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Approximation of functions belonging to $\mathrm{Lip}(\xi(t), r)$ class by $(N, p_n)(E, q)$ summability of conjugate series of Fourier series

Vishnu Narayan Mishra[1*], Kejal Khatri[1] and Lakshmi Narayan Mishra[2]

*Correspondence: vishnunarayanmishra@gmail.com
[1]Department of Mathematics, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Road, Surat, Gujarat 395007, India
Full list of author information is available at the end of the article

Abstract
In this paper, a new theorem concerning the degree of approximation of the conjugate of a function belonging to $\mathrm{Lip}(\xi(t), r)$ class by $(N, p_n)(E, q)$ summability of its conjugate series of a Fourier series has been proved. Here the product of Euler (E, q) summability method and Nörlund (N, p_n) method has been taken.

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1 Introduction

Khan [1, 2] has studied the degree of approximation of a function belonging to $\mathrm{Lip}(\alpha, r)$-class by Nörlund means. Generalizing the results of Khan [1, 2], many interesting results have been proved by various investigators like Mittal *et al.* [3–5], Mittal, Rhoades and Mishra [6], Mittal and Singh [7], Rhoades *et al.* [8], Mishra *et al.* [9, 10] and Mishra and Mishra [11] for functions of various classes $\mathrm{Lip}\,\alpha$, $\mathrm{Lip}(\alpha, r)$, $\mathrm{Lip}(\xi(t), r)$ and $W(L_r, \xi(t))$, $(r \geq 1)$ by using various summability methods. But till now, nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function using $(N, p_n)(E, q)$ product summability method of its conjugate series of Fourier series. In this paper, we obtain a new theorem on the degree of approximation of a function $\tilde{f}$, conjugate to a periodic function $f \in \mathrm{Lip}(\xi(t), r)$-class, by $(N, p_n)(E, q)$ product summability means.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its nth partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative generating sequence of constants, real or complex, and let us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \quad \forall n \geq 0, \qquad p_{-1} = 0 = P_{-1} \quad \text{and} \quad P_n \to \infty \quad \text{as } n \to \infty.$$

The conditions for regularity of Nörlund summability are easily seen to be

(1) $\lim_{n\to\infty} \frac{p_n}{P_n} \to 0$ and
(2) $\sum_{k=0}^{\infty} |p_k| = O(P_n)$, as $n \to \infty$.

The sequence-to-sequence transformation

$$t_n^N = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \tag{1.1}$$

defines the sequence $\{t_n^N\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable (N,p_n) to the sum s if $\lim_{n\to\infty} t_n^N$ exists and is equal to s.

The (E,q) transform is defined as the nth partial sum of (E,q) summability, and we denote it by E_n^q. If

$$E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}s_k \to s \quad \text{as } n\to\infty, \tag{1.2}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E,q) to the sum s Hardy [12]. The (N,p_n) transform of the (E,q) transform defines $(N,p_n)(E,q)$ product transform and denotes it by t_n^{NE}. This is if

$$t_n^{NE} = \frac{1}{P_n}\sum_{k=0}^{n}\frac{p_{n-k}}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}s_\nu. \tag{1.3}$$

If $t_n^{NE}\to s$ as $n\to\infty$, then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable $(N,p_n)(E,q)$ to the sum s.

$$\begin{aligned}
s_n \to s \quad &\Rightarrow \quad (E,q)(s_n) = E_n^q = (1+q)^{-n}\sum_{k=0}^{n}\binom{k}{n} q^{n-k}s_k \to s,\\
&\qquad \text{as } n\to\infty,\ (E,q) \text{ method is regular},\\
&\Rightarrow \quad \big((N,p_n)(E,q)(s_n)\big) = t_n^{NE} \to s, \quad \text{as } n\to\infty,\ (N,p_n) \text{ method is regular},\\
&\Rightarrow \quad (N,p_n)(E,q) \text{ method is regular}.
\end{aligned}$$

A function $f(x) \in \operatorname{Lip}\alpha$ if

$$f(x+t) - f(x) = O\big(|t^\alpha|\big) \quad \text{for } 0<\alpha\le 1, t>0$$

and $f(x) \in \operatorname{Lip}(\alpha,r)$, for $0\le x\le 2\pi$ if

$$\left(\int_0^{2\pi}\big|f(x+t)-f(x)\big|^r\,dx\right)^{1/r} = O\big(|t|^\alpha\big), \quad 0<\alpha\le 1, r\ge 1, t>0.$$

Given a positive increasing function $\xi(t), f(x)\in \operatorname{Lip}(\xi(t),r)$, [2] if

$$\omega_r(t;f) = \left(\int_0^{2\pi}\big|f(x+t)-f(x)\big|^r\,dx\right)^{1/r} = O\big(\xi(t)\big), \quad r\ge 1, t>0, \tag{1.4}$$

we observe that

$$\operatorname{Lip}\big(\xi(t),r\big) \xrightarrow{\xi(t)=t^\alpha} \operatorname{Lip}(\alpha,r) \xrightarrow{r\to\infty} \operatorname{Lip}\alpha \quad \text{for } 0<\alpha\le 1, r\ge 1, t>0.$$

L_r-norm of a function $f : R \to R$ is defined by

$$\|f\|_r = \left(\int_0^{2\pi} \left|f(x)\right|^r dx \right)^{1/r}, \quad r \ge 1. \tag{1.5}$$

L_∞-norm of a function $f : R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$.

A signal (function) f is approximated by trigonometric polynomials t_n of order n and the degree of approximation $E_n(f)$ is given by Zygmund [13]

$$E_n(f) = \min_n \left\| f(x) - t_n(f;x) \right\|_r \tag{1.6}$$

in terms of n, where $t_n(f;x)$ is a trigonometric polynomial of degree n. This method of approximation is called Trigonometric Fourier Approximation (TFA) [6].

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\| \ \|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\left\{ \left|t_n(x) - f(x)\right| : x \in R \right\}.$$

Let $f(x)$ be a 2π-periodic function and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.7}$$

with nth partial sum $s_n(f;x)$.

The conjugate series of Fourier series (1.7) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{1.8}$$

Particular cases:

(1) $(N,p_n)(E,q)$ means reduces to $(N, \frac{1}{n+1})(E,q)$ means if $p_n = \frac{1}{n+1}$.
(2) $(N,p_n)(E,q)$ means reduces to $(N, \frac{1}{n+1})(E,1)$ means if $p_n = \frac{1}{n+1}$ and $q_n = 1\ \forall n$.
(3) $(N,p_n)(E,q)$ means reduces to $(N,p_n)(E,1)$ means if $q_n = 1\ \forall n$.
(4) $(N,p_n)(E,q)$ means reduces to $(C,\delta)(E,q)$ means if $p_n = \binom{n+\delta-1}{\delta-1}$, $\delta > 0$.
(5) $(N,p_n)(E,q)$ means reduces to $(C,\delta)(E,1)$ means if $p_n = \binom{n+\delta-1}{\delta-1}$, $\delta > 0$ and $q_n = 1\ \forall n$.
(6) $(N,p_n)(E,q)$ means reduces to $(C,1)(E,1)$ means if $p_n = 1$ and $q_n = 1\ \forall n$.

We use the following notations throughout this paper:

$$\psi(t) = f(x+t) - f(x-t),$$

$$\tilde{G}_n(t) = \frac{1}{2\pi P_n} \left[\sum_{k=0}^{n} \frac{p_{n-k}}{(1+q)^k} \sum_{\nu=0}^{k} \left(\binom{k}{\nu} q^{k-\nu} \frac{\cos(\nu + 1/2)t}{\sin t/2} \right) \right].$$

2 Main result

The approximation of a function $\tilde{f}$, conjugate to a periodic function $f \in \mathrm{Lip}(\xi(t), r)$ using product $(N,p_n)(E,q)$ summability, has not been studied so far. Therefore, the purpose of the present paper is to establish a quite new theorem on the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to $\mathrm{Lip}(\xi(t),r)$-class, by $(N,p_n)(E,q)$ means of conjugate series of Fourier series. In fact, we prove the following theorem.

Theorem 2.1 *If $\tilde{f}(x)$ is conjugate to a 2π-periodic function f belonging to* $\mathrm{Lip}(\xi(t),r)$*-class, then its degree of approximation by $(N,p_n)(E,q)$ product summability means of conjugate series of Fourier series is given by*

$$\|\tilde{t}_n^{NE}-\tilde{f}\|_r=\mathrm{O}\left\{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right\} \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left(\int_0^{\pi/n+1}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r dt\right)^{1/r}=\mathrm{O}\left((n+1)^{-1}\right) \tag{2.2}$$

and

$$\left(\int_{\pi/n+1}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right)^{1/r}=\mathrm{O}(n+1)^{\delta}, \tag{2.3}$$

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $r^{-1}+s^{-1}=1$, $1\le r\le\infty$, conditions (2.2) *and* (2.3) *hold uniformly in x and $\tilde{t}_n^{NE}$ is $(N,\widetilde{p_n)(E},q)$ means of the series* (1.8), *and the conjugate function $\tilde{f}(x)$ is defined for almost every x by*

$$\tilde{f}(x)=-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot(t/2)\,dt=\lim_{h\to 0}\left(-\frac{1}{2\pi}\int_h^{\pi}\psi(t)\cot(t/2)\,dt\right). \tag{2.4}$$

Note 2.2 $\xi(\frac{\pi}{n+1})\le\pi\xi(\frac{1}{n+1})$, for $(\frac{\pi}{n+1})\ge(\frac{1}{n+1})$.

Note 2.3 The product transform plays an important role in signal theory as a double digital filter [7] and the theory of machines in mechanical engineering.

3 Lemmas

For the proof of our theorem, the following lemmas are required.

Lemma 3.1 $|\tilde{G}_n(t)|=\mathrm{O}[1/t]$ *for* $0<t\le\pi/(n+1)$.

Proof For $0<t\le\pi/(n+1)$, $\sin(t/2)\ge(t/\pi)$ and $|\cos nt|\le 1$,

$$\begin{aligned}
\left|\tilde{G}_n(t)\right| &= \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos(\nu+1/2)t}{\sin t/2}\right]\right| \\
&\le \frac{1}{2\pi P_n}\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{|\cos(\nu+1/2)t|}{|\sin t/2|}\right] \\
&\le \frac{1}{2tP_n}\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right], \quad \text{since } \sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}=(1+q)^k
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2tP_n}\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}(1+q)^k\right] \\
&= \frac{1}{2tP_n}\left[\sum_{k=0}^{n}p_{n-k}\right] \\
&= \mathrm{O}[1/t], \quad \text{since } \sum_{k=0}^{n}p_{n-k} = P_n.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2 $|\tilde{G}_n(t)| = \mathrm{O}[1/t]$ *for* $0 < \pi/(n+1) \le t \le \pi$ *and any* n.

Proof For $0 < \pi/(n+1) \le t \le \pi$, $\sin(t/2) \ge (t/\pi)$,

$$\begin{aligned}
\left|\tilde{G}_n(t)\right| &= \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos(\nu+1/2)t}{\sin t/2}\right]\right| \\
&\le \frac{1}{2tP_n}\left|\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\text{Real part of }\binom{k}{\nu}q^{k-\nu}e^{i(\nu+1/2)t}\right\}\right]\right| \\
&\le \frac{1}{2tP_n}\left|\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\text{Real part of }\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|\left|e^{it/2}\right| \\
&= \frac{1}{2tP_n}\left|\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\text{Real part of }\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right| \\
&= \frac{1}{2tP_n}\left|\sum_{k=0}^{\tau-1}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\text{Real part of }\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right| \\
&\quad + \frac{1}{2tP_n}\left|\sum_{k=\tau}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\text{Real part of }\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right|. \qquad (3.1)
\end{aligned}$$

Now, considering the first term of equation (3.1),

$$\begin{aligned}
&\frac{1}{2tP_n}\left|\sum_{k=0}^{\tau-1}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\text{Real part of }\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right| \\
&\quad\le \frac{1}{2tP_n}\left|\sum_{k=0}^{\tau-1}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right\}\right]\right|\left|e^{i\nu t}\right| \\
&\quad\le \frac{1}{2tP_n}\left|\sum_{k=0}^{\tau-1}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right\}\right]\right| \\
&\quad= \frac{1}{2tP_n}\left|\sum_{k=0}^{\tau-1}p_{n-k}\right|. \qquad (3.2)
\end{aligned}$$

Now, considering the second term of equation (3.1) and using Abel's lemma

$$\begin{aligned}
&\frac{1}{2tP_n}\left|\sum_{k=\tau}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\left\{\sum_{\nu=0}^{k}\text{Real part of}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right\}\right]\right| \\
&\quad\le\frac{1}{2tP_n}\sum_{k=\tau}^{n}\frac{p_{n-k}}{(1+q)^k}\max_{0\le m\le k}\left|\sum_{\nu=0}^{m}\binom{k}{\nu}q^{k-\nu}e^{i\nu t}\right| \\
&\quad\le\frac{1}{2tP_n}\sum_{k=\tau}^{n}\frac{p_{n-k}}{(1+q)^k}\max_{0\le m\le k}\sum_{\nu=0}^{m}\binom{k}{\nu}q^{k-\nu}\left|e^{i\nu t}\right| \\
&\quad=\frac{1}{2tP_n}\sum_{k=\tau}^{n}\frac{p_{n-k}}{(1+q)^k}\max_{0\le m\le k}\sum_{\nu=0}^{m}\binom{k}{\nu}q^{k-\nu} \\
&\quad\le\frac{1}{2tP_n}\sum_{k=\tau}^{n}\frac{p_{n-k}}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}=\frac{1}{2tP_n}\sum_{k=\tau}^{n}p_{n-k}. \qquad (3.3)
\end{aligned}$$

On combining (3.1), (3.2) and (3.3), we have

$$\begin{aligned}
&\left|\tilde{G}_n(t)\right|\le\frac{1}{2tP_n}\sum_{k=0}^{\tau-1}p_{n-k}+\frac{1}{2tP_n}\sum_{k=\tau}^{n}p_{n-k}, \\
&\left|\tilde{G}_n(t)\right|=\mathrm{O}[1/t].
\end{aligned}$$

This completes the proof of Lemma 3.2. □

4 Proof of theorem

Let $\tilde{s}_n(x)$ denote the partial sum of series (1.8), we have

$$\tilde{s}_n(x)-\tilde{f}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+1/2)t}{\sin t/2}\,dt.$$

Therefore, using (1.2), the (E,q) transform E_n^q of $\tilde{s}_n$ is given by

$$\tilde{E}_n^q(x)-\tilde{f}(x)=\frac{1}{2\pi(1+q)^k}\int_0^{\pi}\frac{\psi(t)}{\sin t/2}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cos(k+1/2)t\right\}dt.$$

Now, denoting $\widetilde{(N,p_n)}(E,q)$ transform of $\tilde{s}_n$ as $\tilde{t}_n^{NE}$, we write

$$\begin{aligned}
\tilde{t}_n^{NE}(x)-\tilde{f}(x)&=\frac{1}{2\pi P_n}\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\int_0^{\pi}\frac{\psi(t)}{\sin t/2}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\cos(\nu+1/2)t\right\}dt\right] \\
&=\int_0^{\pi}\psi(t)\tilde{G}_n(t)\,dt \\
&=\left[\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi}\right]\psi(t)\tilde{G}_n(t)\,dt \\
&=I_1+I_2\quad\text{(say)}. \qquad (4.1)
\end{aligned}$$

We consider

$$|I_1| \le \int_0^{\pi/(n+1)} \left|\psi(t)\right|\left|\tilde{G}_n(t)\right| dt.$$

Using Hölder's inequality, equation (2.2) and Lemma (3.1), we get

$$\begin{aligned}
|I_1| &\le \left[\int_0^{\pi/(n+1)} \left(\frac{t|\psi(t)|}{\xi(t)}\right)^r dt\right]^{1/r} \left[\lim_{h\to 0}\int_h^{\pi/(n+1)} \left(\frac{\xi(t)|\tilde{G}_n(t)|}{t}\right)^s dt\right]^{1/s} \\
&= \mathrm{O}\left(\frac{1}{n+1}\right)\left[\lim_{h\to 0}\int_h^{\pi/(n+1)} \left(\frac{\xi(t)|\tilde{G}_n(t)|}{t}\right)^s dt\right]^{1/s} \\
&= \mathrm{O}\left(\frac{1}{n+1}\right)\left[\lim_{h\to 0}\int_h^{\pi/(n+1)} \left(\frac{\xi(t)}{t^2}\right)^s dt\right]^{1/s}.
\end{aligned}$$

Since $\xi(t)$ is a positive increasing function, using the second mean value theorem for integrals,

$$\begin{aligned}
I_1 &= \mathrm{O}\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{\pi}{n+1}\right)\right\}\left[\lim_{h\to 0}\int_h^{\pi/(n+1)} \left(\frac{1}{t^2}\right)^s dt\right]^{1/s} \\
&= \mathrm{O}\left\{\left(\frac{1}{n+1}\right)\pi\xi\left(\frac{1}{n+1}\right)\right\}\left[\lim_{h\to 0}\int_h^{\pi/(n+1)} t^{-2s}\, dt\right]^{1/s}, \quad \text{in view of note (2.2)} \\
&= \mathrm{O}\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-2s+1}}{-2s+1}\right\}_h^{\pi/(n+1)}\right]^{1/s}, \quad h\to 0 \\
&= \mathrm{O}\left[\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)(n+1)^{2-1/s}\right] \\
&= \mathrm{O}\left[\xi\left(\frac{1}{n+1}\right)(n+1)^{1-1/s}\right] \\
&= \mathrm{O}\left[\xi\left(\frac{1}{n+1}\right)(n+1)^{1/r}\right] \quad \because r^{-1}+s^{-1}=1, 1\le r\le\infty.
\end{aligned} \tag{4.2}$$

Now, we consider

$$|I_2| \le \int_{\pi/(n+1)}^{\pi} \left|\psi(t)\right|\left|\tilde{G}_n(t)\right| dt.$$

Using Hölder's inequality, equation (3.2) and Lemma 3.2, we have

$$\begin{aligned}
|I_2| &\le \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right]^{1/r} \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)|\tilde{G}_n(t)|}{t^{-\delta}}\right)^s dt\right]^{1/s} \\
&= \mathrm{O}\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)|\tilde{G}_n(t)|}{t^{-\delta}}\right)^s dt\right]^{1/s} \\
&= \mathrm{O}\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{-\delta}t}\right)^s dt\right]^{1/s} \\
&= \mathrm{O}\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{-\delta+1}}\right)^s dt\right]^{1/s}.
\end{aligned}$$

Now, putting $t = 1/y$,

$$I_2 = \mathrm{O}\{(n+1)^\delta\}\left[\int_{1/\pi}^{(n+1)/\pi}\left(\frac{\xi(1/y)}{y^{\delta-1}}\right)^s\frac{dy}{y^2}\right]^{1/s}.$$

Since $\xi(t)$ is a positive increasing function, so $\frac{\xi(1/y)}{1/y}$ is also a positive increasing function and using the second mean value theorem for integrals, we have

$$\begin{aligned}
I_2 &= \mathrm{O}\left\{(n+1)^\delta\frac{\xi(\pi/n+1)}{\pi/n+1}\right\}\left[\int_{1/\pi}^{(n+1)/\pi}\frac{dy}{y^{\delta s+2}}\right]^{1/s}\\
&= \mathrm{O}\left\{(n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right\}\left\{\left[\frac{y^{-\delta s-2+1}}{-\delta s-2+1}\right]_{1/\pi}^{(n+1)/\pi}\right\}^{1/s}\\
&= \mathrm{O}\left\{(n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right\}\left\{\left[y^{-\delta s-1}\right]_{1/\pi}^{(n+1)/\pi}\right\}^{1/s}\\
&= \mathrm{O}\left\{(n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right\}(n+1)^{-\delta-1/s}\\
&= \mathrm{O}\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\delta+1-\delta-1/s}\right\}\\
&= \mathrm{O}\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{1/r}\right\} \quad \because r^{-1}+s^{-1}=1, 1\le r\le\infty.
\end{aligned} \tag{4.3}$$

Combining I_1 and I_2 yields

$$\left|\tilde{t}_n^{NE}-\tilde{f}\right| = \mathrm{O}\left\{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right\}. \tag{4.4}$$

Now, using the L_r-norm of a function, we get

$$\begin{aligned}
\left\|\tilde{t}_n^{NE}-\tilde{f}\right\|_r &= \left\{\int_0^{2\pi}\left|\tilde{t}_n^{NE}-\tilde{f}\right|^r dx\right\}^{1/r}\\
&= \mathrm{O}\left\{\int_0^{2\pi}\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right)^r dx\right\}^{1/r}\\
&= \mathrm{O}\left\{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\left(\int_0^{2\pi}dx\right)^{1/r}\right\}\\
&= \mathrm{O}\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 2.1.

5 Applications

The study of the theory of trigonometric approximation is of great mathematical interest and of great practical importance. The following corollaries can be derived from our main Theorem 2.1.

Corollary 5.1 *If* $\xi(t) = t^\alpha$, $0 < \alpha \le 1$, *then the class* $\mathrm{Lip}(\xi(t), r)$, $r \ge 1$ *reduces to the class* $\mathrm{Lip}(\alpha, r)$, $1/r < \alpha \le 1$ *and the degree of approximation of a function* $\tilde{f}(x)$, *conjugate to a*

2π-periodic function f belonging to the class $\mathrm{Lip}(\alpha, r)$, *by* $(N, p_n)(E, q)$*-means is given by*

$$\left|\tilde{t}_n^{NE} - \tilde{f}\right| = \mathrm{O}\left(\frac{1}{(n+1)^{\alpha-1/r}}\right). \tag{5.1}$$

Proof We have

$$\begin{aligned}\left\|\tilde{t}_n^{NE} - \tilde{f}\right\|_r &= \left\{\int_0^{2\pi} \left|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right|^r dx\right\}^{1/r} = \mathrm{O}\left((n+1)^{1/r}\xi\left(1/(n+1)\right)\right) \\ &= \mathrm{O}\left((n+1)^{-\alpha+1/r}\right).\end{aligned}$$

Thus, we get

$$\left|\tilde{t}_n^{NE} - \tilde{f}\right| \le \left\{\int_0^{2\pi} \left|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right|^r dx\right\}^{1/r} = \mathrm{O}\left((n+1)^{-\alpha+1/r}\right), \quad r \ge 1.$$

This completes the proof of Corollary 5.1. □

Corollary 5.2 *If* $\xi(t) = t^\alpha$ *for* $0 < \alpha < 1$ *and* $r = \infty$ *in Corollary* 5.1, *then* $f \in \mathrm{Lip}\,\alpha$ *and*

$$\left|\tilde{t}_n^{NE} - \tilde{f}\right| = \mathrm{O}\left(\frac{1}{(n+1)^{\alpha}}\right). \tag{5.2}$$

Proof For $r \to \infty$, we get

$$\left\|\tilde{t}_n^{NE} - \tilde{f}\right\|_\infty = \sup_{0 \le x \le 2\pi} \left|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right| = \mathrm{O}\left((n+1)^{-\alpha}\right).$$

Thus, we get

$$\begin{aligned}\left|\tilde{t}_n^{NE} - \tilde{f}\right| &\le \left\|\tilde{t}_n^{NE} - \tilde{f}\right\|_\infty \\ &= \sup_{0 \le x \le 2\pi} \left|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right| \\ &= \mathrm{O}\left((n+1)^{-\alpha}\right).\end{aligned}$$

This completes the proof of Corollary 5.2. □

Corollary 5.3 *If* $\xi(t) = t^\alpha$, $0 < \alpha \le 1$, *then the class* $\mathrm{Lip}(\xi(t), r)$, $r \ge 1$, *reduces to the class* $\mathrm{Lip}(\alpha, r)$, $1/r < \alpha \le 1$ *and if* $q = 1$, *then* (E, q) *summability reduces to* $(E, 1)$ *summability and the degree of approximation of a function* $\tilde{f}(x)$, *conjugate to a* 2π*-periodic function* f *belonging to the class* $\mathrm{Lip}(\alpha, r)$, *by* $(N, p_n)(E, 1)$*-means is given by*

$$\left\|\tilde{t}_n^{NE} - \tilde{f}\right\|_r = \mathrm{O}\left(\frac{1}{(n+1)^{\alpha-1/r}}\right). \tag{5.3}$$

Corollary 5.4 *If* $\xi(t) = t^\alpha$ *for* $0 < \alpha < 1$ *and* $r = \infty$ *in Corollary* 5.3, *then* $f \in \mathrm{Lip}\,\alpha$ *and*

$$\left\|\tilde{t}_n^{NE} - \tilde{f}\right\|_\infty = \mathrm{O}\left(\frac{1}{(n+1)^{\alpha}}\right). \tag{5.4}$$

Remark An independent proof of above Corollary 5.3 can be obtained along the same line of our main theorem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
VNM, KK and LNM contributed equally to this work. All the authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Road, Surat, Gujarat 395007, India. [2]Dr. Ram Manohar Lohia Avadh University, Hawai Patti Allahabad Road, Faizabad, Uttar Pradesh 224 001, India.

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