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Approximation of Signals by General Matrix Summability with Effects of Gibbs Phenomenon

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Abstract: In the proposed paper the degree of approximation of signals (functions) belonging to $Lip(\alpha, p_n)$ class has been obtained using general sub-matrix summability and a new theorem is established that generalizes the results of Mittal and Singh [10] (see [M. L. Mittal and Mradul Veer Singh, Approximation of signals (functions) by trigonometric polynomials in L_p-norm, *Int. J. Math. Math. Sci.* **2014** (2014), Article ID 267383, 1-6]). Furthermore, as regards to the convergence of Fourier series of the signals, the effect of the Gibbs Phenomenon has been presented with a comparison among different means that are generated from sub-matrix summability mean together with the partial sum of Fourier series of the signals.

Key Words: Trigonometric approximation, Signal functions, Gibbs Phenomenon, L_p-norm.

Contents

1. Introduction, Definitions and Motivation

The study of the Theory of Approximation, is an exceptionally broad field and the investigation of the hypothesis of trigonometric estimation is of incredible scientific interest and of incredible functional significance. As mentioned in [14], the L_p-space in general, and L_2 and L_1 specifically assume an essential part of the hypothesis of signals. It is believed that the Theory of Approximation which started from a surely understood hypothesis of Weierstrass, has turned into an exciting interdisciplinary field of study for the last 130 years. These approximations have expected imperative new measurements because of their wide applications in Signal Analysis (see [13]), in general and specifically in Digital Signal Processing. Investigation of signals or time capacities is of awesome significance, since it passes on data

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or characteristics of some phenomenon. The Engineers and Scientists use properties of Fourier approximation for outlining digital filters and signals. Particularly, Psarakis and Moustakides [14] exhibited another L_2 based technique for outlining the Finite Impulse Response digital filters and get comparing optimum approximations having enhanced execution. Recently, Diger *et al.* [4], and Mittal and Singh [10] have obtained numerous nice results on Theory of Approximation utilizing sub-Nörlund, sub-Riesz mean of summability techniques with monotonicity on the rows of the corresponding matrix T (a digital filter) by using sub-Cesàro mean of summability method presented earlier by Armitage and Maddox (see [1]). Till now, nothing appears to have been done for obtaining the degree of approximation of signals (functions) using general sub-matrix mean of summability method. The purpose of the present study is to establish certain new theorems in this direction that will generalize some existing results.

Let $f(x) \in L_p[0, 2\pi]$ $(p \geq 1)$ be a signal function with period 2π, then the Fourier series of f is given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx). \tag{1.1}$$

Let $s_n(f)$ be the n^{th} partial sum of the Fourier series (1.1), then

$$s_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx). \tag{1.2}$$

The integral modulus of continuity of f is defined by

$$\omega_p(f;\delta) = \sup_{0<|h|\leq\delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p dx \right\}^{\frac{1}{p}}. \tag{1.3}$$

If $\omega_p(f;\delta) = O(\delta^\alpha)$ $(\alpha > 0)$, then we write $f \in Lip(\alpha, p)$ $(p \geq 1)$.

For, $p \to \infty$, $Lip(\alpha, p)$ class reduces to the $Lip(\alpha)$ class.

In this paper throughout, $\|.\|_{L_p}$ will denote L_p-norm and is defined by,

$$\|f\|_{L_p} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right\}^{\frac{1}{p}} \quad (f \in L_p;\ p \geq 1)$$

and L_∞- norm of a function over $\mathbb{R}$ is defined by,

$$\|f\|_{L_\infty} = \sup\{|f(x)| : x \in \mathbb{R}\}.$$

The degree of approximation of a function over $\mathbb{R}$ by trigonometric polynomial (t_n) of degree n under supremum norm $\|.\|_{L_\infty}$ is defined by Zygmund (see [16]) and given us,

$$\|t_n - f\|_{L_\infty} = \sup\{|t_n - f(x)| : x \in \mathbb{R}\}$$

and error E_n of a function $f \in L_p$ is defined by

$$E_n = \min_n \|t_n - f\|_{L_p}.$$

The formula of Abel's Transformation is given by,

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k(v_k - v_{k+1}) - U_{m-1}v_m + U_n v_n \ (0 \leq m \leq n), \tag{1.4}$$

$$U_k = u_0 + u_1 + ... + u_k,\ k \geq 0,\ U_{-1} = 0,$$

which can be verified, is known as Abel's transformation and will be used extensively in what follows.

If $v_m, v_{m+1}, ..., v_n$ are non-negative and non-increasing, the left hand side of (1.4) does not exceed

$$2v_m \max_{m-1 \leq k \leq n} |U_k|$$

in absolute value. In fact,

$$\left|\sum_{k=m}^{n} u_k v_k\right| = \max |U_k| \left\{\sum_{k=m}^{n-1} (v_k - v_{k+1}) - v_m + v_n\right\} = 2v_m \max |U_k|. \tag{1.5}$$

A non-negative sequence (c_n) is known as almost monotone decreasing (increasing) if there exists a constant $K = K(c)$, depending on the sequence c only, such that,

$$c_n \leq Kc_m \ (c_m \leq Kc_n) \ (\forall\ n \geq m).$$

A non-negative sequence (c_n) which is either almost increasing sequence or almost decreasing sequence is called an almost monotone sequence.

Let $F \subset N$ be infinite and it be the range of strictly increasing sequence of positive integers of the form $F = (\lambda(n))_{n=1}^{\infty}$. The method of sub-Cesàro summability (C_λ) is defined by

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k \qquad (n = 1, 2, 3...),$$

where (x_k) is a real sequence. Therefore, C_λ summability method is a subsequence of the Cesàro (C_1) summability method and hence it is regular for any λ. Also C_λ is obtained by deleting a set of rows from Cesàro matrix. The reader will be known about the most fundamental properties of C_λ method see (Armitage and Maddox [1], Osikiewicz [12]). In the present paper, to determine the degree of approximation of signals $f \in Lip(\alpha, p)$ by imposing n^{th} degree trigonometric polynomial $(T_n^\lambda(f))$, we first set

$$T_n^\lambda(f) = \sum_{k=0}^{\lambda(n)} (a_{\lambda(n),k}) s_k(f), \tag{1.6}$$

where

$$s_k(f) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)D_n(t)dt$$

with

$$D_n(t) = \frac{\sin(\frac{n+1}{2})t}{2\sin(t/2)}.$$

Here, throughout the paper $T = (a_{\lambda(n),k})$ will denote a lower triangular infinite matrix of real numbers such that,

$$a_{\lambda(n),k} \geq 0 \ (\lambda(n) \geq k),\ \ a_{\lambda(n),k} = 0 \ (\lambda(n) < k)$$

and

$$\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} = 1 \ (\lambda(n), v = 0, 1, 2, 3, ...).$$

We shall also use the notations,

$$\Delta_k(a_{\lambda(n),k}) = (a_{\lambda(n),k} - a_{\lambda(n),k+1}).$$

The result in equation (1.6) is the generalization of the following known results:

(a) for $a_{\lambda(n),k} = \frac{p_{\lambda(n)-k}}{P_{\lambda(n)}}$, the trigonometric polynomial $T_m^{\lambda}(f)$ is reduced to the trigonometric polynomial $N_m^{\lambda}(f)$ (see [10]). In this case, we write

$$N_n^{\lambda}(f) = \frac{1}{P_{\lambda(n)}}\sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k}s_k(f); \tag{1.7}$$

where,

$$P_{\lambda(n)} = p_0 + p_0 + ... + p_{\lambda(n)}.$$

(b) for $a_{\lambda(n),k} = \frac{p_k}{P_{\lambda(n)}}$, the trigonometric polynomial $T_n^{\lambda}(f)$ is reduced to the trigonometric polynomial $R_n^{\lambda}(f)$ (see [10]). In this case, we write

$$R_n^{\lambda}(f) = \frac{1}{P_{\lambda(n)}}\sum_{k=0}^{\lambda(n)} p_k s_k(f), \tag{1.8}$$

where,

$$P_{\lambda(n)} = p_0 + p_0 + ... + p_{\lambda(n)}.$$

(c) for $a_{\lambda(n),k} = \frac{1}{\lambda(n)+1}$, the trigonometric polynomial $T_n^{\lambda}(f)$ is reduced to the trigonometric polynomial $C_n^\lambda(f)$ (see [10]). In this case, we write

$$C_n^\lambda(f) = \frac{1}{\lambda(n)+1}\sum_{k=0}^{\lambda(n)} s_k(f). \tag{1.9}$$

Next, we define the product of sub-Cesàro summability $C_n^\lambda(f)$ with a sub-Nörlund summability $N_n^\lambda(f)$ denoted by $A_{\lambda(n),k}(f)$-summability and it has the mean given by,

$$\begin{aligned} A_{\lambda(n),k}(f) &= \frac{1}{\lambda(n)+1}\sum_{\lambda(k)=0}^{\lambda(n)} \frac{1}{P_{\lambda(n)}}\sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k}s_k(f) \\ &= \frac{1}{\lambda(n)+1}\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} \ (0 \le k \le \lambda(n)). \end{aligned} \tag{1.10}$$

Here,

$$A_{\lambda(n),k} = \frac{1}{\lambda(n)+1}\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}$$

and the matrix $A_{\lambda(n),k}$ is said to be almost row monotone for each $0 \le k \le \lambda(n)$, whenever, $T = (a_{\lambda(n),k})$ is either almost increasing or almost decreasing in k and $0 \le k \le \lambda(n)$.

Remark 1.1. The product transforms $A_n^\lambda(f)$ of this form plays an important role as a double digital filter [5, 6] in signal theory as well as the theory of Machines in Mechanical Engineering (see [5]).

Many researchers like, Quade [15], Mohapatra and Russell [11], Chandra [2] and a few others used different summability means to determine the degree of approximations of trigonometric polynomials. Further Mittal and Rhoades [7, 8] estimate the error of trigonometric polynomials by Fourier series expansion. In 2002, Chandra [3] has established a result of the degree of approximation of the trigonometric polynomial using (N, p_m) matrix. After that Mittal *et al.* [9] proved a theorem on the degree approximation of the trigonometric polynomial using lower triangular infinite matrix. Very recently, Deger *et al.* [4] and Mittal and Singh [10] used a more general sub-Cesàro summability mean (C_λ) (see Armitage and Maddox [1]) to establish a result of the approximation of signals by trigonometric polynomials in L_p- norm. In order to have some advance study in this direction, in the proposed paper, we have established a new theorem on the degree of approximation of signals $(f \in Lip(\alpha, p))$ under some weaker conditions by using general sub-matrix mean $T_n^\lambda(f)$ (that is, weakening the conditions of the filter, we enhance the quality of digital filter) that generalizes some known theorems. Further, we have established a new result on the approximation of signals of $(f \in Lip(\alpha, p))$ class by the product of sub-Cesàro mean and sub-Nörlund mean $(A_{\lambda(n),k}(f))$. Next as regards to convergence, the signals for n^{th} partial sum of Fourier series and signals or $C_n^\lambda(f)$ mean, $(N_n^\lambda(f))$ mean and $(A_{\lambda(n),k}(f))$ means are plotted by using Matlab and are compared in a suitable example.

2. Known Results

Dealing with a degree of approximations, Deger *et al.* [4] and Mittal and Singh [10] in the year 2012 and 2014 respectively established the following theorems.

Theorem 2.1. (see [4]) *Let $f \in Lip(\alpha, p)$ and let (p_n) be a positive sequence such that*

$$(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}).$$

If one of the conditions hold true,

(i) $p > 1,\ \alpha \in (0,1]$ and p_n *is monotonic sequence;*

(ii) $p = 1,\ \alpha \in (0,1)$ and p_n *is monotonic increasing sequence, then*

$$\|N_n^\lambda(f) - f\|_{L_p} = O\left(\frac{1}{n^\alpha}\right).$$

Theorem 2.2. (see [4]) *Let $f \in Lip(\alpha, 1),\ \alpha \in (0,1)$.*

If the positive sequence (p_n) satisfies

$$(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}),$$

and (p_n) is a monotonic increasing sequence, then

$$\|R_n^\lambda(f) - f\|_1 = O\left(\frac{1}{n^\alpha}\right).$$

Theorem 2.3. (see [10]) *If $f \in Lip(\alpha, p)$ and (p_n) is positive and if one the following conditions*

(i) $p > 1,\ \alpha \in (0,1)$ *and p_n is almost decreasing sequence;*

(ii) $p > 1,\ \alpha \in (0,1),\ p_n$ *is almost decreasing sequence and $(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)})$ holds;*

(iii) $p > 1,\ \alpha = 1,$ *and* $\displaystyle\sum_{v=1}^{\lambda(n)-1} v|\alpha p_v| = O(P_{\lambda(n)});$

(iv) $p > 1,\ \alpha = 1,\ \displaystyle\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$, and $(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)})$

holds;

(v) $p = 1,\ \delta \in (0, 1]$ and $\sum_{k=-1}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$, *then*

$$\|N_m^{\lambda}(f) - f\|_{L_p} = O\left(\frac{1}{(\lambda(n))^{\alpha}}\right).$$

Theorem 2.4. (see [10] *Let $f \in Lip(\alpha, 1)$, $\alpha \in (0, 1)$. If the positive (p_n) satisfies*

$$(\lambda(n) + 1)p_{\lambda(n)} = O(P_{\lambda(n)}),$$

and the condition

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$$

holds, then

$$\|R_n^{\lambda}(f) - f\|_{L_1} = O\left(\frac{1}{(\lambda(n))^{\alpha}}\right).$$

3. Main Results

Theorem 3.1. *Let $f \in Lip(\alpha, p)$, if one of the conditions holds true*

(i) $p > 1,\ \alpha \in (0, 1)$, $(a_{\lambda(n),k})$ *is almost decreasing sequence and* $(\lambda(n)+1)a_{\lambda(n),0} = O(1)$;

(ii) $p > 1,\ \alpha \in (0, 1)$ *and* $(a_{\lambda(n),k})$ *is almost increasing sequence;*

(iii) $p > 1,\ \alpha = 1$ *and* $\sum_{k=0}^{\lambda(n)-1} |\Delta_k A_{\lambda(n),k}| = O\left(\frac{1}{\lambda(n)}\right)$;

(iv) $p = 1,\ \alpha \in (0, 1)$, $\sum_{v=0}^{\lambda(n)-1} |\Delta_k a_{\lambda(n),k}| = O\left(\frac{1}{\lambda(n)}\right)$ *and* $(\lambda(n) + 1)a_{\lambda(n),\lambda(n)} = O(1)$, *then*

$$\|T_n^{\lambda}(f) - f\|_{L_p} = O\left(\frac{1}{(\lambda(n))^{\alpha}}\right). \tag{3.1}$$

Each of the following Lemmas will be needed in our present work.

Lemma 3.2. (see [15]) *If $f \in Lip(\alpha, p)$, for $\alpha \in (0,1]$ and $p > 1$, then*

$$\|s_n(f) - f\|_{L_p} = O\left(\frac{1}{n^\alpha}\right). \tag{3.2}$$

Lemma 3.3. (see [15]) *If $f \in Lip(1, p)$, for $p > 1$, then*

$$\|\sigma_n(f) - s_n(f)\|_{L_p} = O\left(\frac{1}{n}\right). \tag{3.3}$$

Lemma 3.4. (see [15]) *If $f \in Lip(\alpha, 1)$, $\alpha \in (0,1)$, then*

$$\|\sigma_n(f) - f\|_1 = O\left(\frac{1}{n^\alpha}\right). \tag{3.4}$$

Lemma 3.5. *Let $a_{\lambda(n),k} \geq 0$ $(\lambda(n) \geq k)$ and $a_{\lambda(n),k} = 0$ $(\lambda(n) < k)$, such that $\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} = 1$.*

If $(a_{\lambda(n),k})$ is almost increasing sequence or almost decreasing sequence, and

$$(1 + \lambda(n))(a_{\lambda(n),0}) = O(1),$$

then

$$\sum_{v=0}^{\lambda(n)} \frac{1}{(1+k)^\alpha}(a_{\lambda(n),k}) = O\left(\frac{1}{(1+\lambda(n))^\alpha}\right) \ (\alpha \in (0,1)). \tag{3.5}$$

Proof. Suppose $q = \left[\frac{\lambda(n)}{2}\right]$, $a_{\lambda(n),k} \geq 0$ $(\lambda(n) \geq k)$ and $a_{\lambda(n),k} = 0$ $(\lambda(n) < k)$, such that $\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} = 1$,

by Abel's transformations, we have

$$\begin{aligned}\sum_{k=0}^{\lambda(n)} \frac{1}{(1+k)^\alpha}(a_{\lambda(n),k}) &\leq \sum_{k=0}^{q} \frac{1}{(1+k)^\alpha}(a_{\lambda(n),k}) + \frac{1}{(1+q)^\alpha} \sum_{k=q+1}^{\lambda(n)} (a_{\lambda(n),k}) \\ &\leq \sum_{k=0}^{q} \frac{1}{(1+v)^\delta}(a_{\lambda(m),v}) + \frac{1}{(1+q)^\delta}.\end{aligned}$$

$$\sum_{k=0}^{\lambda(n)} \frac{1}{(1+k)^{\alpha}}(a_{\lambda(n),k}) \leq \sum_{k=0}^{q-1} \left\{ \frac{1}{(1+k)^{\alpha}} - \frac{1}{(2+k)^{\alpha}} \right\} \sum_{j=0}^{k} (a_{\lambda(n),j})$$

$$+ \frac{1}{(1+q)^{\alpha}} \sum_{k=0}^{q} (a_{\lambda(n),k}) + \frac{1}{(1+q)^{\alpha}}$$

$$\leq \sum_{k=0}^{q-1} \left(\frac{(k+2)^{\alpha} - (k+1)^{\alpha}}{(k+1)^{\alpha-1}(k+2)^{\alpha}} \right) A_{\lambda(n),k} + \frac{1}{(1+q)^{\alpha}}.$$

Using Lagrange's mean value theorem to the function $f(x) = x^{\alpha}$ ($\alpha \in (0,1)$) on the interval $(k+1, k+2)$, we obtain

$$\sum_{k=0}^{\lambda(n)} \frac{1}{(1+k)^{\alpha}}(a_{\lambda(n),k}) \leq \sum_{k=0}^{q-1} \frac{\alpha}{(k+2)^{\alpha}}(A_{\lambda(n),k}) + \frac{1}{(1+q)^{\alpha}}.$$

When, $(a_{\lambda(n),k})$ is almost decreasing sequence and $(1+\lambda(n))(a_{\lambda(n),0}) = O(1)$ we get,

$$\sum_{k=0}^{\lambda(n)} \frac{1}{(1+k)^{\alpha}}(a_{\lambda(n),k}) \leq (A_{\lambda(n),0}) \sum_{k=0}^{q-1} \left(\frac{1}{(k+2)^{\alpha}} + \frac{1}{(1+q)^{\alpha}} \right)$$

$$\leq (q+1)^{1-\alpha}(a_{\lambda(n),0}) + \frac{1}{(1+q)^{\alpha}}$$

$$\leq \frac{1}{(1+\lambda(n))^{\alpha}}.$$

Again, if $(a_{\lambda(n),k})$ almost increasing sequence, then

$$\sum_{k=0}^{\lambda(n)} \frac{1}{(1+k)^{\alpha}}(a_{\lambda(n),k}) \leq (A_{\lambda(n),q}) \sum_{k=0}^{q-1} \left(\frac{1}{(k+2)^{\alpha}} + \frac{1}{(1+q)^{\alpha}} \right)$$

$$\leq \frac{1}{(1+q)^{\alpha}} \sum_{k=0}^{q} (a_{\lambda(n),k}) + \frac{1}{(1+q)^{\alpha}}$$

$$\leq \frac{1}{(1+\lambda(n))^{\alpha}}.$$

This completes proof of the Lemma 3.5. □

4. Proof of the Theorem 3.1

Initially, we wish to prove cases (i) and (ii) together, we have

$$T_n^{\lambda}((f) - f = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(s_k(f) - f)$$

$$\begin{aligned}
\|T_n^{\lambda}((f)-f\|_{L_p} &\leq \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}\|(s_k(f)-f)\|_p \\
&\leq \sum_{k=0}^{\lambda(n)} \frac{1}{(1+k)^{\alpha}}(a_{\lambda(n),k}) \text{ (by Lemma 3.2)} \\
&= O\left(\frac{1}{(\lambda(n))^{\alpha}}\right) \text{ (by Lemma 3.5).}
\end{aligned}$$

Next, under the condition (iii), we have

$$T_n^{\lambda}(f)-f = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(s_k(f)-f).$$

By Abel's transformation, we have

$$\begin{aligned}
T_n^{\lambda}((f)-f = &\sum_{k=0}^{\lambda(n)-1}(s_k(f)-s_{k+1}(f))\sum_{j=0}^{k} a_{\lambda(n),j} + s_{\lambda(n)}(f)-f \\
&= s_n(f)-f-\sum_{k=0}^{\lambda(n)-1}(1+k)U_{k+1}(f)A_{\lambda(n),k} \\
&= s_n(f)-f-\sum_{k=0}^{\lambda(n)-2}(A_{\lambda(n),k}-A_{\lambda(n),k+1})\sum_{j=0}^{k}(j+1)U_{j+1}(f) \\
&\qquad -A_{\lambda(n),\lambda(n)-1}\sum_{k=0}^{\lambda(n)-1}(k+1)U_{k+1}(f) \\
&= s_n(f)-f-\sum_{k=0}^{\lambda(n)-2}(A_{\lambda(n),k}-A_{\lambda(n),k+1})\sum_{j=0}^{k}(j+1)U_{j+1}(f) \\
&\quad -\frac{1}{\lambda(n)}\sum_{j=0}^{\lambda(n)-1} a_{\lambda(n),j}\sum_{k=0}^{\lambda(n)-1}(k+1)U_{k+1}(f), \ \left(\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}=1\right).
\end{aligned}$$

Now by Triangle inequality,

$$\|T_n^\lambda(f)-f\|_{L_p} \le \|s_n(f)-f\|_{L_p} + \sum_{k=0}^{\lambda(n)-2} \left|A_{\lambda(n),k} - A_{\lambda(n),k+1}\right| \left\|\sum_{j=1}^{k+1} jU_j(f)\right\|_{L_p}$$
$$+\frac{1}{\lambda(n)} \left\|\sum_{k=1}^{n} kU_k(f)\right\|_{L_p}. \qquad (4.1)$$

Also,

$$\sigma_n^\lambda(f) - s_n(f) = \frac{1}{(1+\lambda(n))} \sum_{k=1}^{\lambda(n)} kU_k(f).$$

Since,

$$\left\|\sum_{k=1}^{\lambda(n)} kU_k(f)\right\|_{L_p} = (\lambda(n)+1)\|\sigma_n^\lambda(f) - s_n(f)\| = O(1). \text{ (By Lemma 3.3).} \qquad (4.2)$$

From (4.1) and (4.2), we have

$$\begin{aligned} \|T_n^\lambda((f)-f\|_{L_p} &\le \frac{1}{\lambda(n)} + \sum_{k=0}^{\lambda(n)-2} |A_{\lambda(n),k} - A_{\lambda(n),k+1}| \\ &= O\left(\frac{1}{\lambda(n)}\right) \text{ (by condition (iii).} \end{aligned}$$

Finally, for the condition (iv),

$$\begin{aligned} T_n^\lambda(f) - f &= \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(s_k(f)-f) \\ &= \sum_{k=0}^{\lambda(n)-1} (a_{\lambda(n),k} - a_{\lambda(n),k+1}) \sum_{j=0}^{k} (s_j(f)-f) \\ &\quad + a_{\lambda(n),\lambda(n)} \sum_{k=0}^{\lambda(n)} (s_k(f)-f) \\ &= \sum_{k=0}^{\lambda(n)-1} (a_{\lambda(n),k} - a_{\lambda(n),k+1})(k+1)(\sigma_k^\lambda(f)-f) \\ &\quad + a_{\lambda(n),\lambda(n)}(1+\lambda(n))(\sigma_k^\lambda(f)-f). \end{aligned}$$

$$\|T_n^{\lambda}((f)-f)\|_{L_1} \leq \sum_{k=0}^{\lambda(n)-1} (a_{\lambda(n),k} - a_{\lambda(n),k+1})(k+1)\|\sigma_k^{\lambda}(f)-f\|_{L_1}$$

$$+a_{\lambda(n),\lambda(n)}(1+\lambda(n))\|\sigma_k^{\lambda}(f)-f\|_1$$

$$\leq \sum_{k=0}^{\lambda(n)-1} |a_{\lambda(n),k} - a_{\lambda(n),k+1}|(1+k)^{1-\alpha}$$

$$+a_{\lambda(n),\lambda(n)}(1+\lambda(n))^{1-\alpha} \text{ (by Lemma 3.4).}$$

$$\leq (1+\lambda(n))^{1-\alpha}\left(\sum_{k=0}^{\lambda(n)-1} |a_{\lambda(n),k} - a_{\lambda(n),k+1}| + a_{\lambda(n),\lambda(n)}\right)$$

$$\|T_n^{\lambda}(f)-f\|_{L_1} = O\left(\frac{1}{\lambda(n)^{\alpha}}\right).$$

This completes the proof of Theorem 3.1. □

Corollary 4.1. *Let $f \in Lip(\alpha,1)$ $(0<\alpha<1)$. If $\lambda(n)=n$ and the conditions (iv) of Theorem 3.1, that is,*

$$\sum_{k=0}^{n-1} |\Delta_k a_{n,k}| = O\left(\frac{1}{n}\right) \text{ and } (n+1)a_{n,n} = O(1) \text{ holds,}$$

then

$$\|T_n(f)-f)\|_{L_1} = O\left(\frac{1}{n^{\alpha}}\right).$$

Proof. We have,

$$\begin{aligned}
T_n(f)-f &= \sum_{k=0}^{n} a_{n,k}(s_k(f)-f) \\
&= \sum_{k=0}^{n-1}(a_{n,k}-a_{n,k+1})\sum_{j=0}^{k}(s_j(f)-f) + a_{n,n}\sum_{k=0}^{n}(s_k(f)-f) \\
&= \sum_{k=0}^{n-1}(a_{n,k}-a_{n,k+1})(k+1)(\sigma_k^{\lambda}(f)-f) + a_{n,n}(n)(\sigma_k^{\lambda}(f)-f).
\end{aligned}$$

$$\begin{aligned}
\|T_n(f) - f\|_{L_1} \leq & \sum_{k=0}^{n-1}(a_{n,k} - a_{n,k+1})(k+1)\|\sigma_k(f) - f\|_{L_1} \\
& +a_{n,n}(1+n)\|\sigma_k(f) - f\|_{L_1} \\
\leq & \sum_{k=0}^{n-1}|a_{n,k} - a_{n,k+1}|(1+k)^{1-\alpha} + a_{n,n}(n)^{1-\alpha} \text{ (by Lemma 3)} \\
= & (1+n)^{1-\alpha}\left(\sum_{k=0}^{n-1}|a_{n,k} - a_{n,k+1}| + a_{n,n}\right) \\
\|T_n(f) - f\|_{L_1} = & O\left(\frac{1}{n^\alpha}\right).
\end{aligned}$$

This completes the proof of Corollary 4.1. □

Corollary 4.2. *If $p \to \infty$ $(0 < \alpha < 1)$, then the generalized $Lip(\alpha, p)$ reduces to the class $Lip(\alpha)$, and the degree of approximation of a function (f) belonging to the $Lip(\alpha)$-class, given by*

$$\|T_n^\lambda(f) - f\|_{L_\infty} = O\left(\frac{1}{\lambda(n)^\alpha}\right).$$

Proof. For $p \to \infty$ $(0 < \alpha < 1)$, we have

$$\begin{aligned}
\|T_n^\lambda(f) - f\|_{L_\infty} &= \sup\left\{\left|T_n^\lambda(f) - f\right| : 0 \leq x \leq 2\pi\right\} \\
&= O\left(\frac{1}{\lambda(n)^\alpha}\right).
\end{aligned}$$

This establishes of Corollary 4.2. □

Remark 4.3. In Theorem 3.1, as well as in Corollary 4.1 and 4.2, as $(\lambda(n))^{-\alpha} \leq (n)^{-\alpha}$ $(0 < \alpha \leq 1)$, so our result for sub matrix summability gives better estimates (that is, minimizes the error) in comparison to the earlier existing results for general matrix summability methods.

Corollary 4.4. *Let $f \in Lip(\alpha, 1)$ $(0 < \alpha < 1)$. If the conditions,*

$$\sum_{k=0}^{\lambda(n)-1}|\Delta_k a_{\lambda(n),k}| = O\left(\frac{1}{\lambda(n)}\right) \text{ and } (\lambda(n)+1)a_{\lambda(n),\lambda(n)} = O(1) \text{ holds true,}$$

then

$$\|A_{\lambda(n),k}(f) - f\|_{L_1} = O\left(\frac{1}{(\lambda(n))^{1+\alpha}}\right),$$

where $A_{\lambda(n),k}(f)$ *is the mean for the product* $(C_m^\lambda.N_m^\lambda)$.

Proof. Using the conditions we have,

$$\begin{aligned}
A_{\lambda(n),k}(f)-f &= \frac{1}{1+\lambda(n)}\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(s_k(f)-f)\\
&= \frac{1}{1+\lambda(n)}\sum_{k=0}^{\lambda(n)-1}(a_{\lambda(n),k}-a_{\lambda(n),k+1})\sum_{j=0}^{k}(s_j(f)-f)\\
&\qquad +a_{\lambda(n),\lambda(n)}\sum_{k=0}^{\lambda(n)}(s_k(f)-f)\\
&= \frac{1}{1+\lambda(n)}\sum_{k=0}^{\lambda(n)-1}(a_{\lambda(n),k}-a_{\lambda(n),k+1})(k+1)(\sigma_k^\lambda(f)-f)\\
&\qquad +a_{\lambda(n),\lambda(n)}(1+\lambda(n))(\sigma_k^\lambda(f)-f).
\end{aligned}$$

$$\begin{aligned}
\|A_{\lambda(n),k}(f)-f\|_{L_1} &\le \frac{1}{1+\lambda(n)}\sum_{k=0}^{\lambda(n)-1}(a_{\lambda(n),k}-a_{\lambda(n),k+1})(k+1)\|\sigma_k^\lambda(f)-f\|_{L_1}\\
&\quad +a_{\lambda(n),\lambda(n)}(1+\lambda(n))\|\sigma_k^\lambda(f)-f\|_{L_1}\\
&\le \frac{1}{1+\lambda(n)}\sum_{k=0}^{\lambda(n)-1}|a_{\lambda(n),k}-a_{\lambda(n),k+1}|(1+k)^{1-\alpha}\\
&\quad +a_{\lambda(n),\lambda(n)}(1+\lambda(n))^{1-\alpha} \text{ (by lemma 3.4)}
\end{aligned}$$

$$\le \frac{(1+\lambda(n))}{(1+\lambda(n))^{1+\alpha}}\left(\sum_{k=0}^{\lambda(n)-1}|a_{\lambda(n),k}-a_{\lambda(n),k+1}|+a_{\lambda(n),\lambda(n)}\right)$$

$$\|A_{\lambda(n),k}(f)-f\|_{L_1}=O\left(\frac{1}{(\lambda(n))^{1+\alpha}}\right).$$

This establishes Corollary 4.4. □

Remark 4.5 From Corollary 4.4, as $1+\alpha \ge \alpha$, $\alpha \in (0,1)$, so it gives still sharper estimates. Thus, as regards to convergence of $f(x)$, the product summability $(C_m^\lambda.N_m^\lambda)$. gives better estimates than the individuals.

5. Effects of Gibbs Phenomenon and Applications

As regards to the effect of the Gibbs Phenomenon in the following example, we will see how the sub-Cesàro mean $C_n^\lambda(f)$, sub-Nörlund mean $N_n^\lambda(f)$ that are generated for sub-matrix mean $T_n^\lambda(f)$ as mentioned by the authors and the product $A_{\lambda(n),k}(f)$ mean of partial sums of Fourier series of 2π - periodic signal is better behaved than the sequence of partial sums $s_n(x)$ itself.

Consider

$$f(x) = \begin{cases} -1 & (-\pi \le x < 0) \\ 1 & (0 \le x < \pi), \end{cases}$$

be periodic with period 2π. Clearly, it is an odd function. So its Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx = \frac{2}{\pi}\left(\frac{1-(-1)^n}{n}\right).$$

Thus the Fourier series of $f(x)$ is,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx,\ \ x\in[-\pi,\pi]. \tag{5.1}$$

The n^{th} partial sum $s_n(x)$ of Fourier series (5.1), is given by

$$s_{\lambda(n)}(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + ... + \frac{1}{\lambda(n)}\sin nx\right) \tag{5.2}$$

and the average sub-mean of Fourier series (5.1), is given by

$$C_n^\lambda(f) = \frac{2}{\pi}\sum_{k=1}^{\lambda(n)}\left(1-\frac{k}{\lambda(n)}\right)\left(\frac{1-(-1)^k}{k}\right)\sin kx. \tag{5.3}$$

In equation (1.6), if we take $a_{\lambda(n),k} = \frac{p_{\lambda(n)-k}}{P_{\lambda(n)}}$, $p_{\lambda(n)} = \lambda(n)+1$ and $a_{\lambda(n),k} = \frac{1}{\lambda(n)+1}$, then the sub-Nörlund and sub-Cesàro mean are respectively given as

$$N_n^\lambda(f) = \frac{2}{(\lambda(n)+1)(\lambda(n)+2)}\sum_{k=0}^{\lambda(n)}(\lambda(n)-k+1)s_k(f), \tag{5.4}$$

and

$$C_n^{\lambda}(f) = \frac{1}{(\lambda(n)+1)} \sum_{k=0}^{\lambda(n)} s_k(f). \tag{5.5}$$

Finally, in $(C_n^{\lambda}.N_n^{\lambda})$ summability the mean is given by

$$A_{\lambda(n),k}(f) = \frac{2}{(\lambda(n)+1)^2(\lambda(n)+2)} \sum_{k=0}^{\lambda(n)} (\lambda(n)-k+1)s_k(f). \tag{5.6}$$

Now the graphs for the signals, namely graph for n^{th} partial sum $s_n(x)$, sub-Cesàro $C_n^{\lambda}(f)$, sub-Nörlund $N_n^{\lambda}(f)$ and finally for the product sum $A_{\lambda(n),k}(f)$ are plotted in the following figure.

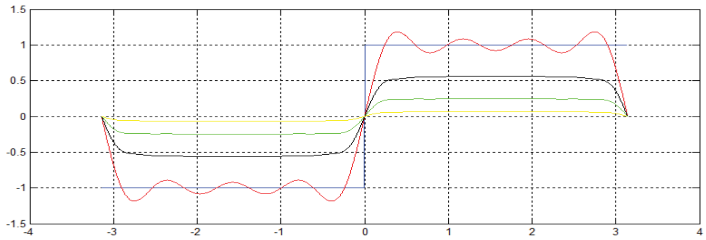

Figure-5(a): The signals

$f(x)$ *(blue)*, $s_m(x)$ *(red)*, $\sigma_m^{\lambda}(f)$ *(black)*, $N_m^{\lambda}(f)$ *(green)*, $A_{\lambda(n),k}(f)$ *(yellow)*, for $\lambda(m)=7$.

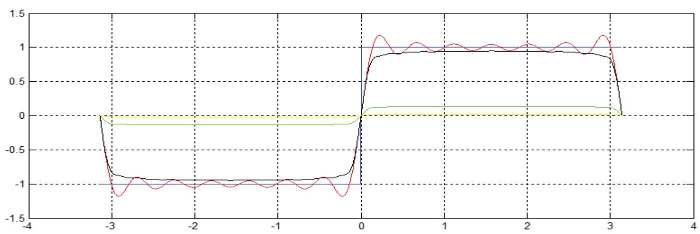

Figure-5(b): The signals

$f(x)$ *(blue)*, $s_m(x)$ *(red)*, $\sigma_m^{\lambda}(f)$ *(black)*, $N_m^{\lambda}(f)$ *(green)*, $A_{\lambda(n),k}(f)$ *(yellow)*, for $\lambda(m)=14$.

From the above graphs we can compare the different signals obtained by summability means with the signal of n^{th} partial sum of Fourier series. Next as regards to Gibbs Phenomenon we conclude the convergence of signals as follows:

According to Gibbs Phenomenon, in the neighborhood of discontinuity, the convergence of Fourier series is not uniform and the sequence of partial sum is over estimated the signal by 18 percent, that is, in the neighborhood of discontinuity overshoots in the peaks of partial sum $s_n(x)$ are noticed closure of the line passing through a point of discontinuity as n- increases.

From the Figure 5(a) and 5(b), we observe that $C_n^\lambda(f)$, $N_m^\lambda(f)$ and $A_{\lambda(n),k}(f)$ converges quickly to $f(x)$ than the sequence of partial sum $s_{\lambda(n)}$ in the interval $[-\pi, \pi]$. We further notice that in the neighborhood of discontinuity that is, in the neighborhood of $-\pi, 0$ and π, the graph of s_7 and s_{14} show overshoots in peaks and move closer the line passing through points of discontinuity as $\lambda(n)$ increases, but in the graph of $C_n^\lambda(f)$, $N_n^\lambda(f)$ and $A_{\lambda(n),k}(f)$, $\lambda(n) = 7, 14$ the peaks become flatter. Clearly, the product summability means of the Fourier series of $f(x)$ overshoot the Gibbs Phenomenon and show the smoothing effect of the method. Thus $C_n^\lambda(f)$, $N_n^\lambda(f)$ and $A_{\lambda(n),k}(f)$ are the better approximates than $s_n(x)$ and product $A_{\lambda(n),k}(f)$ summability is better behaved than the individual $s_{\lambda(n)}, C_n^\lambda$ and N_n^λ summability methods.

References

1. D. H. Armitage and I. J. Maddox, A new type of Cesàro mean, *Anal.* **9** (1989), 195-204.
2. P. Chandra, A note on degree of approximation by Nörlund and Riesz operators, *Mat. Vestnik* **42** (1990), 9-10.
3. P. Chandra, Trigonometric approximation of functions in L_p-norm, *J. Math. Anal. Appl.* **275** (2002), 13-26.
4. U. Deger, İ. Dagadur, and M. Küçükaslan, Approximation by trigonometric polynomials to functions in L_p-norm, *Proc. Jangjeon Math. Soc.* **15** (2012), 203-213.
5. V. N. Mishra, K. Khatri and L. N. Mishra, Product summability transform of conjugate series of Fourier series, *Int. J. Math. Math. Sci.* **2012** (2012), Article ID 298923, 1-14.
6. V. N. Mishra, K. Khatri and L. N. Mishra, Product $(N, p_n)(C, 1)$ summability of a sequence of Fourier coefficients, *Math. Sci.* **6** (2012), Article ID 38, 1-5.
7. M. L. Mittal and B. E. Rhoades, On the degree of approximation of continuous functions by using linear operators on their Fourier series, *Int. J. Math. Game Theory Algebra*, **9** (1999), 259-267.
8. M. L. Mittal and B. E. Rhoades, Degree of approximation to functions in a normed space, *J. Comput. Anal. Appl.* **2** (2000), 1-10.
9. M. L. Mittal, B. E. Rhoades, V. N. Mishra and Uaday Singh: Using infinite matrices to approximate functions of class Lip (α, p) using trigonometric polynomials, *J. Math. Anal. Appl.* **326** (2007), 667-676.
10. M. L. Mittal and Mradul Veer Singh, Approximation of signals (functions) by trigonometric polynomials in L_p-norm, *Int. J. Math. Math. Sci.* **2014** (2014), Article ID 267383, 1-6.
11. R. N. Mohapatra and D. C. Russell: Some direct and inverse theorems in approximation of functions, *J. Austral. Math. Soc. (Ser. A)* **34** (1983), 143-154.
12. J. A. Osikiewicz, Equivalence results for Cesàro submethods, *Anal.* **20** (2000), 35-43.

13. J. G. Proakis, *Digital Communications*, McGraw-Hill, New York 1985.

14. E. Z. Psarakis, G. V. Moustakides, An L_2–based method for the design of 1-D zero phase FIR digital filters, *IEEE Trans. Circuits Syst. I. Fundam. Theor. Appl.* **44** (1997), 591-601.

15. E. S. Quade: Trigonometric approximation in the mean, *Duke Math. J.* **3** (1937), 529-542.

16. A. Zygmund, *Trigonometric Series*, third ed., Cambridge University Press, Cambridge, 2002.

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