Now, applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned}|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| &\leq \|f_x\|_{C(I)}S_{m,p_1}^{(\alpha_1,\beta_1)}((t-x)^2;q_m,x)^{1/2}S_{m,p_1}^{(\alpha_1,\beta_1)}(e_0;q_m,x)^{1/2}\\ &\quad+\|f_y\|_{C(I)}S_{n,p_2}^{(\alpha_2,\beta_2)}((s-y)^2;q_n,y)^{1/2}S_{n,p_2}^{(\alpha_2,\beta_2)}(e_0;q_n,y)^{1/2}\\ &\leq \|f_x\|_{C(I)}\delta_m+\|f_y\|_{C(I)}\delta_n.\end{aligned}$$

This completes the proof of the theorem. □

Theorem 5 *For the function $f \in C(I)$, we have the following inequality:*

$$|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) - f(x,y)| \quad \leq \quad M\left\{\bar{\omega_2}(f;\sqrt{C_{m,n}}) + min\{1, C_{m,n}\}||f||_{C(I)}\right\} + \omega(f;\psi_{m,n}),$$

where

$$\psi_{m,n} = \sqrt{max_{(x,y)\in J}\left\{\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1} - x\right)^2 + \left(\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2} - y\right)^2\right\}},$$

$$C_{m,n} = \delta_m^2 + \delta_n^2 + \psi_{m,n}^2$$

and the constant $M > 0$ is independent of f and $C_{m,n}$.

Proof We introduce the auxiliary operators as follows:

$$S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) = S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) - f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}, \frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right) + f(x,y);$$

then using Lemma 1, we have

$$S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x);q_m,q_n,x,y) = 0$$

and

$$S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}((s-y);q_m,q_n,x,y) = 0.$$

Let $g \in C^2(I)$ and $t, s \in I$. Using Taylor's theorem, we may write

$$\begin{aligned} g(t,s) - g(x,y) &= g(t,y) - g(x,y) + g(t,s) - g(t,y) \\ &= \frac{\partial g(x,y)}{\partial x}(t-x) + \int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &\quad + \frac{\partial g(x,y)}{\partial y}(s-y) + \int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv. \end{aligned}$$

Applying the operator $S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(.,q_m,q_n,x,y)$ on both sides, we get

$$\begin{aligned} S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) - f(x,y) &= S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;q_m,q_n,x,y\right) \\ &\quad + S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;q_m,q_n,x,y\right) \\ &= S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;q_m,q_n,x,y\right) \\ &\quad - \int_x^{\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}} \left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1} - u\right)\frac{\partial^2 g(u,y)}{\partial u^2}du \end{aligned}$$

$$+S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv; q_m, q_n, x, y\right)$$

$$-\int_x^{\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}} \left(\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}-v\right)\frac{\partial^2 g(x,v)}{\partial v^2}dv.$$

Hence

$$\begin{aligned}
&|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| \\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\left|\int_x^t |t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;x,y\right) \\
&+\left|\int_x^{\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}}\left|\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right| \\
&+S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\left|\int_y^s |s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\right) \\
&+\left|\int_x^{\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}}\left|\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right| \\
&\leq \left\{S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x)^2;q_m,q_n,x,y)+\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}-x\right)^2\right\}||g||_{C^2(I)} \\
&+\left\{S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((s-y)^2;q_m,q_n,x,y)+\left(\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}-y\right)^2\right\}||g||_{C^2(I)} \\
&\leq (\delta_m^2+\delta_n^2+\psi_{m,n}^2)||g||_{C^2(I)} \\
&= C_{m,n}||g||_{C^2(I)}
\end{aligned} \tag{3.5}$$

Moreover, using Lemma 1

$$\begin{aligned}
|S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)| &\leq |S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)|+\left|f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right)\right| \\
&+|f(x,y)| \\
&\leq 3||f||_{C(I)}.
\end{aligned} \tag{3.6}$$

Hence, in view of (3.5) and (3.6), we get

$$|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)|$$

$$\begin{aligned}
&= \Bigg|S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y) \\
&+f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right)-f(x,y)\Bigg| \\
&\leq |S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f-g;q_m,q_n,x,y)|+|S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(g;q_m,q_n,x,y)-g(x,y)|
\end{aligned}$$

$$
\begin{aligned}
& +|g(x,y)-f(x,y)| + \left| f\left(\frac{[m+p_1]_{q_m} x + \alpha_1}{[m]_{q_m} + \beta_1}, \frac{[n+p_2]_{q_n} y + \alpha_2}{[n]_{q_n} + \beta_2} \right) - f(x,y) \right| \\
\leq \quad & 4||f-g||_{C(I)} + |S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(g; q_m, q_n, x, y) - g(x,y)| \\
& + \left| f\left(\frac{[m+p_1]_{q_m} x + \alpha_1}{[m]_{q_m} + \beta_1}, \frac{[n+p_2]_{q_n} y + \alpha_2}{[n]_{q_n} + \beta_2} \right) - f(x,y) \right| \\
\leq \quad & \left(4||f-g||_{C(I)} + C_{m,n}||g||_{C^2(I)} \right) + \omega(f; \psi_{m,n}) \\
\leq \quad & 4K_2(f; C_{m,n}) + \omega(f; \psi_{m,n}) \\
\leq \quad & M \left\{ \bar{\omega}_2(f; \sqrt{C_{m,n}}) + min\{1, C_{m,n}\}||f||_{C(I)} \right\} + \omega(f; \psi_{m,n}).
\end{aligned}
$$

Hence, we get the desired result. □

3.2. Voronovskaja-type theorem

In this section, we obtain a Voronovskaja-type asymptotic theorem for the bivariate operators $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}$.

Theorem 6 *Let $f \in C^2(I)$. Then we have*

$$
\lim_{n\to\infty} [n]_{q_n} (S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(f; q_n, x, y) - f(x,y))
$$

$$
= \quad (\alpha_1 - \beta_1 x) f_x(x,y) + (\alpha_2 - \beta_2 y) f_y(x,y) + \frac{f_{xx}(x,y)}{2} x(1-x) + \frac{f_{yy}(x,y)}{2} y(1-y),
$$

uniformly in $(x,y) \in J$.

Proof Let $(x,y) \in J$. By Taylor's theorem, we have

$$
\begin{aligned}
f(t,s) \quad = \quad & f(x,y) + f_x(x,y)(t-x) + f_y(x,y)(s-y) + \frac{1}{2}\{ f_{xx}(x,y)(t-x)^2 + 2f_{xy}(x,y)(t-x)(s-y) \\
& + f_{yy}(x,y)(s-y)^2 \} + \varepsilon(t,s;x,y)\sqrt{(t-x)^4 + (s-y)^4}, \qquad (3.7)
\end{aligned}
$$

for $t, s \in I$, where $\varepsilon(t,s;x,y) \in C(I)$ and $\varepsilon(t,s;x,y) \to 0$ as $(t,s) \to (x,y)$. Applying $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(f; q_n, x, y)$ on both sides of (3.7), we get

$$
\begin{aligned}
S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(f(t,s); q_n, x, y) \quad = \quad & f(x,y) + f_x(x,y) S^{(\alpha_1,\beta_1,)}_{n,p_1}((t-x); q_n, x) + f_y(x,y) S^{(\alpha_2,\beta_2)}_{n,p_2}((s-y); q_n, y) \\
& + \frac{1}{2}\{ f_{xx}(x,y) S^{(\alpha_1,\beta_1)}_{n,p_1}((t-x)^2; q_n, x) \\
& + 2f_{xy}(x,y) S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}((t-x)(s-y); q_n, x, y) \\
& + f_{yy}(x,y) S^{(\alpha_2,\beta_2)}_{n,p_2}((s-y)^2; q_n, y) \} \\
& + S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(\varepsilon(t,s;x,y)\sqrt{(t-x)^4 + (s-y)^4}; q_n, x, y). \qquad (3.8)
\end{aligned}
$$

By using Lemma 3, we may write

$$\lim_{n\to\infty}[n]_{q_n}(S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(f;q_n,x,y)-f(x,y))$$

$$= (\alpha_1-\beta_1 x)f_x(x,y)+(\alpha_2-\beta_2 y)f_y(x,y)+\frac{f_{xx}(x,y)}{2}x(1-x)+\frac{f_{yy}(x,y)}{2}y(1-y)$$
$$+\lim_{n\to\infty}[n]_{q_n}S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(\varepsilon(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};q_n,x,y).$$

Now, applying the Hölder inequality, we have

$$\left|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(\varepsilon^2(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};q_n,x,y)\right|$$

$$\leq \{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(\varepsilon(t,s;x,y);q_n,x,y)\}^{1/2}\{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}((t-x)^4+(s-y)^4;q_n,x,y)\}^{1/2}$$
$$\leq \{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(\varepsilon^2(t,s;x,y)q_n,x,y)\}^{1/2}\{S^{(\alpha_1,\beta_1)}_{n,p_1}((t-x)^4;q_n,x)+S^{(\alpha_2,\beta_2)}_{n,p_2}((s-y)^4;q_n,y)\}^{1/2}. \tag{3.9}$$

Since $\varepsilon^2(t,s;x,y)\to 0$ as $(t,s)\to(x,y)$, applying Theorem 2, we get

$$\lim_{n\to\infty} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{n,n,p_1,p_2}(\varepsilon^2(t,s;x,y),x,y)=0,$$

uniformly in $(x,y)\in J$.

Further, in view of (2.1) and (2.2),

$$[n]_{q_n}\left\{S^{(\alpha_1,\beta_1)}_{n,p_1}((t-x)^4;q_n,x)+S^{(\alpha_2,\beta_2)}_{n,p_1}((s-y)^4;q_n,y)\right\}^{1/2}=O(1),\quad \text{as}\quad n\to\infty \text{ uniformly in } (x,y)\in J.$$

Hence

$$\lim_{n\to\infty}[n]_{q_n}S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(\varepsilon(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};q_n,x,y)=0,\ \text{uniformly in } (x,y)\in J.$$

This completes the proof. □

4. Some approximation properties on the q-GBS–Bernstein–Schurer–Stancu operator

Let $B_b(A)$ denote all B-bounded functions on $A\subset X\times Y\to\mathbb{R}$, equipped with the norm

$$\|f\|_B=\sup_{(x,y),(t,s)\in A}|\Delta f[(t,s);(x,y)]|.$$

We denote by $C_b(A)$ the space of all B-continuous functions on A. $B(A), C(A)$ denote the space of all bounded functions and the space of all continuous (in the usual sense) functions on A endowed with the sup-norm $\|.\|_\infty$. It is known that $C(A)\subset C_b(A)$ ([13], page 52).

A function $f:A\longrightarrow\mathbb{R}$ is called a B-differentiable (Bögel differentiable) function at $(x_0,y_0)\in A$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\Delta f[(x,y);(x_0,y_0)]}{(x-x_0)(y-y_0)}$$

exists and is finite.

The limit is said to be the B-differential of f at the point (x_0, y_0) and is denoted by $D_B(f; x_0, y_0)$ and the space of all B-differentiable functions is denoted by $D_b(A)$.

The mixed modulus of smoothness of $f \in C_b(A)$ is defined as

$$\omega_{mixed}(f; \delta_1, \delta_2) := \sup\left\{|\Delta f[(t,s);(x,y)]| : |x-t| < \delta_1, |y-s| < \delta_2\right\},$$

for all $(x, y), (t, s) \in A$ and for any $(\delta_1, \delta_2) \in (0, \infty) \times (0, \infty)$ with $\omega_{mixed} : [0, \infty) \times [0, \infty) \to \mathbb{R}$. The basic properties of ω_{mixed} were obtained by Badea et al. in [6] and [4], which are similar to the properties of the usual modulus of continuity.

The mixed K-functional is introduced in [3, 14] for improving the measure of smoothness.

Now, for $f \in C_b(I)$, we define the mixed K-functional by

$$K_{mixed}(f; t_1, t_2) = \inf_{g_1, g_2, h}\left\{\|f - g_1 - g_2 - h\|_\infty + t_1\left\|D_B^{2,0} g_1\right\|_\infty + t_2\left\|D_B^{0,2} g_2\right\|_\infty + t_1 t_2\left\|D_B^{2,2} h\right\|_\infty\right\}, \qquad (4.1)$$

where $g_1 \in C_B^{2,0}, g_2 \in C_B^{0,2}, h \in C_B^{2,2}$ and, for $0 \le i, j \le 2, C_B^{i,j}$ denotes the space of the functions $f \in C_b(I)$ with continuous mixed partial derivatives $D_B^{p,q} f$, $0 \le p \le i, 0 \le q \le j$. The partial derivatives are defined as follows:

$$D_x f(x_0, y_0) := D_B^{1,0}(f; x_0, y_0) = \lim_{x \to x_0} \frac{\Delta_x f\{[x_0, x]; y_0\}}{x - x_0},$$

and

$$D_y f(x_0, y_0) := D_B^{0,1}(f; x_0, y_0) = \lim_{y \to y_0} \frac{\Delta_y f\{x_0; [y_0, y]\}}{y - y_0},$$

where $\Delta_x f\{[x_0, x]; y_0\} = f(x, y_0) - f(x_0, y_0)$ and $\Delta_y f\{x_0; [y_0, y]\} = f(x_0, y) - f(x_0, y_0)$. The second order partial derivatives are analogous to the ordinary derivatives. For example, the derivative of $D_x f(x_0, y_0)$ with respect to the variable y at the point (x_0, y_0) is defined by

$$D_y D_x f(x_0, y_0) := D_B^{0,1} D_B^{1,0}(f; x_0, y_0) = \lim_{y \to y_0} \frac{\Delta_y (D_x f)\{x_0; [y_0, y]\}}{y - y_0}.$$

Now let us define the Lipschitz class for B-continuous functions. For $f \in C_b(I)$, the Lipschitz class $Lip_M(\xi, \gamma)$ with $\xi, \gamma \in (0, 1]$ is defined by

$$Lip_M(\xi, \gamma) = \left\{f \in C_b(I) : |\Delta f[(t,s);(x,y)]| \le M|t-x|^\xi |s-y|^\gamma, \text{for } (t,s), (x,y) \in J\right\}.$$

The next theorem gives the degree of approximation for the operators $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ by means of the Lipschitz class of Bögel continuous functions.

Theorem 7 *Let $f \in Lip_M(\xi, \gamma)$; then we have*

$$\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f; q_m, q_n, x, y) - f(x, y)\right| \le M\delta_m^{\xi/2}\delta_n^{\gamma/2},$$

for $M > 0$, $\xi, \gamma \in (0, 1]$,

Proof By the definition of the operator $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)$ and by linearity of the operator $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ given by (1.1), we can write

$$\begin{aligned} U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) &= S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f(x,s)+f(t,y)-f(t,s);q_m,q_n,x,y) \\ &= S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f(x,y)-\Delta f[(t,s);(x,y)];q_m,q_n,x,y) \\ &= f(x,y)\, S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_{00};q_m,q_n,x,y) \\ &\quad - S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\Delta f[(t,s);(x,y)];q_m,q_n,x,y). \end{aligned}$$

By the hypothesis, we get

$$\begin{aligned} \left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)\right| &\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|\Delta f[(t,s);(x,y)]|;q_m,q_n,x,y) \\ &\leq M S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(|t-x|^{\xi}|s-y|^{\gamma};q_m,q_n,x,y\right) \\ &= M S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(|t-x|^{\xi};q_m,x\right) S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|s-y|^{\gamma};q_n,y). \end{aligned}$$

Now, using Hölder's inequality with $u_1=2/\xi, v_1=2/(2-\xi)$, and $u_2=2/\gamma, v_2=2/(2-\gamma)$, we have

$$\begin{aligned} \left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)\right| &\leq M\left(S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(t-x)^2;q_m,x\right)^{\xi/2} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_0;q_m,x)^{(2-\xi)/2} \\ &\quad \times S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left((s-y)^2;y\right)^{\gamma/2} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_0;q_n,y)^{(2-\gamma)/2}. \end{aligned}$$

Considering Lemma 1, we obtain the degree of local approximation for B-continuous functions belonging to $Lip_M(\xi,\gamma)$. □

Theorem 8 *Let the function* $f\in D_b(I)$ *with* $D_Bf\in B(I)$. *Then, for each* $(x,y)\in J$, *we have*

$$|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| \leq \frac{C}{[m]_{q_m}^{1/2}[n]_{q_n}^{1/2}}\left\{||D_Bf||_{\infty}+\omega_{mixed}(D_Bf;[m]_{q_m}^{-1/2}[n]_{q_n}^{-1/2})\right\}.$$

Proof Since $f\in D_b(I)$, we have the identity

$$\Delta f[(t,s);(x,y)]=(t-x)(s-y)D_Bf(\xi,\eta),\ \textit{with}\ x<\xi<t;\ y<\eta<s.$$

It is clear that

$$D_Bf(\xi,\eta)=\Delta D_Bf(\xi,\eta)+D_Bf(\xi,y)+D_Bf(x,\eta)-D_Bf(x,y).$$

Since $D_B f \in B(I)$, by the above relations, we can write

$$\begin{aligned}
|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(\Delta f[(t,s);(x,y)];q_m,q_n,x,y)| &= |S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)(s-y)D_B f(\xi,\eta);q_m,q_n,x,y)| \\
&\leq S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|t-x||s-y||\Delta D_B f(\xi,\eta)|;q_m,q_n,x,y) \\
&\quad +S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|t-x||s-y|(|D_B f(\xi,y)| \\
&\quad +|D_B f(x,\eta)|+|D_B f(x,y)|);q_m,q_n,x,y) \\
&\leq S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|t-x||s-y|\omega_{mixed}(D_B f;|\xi-x|,|\eta-y|);q_m,q_n,x,y) \\
&\quad +3\,||D_B f||_\infty\, S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|t-x||s-y|;q_m,q_n,x,y).
\end{aligned}$$

Since the mixed modulus of smoothness ω_{mixed} is nondecreasing, we have

$$\begin{aligned}
\omega_{mixed}(D_B f;|\xi-x|,|\eta-y|) &\leq \omega_{mixed}(D_B f;|t-x|,|s-y|) \\
&\leq (1+\delta_m^{-1}|t-x|)(1+\delta_n^{-1}|s-y|)\ \omega_{mixed}(D_B f;\delta_m,\delta_n).
\end{aligned}$$

Substituting in the above inequality, using the linearity of $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}$ and applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&|U^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)| \\
&= |S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\Delta f[(t,s);(x,y)];q_m,q_n,x,y| \\
&\leq 3||D_B f||_\infty \sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2(s-y)^2;q_m,q_n,x,y)} \\
&\quad +\Big(S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|t-x||s-y|;q_m,q_n,x,y) \\
&\quad +\delta_m^{-1}S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2|s-y|;q_m,q_n,x,y) \\
&\quad +\delta_n^{-1}S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|t-x|(s-y)^2;q_m,q_n,x,y) \\
&\quad +\delta_m^{-1}\delta_n^{-1}S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2(s-y)^2;q_m,q_n,x,y)\Big)\omega_{mixed}(D_B f;\delta_m,\delta_n) \\
&\leq 3||D_B f||_\infty \sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2(s-y)^2;q_m,q_n,x,y)} \\
&\quad +\Big(\sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2(s-y)^2;q_m,q_n,x,y)} \\
&\quad +\delta_m^{-1}\sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^4(s-y)^2;q_m,q_n,x,y)} \\
&\quad +\delta_n^{-1}\sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2(s-y)^4;q_m,q_n,x,y)} \\
&\quad +\delta_m^{-1}\delta_n^{-1}S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((t-x)^2(s-y)^2;q_m,q_n,x,y)\Big)\omega_{mixed}(D_B f;\delta_m,\delta_n).
\end{aligned} \tag{4.2}$$

It is observed that for $(x,y),(t,s)\in J$ and $i,j\in\{1,2\}$

$$S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x)^{2i}(s-y)^{2j};q_m,q_n,x,y)=S_{m,p_1}^{(\alpha_1,\beta_1)}((t-x)^{2i};q_m,x,y)S_{n,p_2}^{(\alpha_2,\beta_2)}((s-y)^{2j};q_n,x,y). \tag{4.3}$$

Hence choosing $\delta_m=\frac{1}{[m]_{q_m}^{1/2}}$, $\delta_n=\frac{1}{[n]_{q_n}^{1/2}}$. and using Lemma 3, we get the required result. □

Example 3 *In Figures 3 and 4, respectively, for* $m,n=10$, $\alpha_1,\alpha_2=1,\beta_1,\beta_2=2$, $p_1,p_2=1$ *and for* $m,n=5,\alpha_1=0.4$, $\beta_1=0.7$, $\alpha_2=0.5$, $\beta_2=0.9$, $p_1,p_2=2$, *the comparison of convergence of the operators* $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)$ *(green) and its GBS type operators* $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)$ *(pink) to* $f(x,y)=x\sin(\pi x)\,y$*; (yellow) with* $q_m=m/(m+1),q_n=1-1/\sqrt{n}$ *is illustrated. It is clearly seen that the operator* $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ *gives a better approximation than the operator* $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$.

For the order of approximation of the sequence $\left\{U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f)\right\}$ *to the function* $f\in C_b(I)$, *we present an estimate in terms of the mixed* K*-functional given by (4.1).*

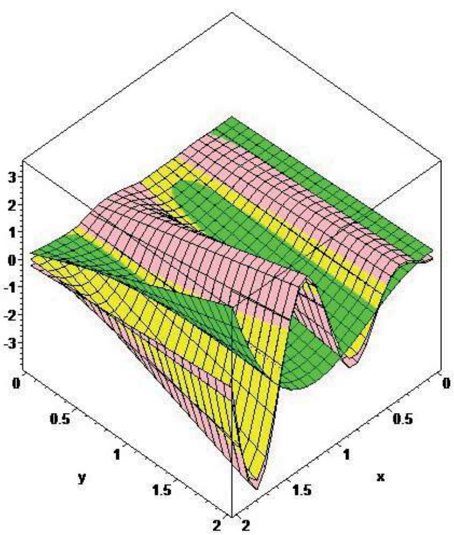

Figure 3

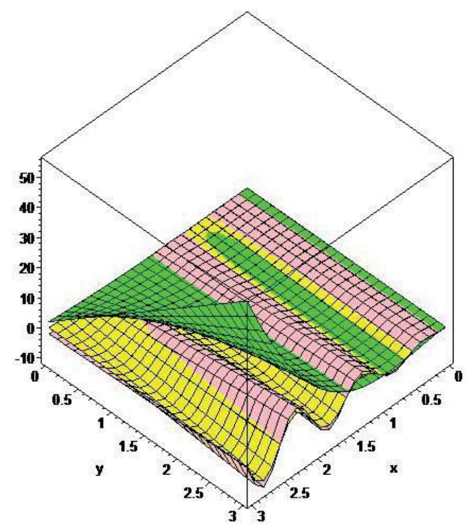

Figure 4

Theorem 9 *Let* $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ *be the GBS operator of* $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ *given by (1.2). Then*

$$\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)\right|\leq 2K_{mixed}\left(f;\delta_m^2,\delta_n^2\right)$$

for each $f\in C_b(I)$.

Proof From Taylor's formula for the function $g_1\in C_B^{2,0}(I)$, we get

$$g_1(t,s)=g_1(x,y)+(t-x)D_B^{1,0}g_1(x,y)+\int_x^t(t-u)D_B^{2,0}g_1(u,y)\,du$$

([13], page 67-69). Since the operator $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ is a linear operator

$$U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(g_1;q_m,q_n,x,y) = g_1(x,y) + U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_x^t (t-u)\,D_B^{2,0}g_1(u,y)\,du;q_m,q_n,x,y\right),$$

and by the definition of $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$

$$\begin{aligned}
&\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(g_1;q_m,q_n,x,y) - g_1(x,y)\right| \\
&= \left|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_x^t (t-u)\left[D_B^{2,0}g_1(u,y) - D_B^{2,0}g_1(u,s)\right]du;q_m,q_n,x,y\right)\right| \\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\left|\int_x^t |t-u|\left|D_B^{2,0}g_1(u,y) - D_B^{2,0}g_1(u,s)\right|du;q_m,q_n,x,y\right|\right) \\
&\leq \left\|D_B^{2,0}g_1\right\|_\infty S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left((t-x)^2;q_m,q_n,x,y\right) \\
&\leq \left\|D_B^{2,0}g_1\right\|\delta_m^2.
\end{aligned}$$

Similarly, we can write

$$\begin{aligned}
\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(g_2;q_m,q_n,x,y) - g_2(x,y)\right| &\leq \left\|D_B^{0,2}g_2\right\|_\infty S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left((s-y)^2;q_m,q_n,x,y\right) \\
&\leq \left\|D_B^{0,2}g_2\right\|_\infty \delta_n^2,
\end{aligned}$$

for $g_2 \in C_B^{0,2}(I)$.

For $h \in C_B^{2,2}(I)$,

$$h(t,s) = h(x,y) + (t-x)\,D_B^{1,0}h(x,y) + (s-y)\,D_B^{0,1}h(x,y) + (t-x)(s-y)\,D_B^{1,1}h(x,y)$$

$$+\int_x^t (t-u)\,D_B^{2,0}h(u,y)\,du + \int_y^s (s-v)\,D_B^{0,2}h(x,v)\,dv$$

$$+\int_x^t (s-y)(t-u)\,D_B^{2,1}h(u,y)\,du + \int_y^s (t-x)(s-v)\,D_B^{1,2}h(x,v)\,dv$$

$$+\int_x^t\int_y^s (t-u)(s-v)\,D_B^{2,2}h(u,v)\,dvdu.$$

Taking into account the definition of the operator $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ and by using $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((t-x);q_m,q_n,x,y)=0,\ U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((s-y);q_m,q_n,x,y)=0,$ we get

$$\begin{aligned}
\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(h;q_m,q_n,x,y)-h(x,y)\right| &\leq \left|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_x^t\int_y^s(t-u)(s-v)D_B^{2,2}h(u,v)\,dvdu;q_m,q_n,x,y\right)\right| \\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\left|\int_x^t\int_y^s(t-u)(s-v)D_B^{2,2}h(u,v)\,dvdu\right|;q_m,q_n,x,y\right) \\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\int_x^t\int_y^s|t-u||s-v|\left|D_B^{2,2}h(u,v)\right|dvdu;q_m,q_n,x,y\right) \\
&\leq \frac{1}{4}\left\|D_B^{2,2}h\right\|_\infty S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left((t-x)^2(s-y)^2;q_m,q_n,x,y\right) \\
&\leq \frac{1}{4}\left\|D_B^{2,2}h\right\|_\infty \delta_m^2\delta_n^2.
\end{aligned}$$

Therefore, for $f\in C_b(I)$, we obtain

$$\begin{aligned}
\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)\right| &\leq |(f-g_1-g_2-h)(x,y)|+\left|\left(g_1-U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}g_1\right)(x,y)\right| \\
&+ \left|\left(g_2-U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}g_2\right)(x,y)\right|+\left|\left(h-U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}h\right)(x,y)\right| \\
&+ \left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((f-g_1-g_2-h);q_m,q_n,x,y)\right| \\
&\leq 2\|f-g_1-g_2-h\|_\infty+\frac{1}{4}\left\|D_B^{2,0}g_1\right\|_\infty\delta_m^2 \\
&+ \frac{1}{4}\left\|D_B^{0,2}g_2\right\|_\infty\delta_n^2+\frac{1}{4}\left\|D_B^{2,2}h\right\|_\infty\delta_m^2\delta_n^2.
\end{aligned}$$

Taking the infimum over all $g_1\in C_B^{2,0},g_2\in C_B^{0,2},h\in C_B^{2,2},$ we obtain the desired result. □

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