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Approximation of common fixed points in 2-Banach spaces with applications

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Abstract

The purpose of this paper is to establish the existence and uniqueness of common fixed points of a family of self-mappings satisfying generalized rational contractive condition in 2-Banach spaces. An example is included to justify our results. We approximate the common fixed point by Mann and Picard type iteration schemes. Further, an application to well-posedness of the common fixed point problem is given. The presented results generalize many known results on 2-Banach spaces.

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1. Introduction

The Banach Contraction Principle [5] is one of the pivotal results in nonlinear analysis which guarantees the existence and uniqueness of fixed points of mappings. There is a great number of extensions of the Banach contraction principle using different forms of contractive conditions in various generalized metric spaces, we refer the reader to [1, 2, 6, 17, 13, 14, 20, 23, 24, 25, 27, 28, 29, 30]. Some of such generalizations are obtained through rational contractive conditions. Das and Gupta [8] studied the extention of Banach contraction in term

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of rational inequality and derived a fixed point result for a self mapping which runs as follows:

Theorem 1.1. *Let f be a mapping of a complete metric space X into itself such that*

(i)
$$d(f(x), f(y)) \leq \alpha \frac{d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta d(x, y),$$
for all $x, y \in X, \alpha > 0, \beta > 0, \alpha + \beta < 1$ and

(ii) *for some $x_0 \in X$, the sequence of iterates $\{f^n(x_0)\}$ has a subsequence $\{f_k^n(x_0)\}$ with $\xi = \lim_{n\to\infty} f_k^n(x_0)$.*

Then ξ is a unique fixed point of f.

Meanwhile Pachpatte [19] extended it to a pair of self mappings. Recently, Azam et al. [4] obtained fixed point theorems for a pair of contractive mappings using an analogical rational inequality of [8] in a complex-valued metric space setting. Also, in [18], Nashine et al. studied some common fixed point theorems for a pair of mappings under certain rational contractions in complex valued metric spaces. In [33], Shahkoohi and Razani established the existence of fixed point of a self mapping satisfying rational Geraghty contractive mappings in partially ordered b-metric spaces. For related study [3, 32].

In the 1960's, Gähler [10, 11, 12] generalized the idea of metric space and introduced a new theory of 2-metric space. On the other hand, White [35] started the investigation on the concept of 2-Banach spaces. Since then, many authors have focused on these spaces and presented papers that dealt with fixed point theory for single-valued and multi-valued operators in 2-Banach spaces (see [7, 15]). Recently, Pitchaimani and Ramesh Kumar [21] obtained common fixed points under implicit relation in 2-Banach spaces and proved some common and coincidence fixed point theorems for asymptotically regular mappings in [22].

Inspired by the concept of 2-Banach spaces and notion of rational type conditions, an attempt has been made in this paper to prove the existence and uniqueness of common fixed point of a family of self mappings satisfying the generalized rational contractive condition in 2-Banach spaces with an example which illustrates our result. Further, the approximation of the common fixed point by means of Mann and Picard iteration method is given. Finally, we prove the well-posedness of the common fixed point problem.

2. Preliminaries

In this section, we recall the notions which will be required in the sequel. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ denotes the set of all real numbers.

Definition 2.1. Let X be a real linear space and $\|\cdot, \cdot\|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x, y\| = \|y, x\|$, for all $x, y \in X$;
(iii) $\|x, ay\| = |a|\|x, y\|$, for all $x, y \in X$ and $a \in \mathbb{R}$;
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, for all $x, y, z \in X$;

Then $\|\cdot, \cdot\|$ is called a 2 - norm and the pair $(X, \|\cdot, \cdot\|)$ is called a linear 2-normed space.

Some of the basic properties of 2-norms are that they are non-negative satisfying $\|x, y + ax\| = \|x, y\|$, for all $x, y \in X$ and $a \in \mathbb{R}$.

Definition 2.2. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is called a Cauchy sequence if $\lim\limits_{m,n\to\infty} \|x_m - x_n, y\| = 0$ for all $y \in X$.

Definition 2.3. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is said to converge to a point $x \in X$ if $\lim\limits_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in X$.

Definition 2.4. A linear 2-normed space $(X, \|\cdot, \cdot\|)$ in which every Cauchy sequence is convergent is called a 2-Banach space.

Example 2.5. Let $X = \mathbb{R}^3$ and a 2-norm $\|\cdot, \cdot\|$ be defined as follows:

$$\|x, y\| = |x \times y| = det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X$ and $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along the axes. Note that $(X, \|\cdot, \cdot\|)$ is a 2-Banach space.

Definition 2.6. A sequence $\{x_n\}$ in a 2-Banach space X is said to be asymptotically T- regular if $\lim\limits_{n\to\infty} \|x_n - Tx_n, y\| = 0$ for all $y \in X$.

Definition 2.7. Let X be a nonempty set and $S, T : X \to X$ be self mappings. Then

(i) an element $x \in X$ is said to be a fixed point of T if $x = Tx$.
(ii) If $x = Sx = Tx$ then x is called a common fixed point of S and T.

Let C be a nonempty convex subset of a 2-Banach space $(X, \|\cdot, \cdot\|)$ and $T : X \to X$ be a mapping then

(i) the sequence $\{x_n\}$ defined by

$$x_0 \in C,\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T(x_n), \quad \forall n \geq 0,$$

where $\{\beta_n\}$ satisfies $0 < \beta_n \leq 1$, $\forall n$ and $\sum\limits_{n=0}^{\infty} \beta_n = \infty$, is called the Mann iteration scheme.

(ii) the sequence $\{x_n\}$ defined by

$$x_0 \in C,\ x_{n+1} = T(x_n), \ \ \forall n \geq 0,$$

is called the Picard iteration scheme which is particular case of the Mann iteration scheme.

3. Main Results

In this section, we prove the existence and uniqueness of common fixed point of a family of self mappings.

Theorem 3.1. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $S, T : X \to X$ be two self mappings such that*

$$\begin{aligned} \|Sx - Ty, u\| \leq\ & a\frac{\|y - Ty, u\|\left[1 + \|x - Sx, u\|\right]}{1 + \|x - y, u\|} + b\frac{\|x - Sx, u\|\left[1 + \|y - Ty, u\|\right]}{1 + \|x - y, u\|} \\ & + c\frac{\|x - Ty, u\|\left[1 + \|y - Sx, u\|\right]}{1 + \|x - y, u\|} + d\frac{\|y - Sx, u\|\left[1 + \|x - Ty, u\|\right]}{1 + \|x - y, u\|} \\ & + e\|x - y, u\|, \end{aligned} \tag{3.1}$$

for all $x, y, u \in X$, $x \neq y$, where a, b, c, d, e are non negative constants such that $a + b + c + d + e < 1$. Then S and T have a unique common fixed point in X.

Proof. For $x_0 \in X$, define a sequence as follows:

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}, n = 0, 1, 2, \ldots$$

Now for all $u \in X$, using (3.1), we get

$$\begin{aligned} \|x_{2n+1} - x_{2n}, u\| &= \|Sx_{2n} - Tx_{2n-1}, u\| \\ &\leq a\frac{\|x_{2n-1} - x_{2n}, u\|[1 + \|x_{2n} - x_{2n+1}, u\|]}{1 + \|x_{2n} - x_{2n-1}, u\|} \\ &\quad + b\frac{\|x_{2n} - x_{2n+1}, u\|[1 + \|x_{2n} - x_{2n-1}, u\|]}{1 + \|x_{2n} - x_{2n-1}, u\|} \\ &\quad + c\frac{\|x_{2n} - x_{2n}, u\|[1 + \|x_{2n-1} - x_{2n+1}, u\|]}{1 + \|x_{2n} - x_{2n-1}, u\|} \\ &\quad + d\frac{\|x_{2n-1} - x_{2n+1}, u\|[1 + \|x_{2n} - x_{2n}, u\|]}{1 + \|x_{2n} - x_{2n-1}, u\|} \\ &\quad + e\|x_{2n} - x_{2n-1}, u\|, \end{aligned}$$

which gives

$$\|x_{2n+1} - x_{2n}, u\| \leq k_n \|x_{2n} - x_{2n-1}, u\|,$$

where $k_n = \dfrac{(a + d + e) + e\|x_{2n} - x_{2n-1}, u\|}{(1 - b - d) + (1 - a - b)\|x_{2n} - x_{2n-1}, u\|}$.

Note that $k_n < 1$, as $a+b+c+d+e < 1$. Repeating the same argument several times, we obtain

$$\|x_{2n+1} - x_{2n}, u\| \leq (k_n)^{2n} \|x_1 - x_0, u\|.$$

Now, for $n > m$, we have

$$\begin{aligned} \|x_n - x_m, u\| &\leq \|x_n - x_{n-1}, u\| + \|x_{n-1} - x_{n-2}, u\| + \cdots\cdots + \|x_{m+1} - x_m, u\| \\ &\leq ((k_n)^{n-1} + (k_n)^{n-2} + \cdots\cdots + (k_n)^m)\|x_1 - x_0, u\| \\ &\leq \frac{(k_n)^m}{1 - k_n}\|x_1 - x_0, u\|. \end{aligned}$$

Thus $\|x_n - x_m, u\| \to 0$ as $m, n \to \infty$, since $\dfrac{(k_n)^m}{1-k_n} \to 0$ as $m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence, there exists a point $w \in X$ such that $x_n \to w$ as $n \to \infty$. Further, we get

$$\begin{aligned}
\|w - Tw, u\| &\leq \|w - x_{2n+1}, u\| + \|x_{2n+1} - Tw, u\| \\
&= \|w - x_{2n+1}, u\| + \|Sx_{2n} - Tw, u\| \\
&\leq \|w - x_{2n+1}, u\| + a\frac{\|w - Tw, u\|[1 + \|x_{2n} - x_{2n+1}, u\|]}{1 + \|x_{2n} - w, u\|} \\
&\quad + b\frac{\|x_{2n} - x_{2n+1}, u\|[1 + \|w - Tw, u\|]}{1 + \|x_{2n} - w, u\|} \\
&\quad + c\frac{\|x_{2n} - Tw, u\|[1 + \|w - x_{2n+1}\|]}{1 + \|x_{2n} - w, u\|} \\
&\quad + d\frac{\|w - x_{2n+1}, u\|[1 + \|x_{2n} - Tw, u\|]}{1 + \|x_{2n} - w, u\|} \\
&\quad + e\|x_{2n} - w, u\|.
\end{aligned}$$

Letting $n \to \infty$, we have

$$\|w - Tw, u\| \leq (a + c)\|w - Tw, u\|,$$

which implies that $Tw = w$, as $(a + c) < 1$ for all $u \in X$. Similarly, we get $w = Sw$. Thus w is a common fixed point of S and T.

For the uniqueness, assume that $v \in X$ is another common fixed point of S and T, that is, $v = Sv = Tv$. Then

$$\begin{aligned}
\|w - v, u\| &= \|Sw - Tv, u\| \\
&\leq a\frac{\|v - Tv, u\|[1 + \|w - Sw, u\|]}{1 + \|w - v, u\|} \\
&\quad + b\frac{\|w - Sw, u\|[1 + \|v - Tv, u\|]}{1 + \|w - v, u\|} \\
&\quad + c\frac{\|w - Tv, u\|[1 + \|v - Sw, u\|]}{1 + \|w - v, u\|} \\
&\quad + d\frac{\|v - Sw, u\|[1 + \|w - Tv, u\|]}{1 + \|w - v, u\|} \\
&\quad + e\|w - v, u\|,
\end{aligned}$$

which yields

$$\|w - v, u\| + \|w - v, u\|^2 \leq (c + d + e)\|w - v, u\| + (c + d + e)\|w - v, u\|^2,$$

which is a contradiction, since $c + d + e < 1$. Hence w is the unique common fixed point of S and T. □

Taking $S = T$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $T : X \to X$ be a self mapping such that*

$$\begin{aligned} \|Tx - Ty, u\| \leq\ & a\frac{\|y - Ty, u\|\left[1 + \|x - Tx, u\|\right]}{1 + \|x - y, u\|} + b\frac{\|x - Tx, u\|\left[1 + \|y - Ty, u\|\right]}{1 + \|x - y, u\|} \\ & + c\frac{\|x - Ty, u\|\left[1 + \|y - Tx, u\|\right]}{1 + \|x - y, u\|} + d\frac{\|y - Tx, u\|\left[1 + \|x - Ty, u\|\right]}{1 + \|x - y, u\|} \\ & + e\|x - y, u\|, \qquad (3.2) \end{aligned}$$

for all $x, y, u \in X$, $x \neq y$, where a, b, c, d, e are non negative constants and $a + b + c + d + e < 1$. Then T have a unique fixed point in X.

Proof. The proof follows from Theorem 3.1. □

Remark 3.3. In the case $b = c = d = 0$, Corollary 3.2 reduces to Theorem 1.1.

Now we extend the Theorem 3.1 to the case of pair of mappings S^p and T^q where p and q are some positive integers.

Theorem 3.4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $S, T : X \to X$ be two self mappings such that*

$$\begin{aligned} \|S^p x - T^q y, u\| \leq\ & a\frac{\|y - T^q y, u\|\left[1 + \|x - S^p x, u\|\right]}{1 + \|x - y, u\|} + b\frac{\|x - S^p x, u\|\left[1 + \|y - T^q y, u\|\right]}{1 + \|x - y, u\|} \\ & + c\frac{\|x - T^q y, u\|\left[1 + \|y - S^p x, u\|\right]}{1 + \|x - y, u\|} + d\frac{\|y - S^p x, u\|\left[1 + \|x - T^q y, u\|\right]}{1 + \|x - y, u\|} \\ & + e\|x - y, u\|, \qquad (3.3) \end{aligned}$$

for all $x, y, u \in X$, $x \neq y$, where p and q are some positive integers and a, b, c, d, e are non negative constants with $a + b + c + d + e < 1$. Then S and T have a unique common fixed point in X.

Proof. Note that S^p and T^q satisfy the conditions of Theorem 3.1, so S^p and T^q have a unique common fixed point. Let w be the common fixed point. Then

$$\begin{aligned} S^p w = w \quad &\Rightarrow \quad S(S^p w) = Sw, \\ S^p(Sw) \quad &= \quad Sw. \end{aligned}$$

If $Sw = x_0$ then $S^p(x_0) = x_0$. So, Sw is a fixed point of S^p. Similarly, $T^q(Tw) = Tw$. Now, we have

$$\begin{aligned}\|w - Tw, u\| &= \|S^p w - T^q(Tw), u\| \\ &\leq a\frac{\|Tw - T^q(Tw), u\|[1 + \|w - S^p w, u\|]}{1 + \|w - Tw, u\|} \\ &\quad + b\frac{\|w - S^p w, u\|[1 + \|Tw - T^q(Tw), u\|]}{1 + \|w - Tw, u\|} \\ &\quad + c\frac{\|w - T^q(Tw), u\|[1 + \|Tw - S^p w, u\|]}{1 + \|w - Tw, u\|} \\ &\quad + d\frac{\|Tw - S^p w, u\|[1 + \|w - T^q(Tw), u\|]}{1 + \|w - Tw, u\|} \\ &\quad + e\|w - Tw, u\|,\end{aligned}$$

which implies that

$$\|w-Tw, u\| + \|w-Tw, u\|^2 \leq (c+d+e)\|w-Tw, u\| + (c+d+e)\|w-Tw, u\|^2,$$

which is a contradiction, as we are having $c + d + e < 1$. Thus, $w = Tw$ for all $u \in X$. With the similar arguments, we obtain $w = Sw$.

Finally, in order to prove the uniqueness of v, let $w \neq v$ be another common fixed point of S and T. Then clearly w is also a common fixed point of S^p and T^q which implies $w = v$. Therefore, S and T have a unique common fixed point. □

Hence we have proved that if x_0 is a unique common fixed point of S^p and T^q, where p, q are some positive integers then x_0 is a unique common fixed point of S and T. Next we extend Theorem 3.1 to a case of family of mappings satisfying the condition (3.1).

Theorem 3.5. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $\{F_\alpha\}$ be a family of self mappings on X such that*

$$\begin{aligned}\|F_\alpha x - F_\beta y, u\| \leq\ & a\frac{\|y - F_\beta y, u\|\big[1 + \|x - F_\alpha x, u\|\big]}{1 + \|x - y, u\|} + b\frac{\|x - F_\alpha x, u\|\big[1 + \|y - F_\beta y, u\|\big]}{1 + \|x - y, u\|} \\ & + c\frac{\|x - F_\beta y, u\|\big[1 + \|y - F_\alpha x, u\|\big]}{1 + \|x - y, u\|} + d\frac{\|y - F_\alpha x, u\|\big[1 + \|x - F_\beta y, u\|\big]}{1 + \|x - y, u\|} \\ & + e\|x - y, u\|,\end{aligned}$$

for all $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ and $x, y, u \in X$ with $x \neq y$, where a, b, c, d, e are non negative constants such that $a + b + c + d + e < 1$. Then there exists a unique $w \in X$ satisfying $F_\alpha w = w$ for all $\alpha \in \Lambda$.

Proof. Let us take F_α and F_β in place of S and T respectively in Theorem 3.1, an application of which gives a unique $w \in X$ to satisfy $F_\alpha w = F_\beta w = w$. For any other member F_γ, uniqueness of w gives $F_\gamma w = w$ and this completes the proof. □

Remark 3.6. Theorem 3.5 generalizes and improves the results of [4, 8, 18, 19, 32] in 2-Banach space setting.

Example 3.7. Let $X = \mathbb{R}^2$ and a 2-norm $\|\cdot,\cdot\|$ be defined by $\|x, y\| = |x_1y_2 - x_2y_1|$. Note that $(X, \|\cdot,\cdot\|)$ is a 2-Banach space. Let $S, T : X \to X$ be two self mappings defined as follows:

$$S(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$$

and

$$T(u, v) = \left(\frac{u}{2}, \frac{u}{2}\right).$$

For $a, b, c, d, e \in [\frac{1}{2}, 1)$, it can be easily seen that (3.1) is satisfied. Hence, by Theorem 3.1, S and T have a unique common fixed point in X. Here, $\left(\frac{1}{2}, \frac{1}{2}\right)$ is the unique common fixed point of S and T.

4. Approximation by Mann and Picard Type Iteration

In this section, we approximate the common fixed point of S and T by a Mann and Picard type iteration schemes.

Theorem 4.1. *Let C be a nonempty closed convex subset of a 2-Banach space X and $S, T : C \to C$ be two self mappings such that $(1-k)S(C)+kT(C) \subset S(C)$ for $0 < k \leq 1$ and $S(C)$ is closed. Suppose that S and T satisfy all the conditions of Theorem 3.1, then S and T have a unique common fixed point. Moreover, if, for arbitrary $y_0 \in C$, the sequence $\{y_n\}$ defined by*

$$S(y_{n+1}) = (1 - \beta_n)S(y_n) + \beta_n T(y_n), \quad \forall n \geq 0, \tag{4.1}$$

where $\{\beta_n\}$ satisfies $0 < \beta_n \leq 1$, $\forall n$ and $\sum_{n=0}^{\infty} \beta_n = \infty$, is asymptotically T-regular, then it converges to the unique common fixed point of S and T, with a rate estimated by

$$\|(S(y_{n+1}) - w, u\| \leq \lambda^{n+1} L,$$

where $\lambda \in [0, 1)$ and $L \geq 0$ are some constants.

Proof. From Theorem 3.1, S and T have a unique common fixed point $w \in X$. Let $y_0 \in C$ and the sequence $\{y_n\}$ be defined by (4.1). Then, for all $u \in X$ and $n \geq 0$, we have the following

$$\begin{aligned} \|S(y_{n+1}) - w, u\| &= \|(1 - \beta_n)S(y_n) + \beta_n T(y_n) - w, u\| \\ &\leq (1 - \beta_n)\|S(y_n) - w, u\| + \beta_n \|T(y_n) - w, u\| \end{aligned} \tag{4.2}$$

From (3.1), we get

$$\begin{aligned}\|T(y_n) - w, u\| &= \|T(y_n) - Sw, u\| \\ &\leq a\frac{\|w - Sw, u\|[1 + \|y_n - Ty_n, u\|]}{1 + \|y_n - w, u\|} \\ &\quad + b\frac{\|y_n - Ty_n, u\|[1 + \|w - Sw, u\|]}{1 + \|y_n - w, u\|} \\ &\quad + c\frac{\|y_n - Sw, u\|[1 + \|w - Ty_n, u\|]}{1 + \|y_n - w, u\|} \\ &\quad + d\frac{\|w - Ty_n, u\|[1 + \|y_n - Sw, u\|]}{1 + \|y_n - w, u\|} \\ &\quad + e\|y_n - w, u\|.\end{aligned}$$

Since $\{y_n\}$ is asymptotically T-regular and $Sw = Tw = w$, letting $n \to \infty$, we have

$$\|Ty_n - w, u\| + \|Ty_n - w, u\|^2 \leq (c+d+e)\|Ty_n - w, u\| + (c+d+e)\|Ty_n - w, u\|^2,$$

which is not possible as $c + d + e < 1$. Hence $\lim_{n\to\infty} \|T(y_n) - w, u\| = 0$.

Now it follows from (4.2) that

$$\|S(y_{n+1}) - w, u\| \leq (1 - \beta_n)\|S(y_n) - w, u\|.$$

Continuing in this way, we have

$$\|(S(y_{n+1}) - w, u\| \leq \lambda^{n+1} L, \tag{4.3}$$

where $\lambda = (1 - \beta_n) \in [0, 1)$ and $L = \|Sy_0 - w, u\| \geq 0$ are some constants. Since $\lambda \in [0, 1)$, from (4.3) we get

$$\lim_{n\to\infty} \|(S(y_{n+1}) - w, u\| \to 0.$$

This completes the proof. □

Corollary 4.2. *Let C be a nonempty closed convex subset of a 2-Banach space X and $T : C \to C$ be a mapping satisfying all the conditions of Corollary 3.2. Then T has a unique fixed point $w \in X$. In addition, if for arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T(x_n, \ldots, x_n), \quad \forall n \geq 0, \tag{4.4}$$

where $\{\beta_n\}$ satisfies $0 < \beta_n \leq 1$, $\forall n$ and $\sum_{n=0}^{\infty} \beta_n = \infty$, *is asymptotically T-regular, then it converges to the unique fixed point of T, with a rate estimated by*

$$\|x_n - w, u\| \leq \lambda^n L,$$

where $\lambda \in [0, 1)$ and $L \geq 0$ are some constants.

Remark 4.3. From Corollary 4.2, we obtain an approximation of fixed point of a self mapping by the Mann iteration scheme in 2-Banach spaces. Note that the result holds even if $\{x_n\}$ is asymptotically S-regular.

Corollary 4.4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $S, T : X \to X$ be two mappings such that $T(X) \subset S(X)$ and $S(X)$ is a closed subset of X. Suppose all the conditions of Theorem 3.1 are satisfied, then S and T have a unique common fixed point $w \in X$. Further, if for arbitrary $x_0 \in C$, the sequence $\{y_n\}$ defined by*

$$y_n = S(x_n) = T(x_{n-1}), \quad \forall n \in \mathbb{N}$$

is asymptotically T-regular, then it converges to the unique common fixed point of S and T, with a rate estimated by

$$\|y_n - w, u\| \leq \lambda^n L,$$

where $\lambda \in [0, 1)$ and $L \geq 0$ are some constants.

Corollary 4.5. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $T : X \to X$ be a mapping such that all the conditions of Corollary 3.2 are satisfied. Then T has a unique fixed point X. In addition, if for arbitrary $x_0 \in C$, the sequence $\{z_n\}$ defined by*

$$z_n = T(x_{n-1}), \quad \forall n \in \mathbb{N},$$

is asymptotically T-regular, then it converges to the unique fixed point of T, with a rate estimated by

$$\|z_n - w, u\| \leq \lambda^n L,$$

where $\lambda \in [0, 1)$ and $L \geq 0$ are some constants.

Remark 4.6. From Corollary 4.5, we obtain an approximation of fixed point of self mapping by the Picard iteration scheme in 2-Banach space.

5. Applications

The notion of well-posedness of a fixed point problem was introduced in [9] and has generated much interest to several mathematicians, for example [16, 26, 31]. In this section, we study well-posedness of the common fixed point obtainined in Theorem 3.1.

Definition 5.1. Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T be a self mapping on X. Then the fixed point problem of T is said to be well-posed if

(i) T has a unique fixed point $x_0 \in X$
(ii) for any sequence $\{x_n\} \subset X$ and $\lim_{n\to\infty} \|x_n - Tx_n, u\| = 0$, we have $\lim_{n\to\infty} \|x_n - x_0, u\| = 0$.

Let $CFP(S, T, X)$ denote a common fixed point problem of self mappings T and f on X and $CF(S, T)$ denote the set of all common fixed points of T and f.

Definition 5.2. $CFP(S, T, X)$ is called well- posed if $CF(S, T)$ is singleton and for any sequence $\{x_n\}$ in X with

$$\hat{x} \in CF(S, T) \text{ and } \lim_{n\to\infty} \|x_n - Sx_n, u\| = \lim_{n\to\infty} \|x_n - Tx_n, u\| = 0$$

implies $\hat{x} = \lim_{n\to\infty} x_n$.

Theorem 5.3. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, S and T be two self mappings on X as in* Theorem 3.1. *Then the common fixed point problem of S and T is well posed.*

Proof. From Theorem 3.1, the mappings S and T have a unique common fixed point, say $w \in X$. Let $\{x_n\}$ be a sequence in X and $\lim\limits_{n\to\infty} \|Sx_n - x_n, u\| = \lim\limits_{n\to\infty} \|Tx_n - x_n, u\| = 0$. Without loss of generality, assume that $w \neq x_n$ for any non-negative integer n. Using (3.1) and $Sw = Tw = w$, we get

$$\begin{aligned}
\|w - x_n, u\| &\leq \|Tv - Tx_n, u\| + \|Tx_n - x_n, u\| \\
&= \|Tx_n - x_n, u\| + \|Sw - Tx_n, u\| \\
&\leq \|Tx_n - x_n, u\| + a\frac{\|x_n - Tx_n, u\|[1 + \|w - Sw, u\|]}{1 + \|w - x_n, u\|} \\
&\quad +b\frac{\|w - Sw, u\|[1 + \|x_n - Tx_n, u\|]}{1 + \|w - x_n, u\|} \\
&\quad +c\frac{\|w - Tx_n, u\|[1 + \|x_n - Sw, u\|]}{1 + \|w - x_n, u\|} \\
&\quad +d\frac{\|x_n - Sw, u\|[1 + \|w - Tx_n, u\|]}{1 + \|w - x_n, u\|} \\
&\quad +e\|w - x_n, u\|.
\end{aligned}$$

Taking limit $n \to \infty$, we get

$$\|w - x_n, u\| + \|w - x_n, u\|^2 \leq (c + d + e)\|w - x_n, u\| + (c + d + e)\|w - x_n, u\|^2,$$

which gives the contradiction as $c + d + e < 1$. This completes the proof. □

Corollary 5.4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T be a self mapping on X as in* Corollary 3.2. *Then the fixed point problem of T is well posed.*

Remark 5.5. Notice that well-posedness of the common fixed points obtained in Theorems 3.4 and 3.5 can easily be viewed.

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