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### Bi-Bazilevič functions of complex order involving Ruscheweyh type $q$ -difference operator

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**Abstract.** In this paper, we define a new subclass of bi-univalent functions involving  $q$ -difference operator in the open unit disk. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficients  $|a_2|$  and  $|a_3|$ .

#### 1. Introduction and Preliminaries

Let  $\mathcal{A}$  denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

The convolution or Hadamard product of two functions  $f, h \in \mathcal{A}$  is denoted by  $f * h$  and is defined as  $(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ , where  $f(z)$  is given by (1) and  $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ .

An analytic function  $f$  is subordinate to an analytic function  $h$ , written  $f(z) \prec h(z)$  ( $z \in \Delta$ ), provided there is an analytic function  $w$  defined on  $\Delta$  with  $w(0) = 0$  and  $|w(z)| < 1$  satisfying  $f(z) = h(w(z))$ .

By  $\mathcal{S}$  we shall denote the class of all functions in  $\mathcal{A}$  which are univalent in  $\Delta$ . Some of the important and well-investigated subclasses of the univalent function

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class  $\mathcal{S}$  include (for example) the class  $\mathcal{S}^*(\alpha)$  of starlike functions of order  $\alpha$  in  $\Delta$  and the class  $\mathcal{K}(\alpha)$  of convex functions of order  $\alpha$  in  $\Delta$ .

Ma and Minda [13] unified various subclasses of starlike functions and convex functions which consist of functions  $f \in \mathcal{A}$  satisfying the subordinations

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (2)$$

respectively, here (and throughout this paper)  $\phi$  with positive real part in the unit disk  $\Delta$ ,  $\phi(0) = 1$ ,  $\phi'(0) > 0$  and  $\phi$  maps  $\Delta$  onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots, \quad B_1 > 0.$$

It is well known that every function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$ , defined by

$$f^{-1}(f(z)) = z, \quad z \in \Delta$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4},$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \quad (3)$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $\Delta$  if both  $f$  and  $f^{-1}$  are univalent in  $\Delta$ , in the sense that  $f^{-1}$  has a univalent analytic continuation to  $\Delta$ . Let  $\Sigma$  denote the class of bi-univalent functions in  $\Delta$  given by (1).

A function  $f$  is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both  $f$  and  $f^{-1}$  are respectively Ma-Minda starlike or convex. These classes are denoted respectively by  $\mathcal{S}_\Sigma^*(\phi)$  and  $\mathcal{K}_\Sigma(\phi)$ .

Now we recall here the notion of  $q$ -operator i.e.  $q$ -difference operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of  $q$ -calculus was initiated by Jackson [8], recently Kanas and Răducanu [11] have used the fractional  $q$ -calculus operators in investigations of certain classes of functions which are analytic in  $\mathbb{U}$ .

Let  $0 < q < 1$ . For any non-negative integer  $n$ , the  $q$ -integer number  $n$  is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \dots + q^{n-1}, \quad [0]_q = 0. \quad (4)$$

In general, we will denote

$$[x]_q = \frac{1 - q^x}{1 - q}$$

for a non-integer number  $x$ . Also the  $q$ -number shifted factorial is defined by

$$[n]_q! = [n]_q [n-1]_q \dots [2]_q [1]_q, \quad [0]_q! = 1.$$

Clearly,

$$\lim_{q \rightarrow 1^-} [n]_q = n \quad \text{and} \quad \lim_{q \rightarrow 1^-} [n]_q! = n!.$$

For  $0 < q < 1$ , the Jackson's  $q$ -derivative operator (or  $q$ -difference operator) of a function  $f \in \mathcal{A}$  given by (1) defined as follows [8]

$$\mathcal{D}_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases} \quad (5)$$

$$\mathcal{D}_q^0 f(z) = f(z),$$

$$\mathcal{D}_q^m f(z) = \mathcal{D}_q(\mathcal{D}_q^{m-1} f(z)) \quad \text{for } m \in \mathbb{N} = \{1, 2, \dots\}.$$

From (5), we have

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \in \Delta,$$

where  $[n]_q$  is given by (4).

For a function  $h(z) = z^n$  we obtain

$$D_q h(z) = D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1}$$

and

$$\lim_{q \rightarrow 1^-} \mathcal{D}_q h(z) = \lim_{q \rightarrow 1^-} ([n]_q z^{n-1}) = n z^{n-1} = h'(z),$$

where  $h'$  is the ordinary derivative.

Let  $t \in \mathbb{R}$  and  $n \in \mathbb{N}$ . The  $q$ -generalized Pochhammer symbol is defined by

$$[t; n]_q = [t]_q [t+1]_q [t+2]_q \dots [t+n-1]_q$$

and for  $t > 0$  the  $q$ -gamma function is defined by

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Using the  $q$ -difference operator, Kannas and Raducanu [11] defined the Ruscheweyh  $q$ -differential operator as below. For  $f \in \mathcal{A}$ ,

$$\mathcal{R}_q^\delta f(z) = f(z) * F_{q, \delta+1}(z), \quad \delta > -1, \quad z \in \Delta, \quad (6)$$

where

$$F_{q, \delta+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q! \Gamma_q(1+\delta)} z^n = z + \sum_{n=2}^{\infty} \frac{[\delta+1; n]_q}{[n-1]_q!} z^n. \quad (7)$$

We note that

$$\lim_{q \rightarrow 1^-} F_{q, \delta+1}(z) = \frac{z}{(1-z)^{\delta+1}}, \quad \lim_{q \rightarrow 1^-} \mathcal{R}_q^\delta f(z) = f(z) * \frac{z}{(1-z)^{\delta+1}}.$$

Making use of (6) and (7), we have

$$\mathcal{R}_q^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q! \Gamma_q(1+\delta)} a_n z^n, \quad z \in \Delta. \quad (8)$$

From (8), we note that

$$\begin{aligned}\mathcal{R}_q^0 f(z) &= f(z), \\ \mathcal{R}_q^1 f(z) &= z\mathcal{D}_q f(z), \\ \mathcal{R}_q^m f(z) &= \frac{z\mathcal{D}_q^m(z^{m-1}f(z))}{[m]_q!} \quad \text{for } m \in \mathbb{N}.\end{aligned}$$

Also we have

$$\mathcal{D}_q(\mathcal{R}_q^\delta f(z)) = 1 + \sum_{n=2}^{\infty} \Theta_n(q, \delta) a_n z^{n-1}, \quad (9)$$

where

$$\Theta_n := \Theta_n(q, \delta) = \frac{[n]_q \Gamma_q(n + \delta)}{[n-1]_q! \Gamma_q(1 + \delta)}. \quad (10)$$

For our study, we will use the short presentation

$$\begin{aligned}\Theta_2 &= \Theta_2(q, \delta) = \frac{[2]_q \Gamma_q(2 + \delta)}{\Gamma_q(1 + \delta)}, \\ \Theta_3 &= \Theta_3(q, \delta) = \frac{[3]_q \Gamma_q(3 + \delta)}{[2]_q! \Gamma_q(1 + \delta)}.\end{aligned}$$

Recently there has been triggering interest to study bi-univalent function class  $\Sigma$  and obtained non-sharp coefficient estimates on the first two coefficients  $|a_2|$  and  $|a_3|$  of (1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2, 3\}$$

is still an open problem (see [2, 3, 4, 12, 14, 18]). Many researchers (see [1, 7, 9, 17]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class  $\Sigma$ .

Motivated by the earlier work of Bulut [5], Deniz [6], Inayat Noor [10] and Srivastava et al. [16], in the present paper we introduce new families of Bazilevič functions of complex order of the function class  $\Sigma$ , involving the operator  $\mathcal{D}_q(\mathcal{R}_q^\delta f(z))$ , and find estimates on the coefficients  $|a_2|$  and  $|a_3|$  for functions in the new subclass of function class  $\Sigma$ . Several related classes are also considered, and connection to earlier known results are made.

#### DEFINITION 1.1

A function  $f \in \Sigma$  given by (1) is said to be in the class  $\mathcal{S}_\Sigma^q(\gamma, \lambda, \delta; \phi)$  if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{(\mathcal{R}_q^\delta f(z))^{1-\lambda}} - 1 \right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{w^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{(\mathcal{R}_q^\delta g(w))^{1-\lambda}} - 1 \right) \prec \phi(w),$$

where  $z, w \in \Delta$ ,  $\gamma \in \mathbb{C} \setminus \{0\}$ ,  $\delta > -1$ ,  $\lambda \geq 0$  and the function  $g = f^{-1}$  is given by (3).

## REMARK 1.1

The following special cases of Definition 1.1 are worthy of note:

- (i) A function  $f \in \Sigma$  given by (1) is said to be in the class  $\mathcal{S}_\Sigma^q(\gamma, 0, \delta; \phi) \equiv \mathcal{S}_\Sigma^q(\gamma, \delta; \phi)$  if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left( \frac{z \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{\mathcal{R}_q^\delta f(z)} - 1 \right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{w \mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{\mathcal{R}_q^\delta g(w)} - 1 \right) \prec \phi(w),$$

where  $z, w \in \Delta$ ,  $\gamma \in \mathbb{C} \setminus \{0\}$ ,  $\delta > -1$  and the function  $g = f^{-1}$  is given by (3).

- (ii) A function  $f \in \Sigma$  given by (1) is said to be in the class  $\mathcal{S}_\Sigma^q(\gamma, 1, \delta; \phi) \equiv \mathcal{H}_\Sigma^q(\gamma, \delta; \phi)$  if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} (\mathcal{D}_q(\mathcal{R}_q^\delta f(z)) - 1) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma} (\mathcal{D}_q(\mathcal{R}_q^\delta g(w)) - 1) \prec \phi(w),$$

where  $z, w \in \Delta$ ,  $\gamma \in \mathbb{C} \setminus \{0\}$ ,  $\delta > -1$  and the function  $g = f^{-1}$  is given by (3).

- (iii) If we set  $\phi(z) = \frac{1+Az}{1+Bz}$ ,  $-1 \leq B < A \leq 1$ , then the class  $\mathcal{S}_\Sigma^q(\gamma, \lambda, \delta; \phi) \equiv \mathcal{S}_\Sigma^q(\gamma, \lambda, \delta; A, B)$  which is defined as  $f \in \Sigma$ ,

$$1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{(\mathcal{R}_q^\delta f(z))^{1-\lambda}} - 1 \right) \prec \frac{1+Az}{1+Bz}$$

and

$$1 + \frac{1}{\gamma} \left( \frac{w^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{(\mathcal{R}_q^\delta g(w))^{1-\lambda}} - 1 \right) \prec \frac{1+Aw}{1+Bw},$$

where  $z, w \in \Delta$ ,  $\gamma \in \mathbb{C} \setminus \{0\}$ ,  $\delta > -1$ ,  $\lambda \geq 0$  and the function  $g = f^{-1}$  is given by (3).

- (iv) If we set  $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$ , then the class  $\mathcal{S}_\Sigma^q(\gamma, \lambda, \delta; \phi) \equiv \mathcal{S}_\Sigma^q(\gamma, \lambda, \delta; \beta)$  which is defined as  $f \in \Sigma$ ,

$$\Re \left[ 1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{(\mathcal{R}_q^\delta f(z))^{1-\lambda}} - 1 \right) \right] > \beta$$

and

$$\Re \left[ 1 + \frac{1}{\gamma} \left( \frac{w^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{(\mathcal{R}_q^\delta g(w))^{1-\lambda}} - 1 \right) \right] > \beta,$$

where  $z, w \in \Delta$ ,  $\gamma \in \mathbb{C} \setminus \{0\}$ ,  $\delta > -1$ ,  $\lambda \geq 0$  and the function  $g = f^{-1}$  is given by (3).

On specializing the parameters  $\lambda$  and  $\delta$ , one can state the various new subclasses of  $\Sigma$ .

## 2. Coefficient Bounds for the class $\mathcal{S}_{\Sigma}^q(\gamma, \lambda, \delta; \phi)$

We begin by finding the estimates on the coefficients  $|a_2|$  and  $|a_3|$  for functions in the class  $\mathcal{S}_{\Sigma}^q(\gamma, \lambda, \delta; \phi)$ .

In order to derive our main results, we shall need the following lemma.

LEMMA 2.1 (see [15])

If  $p \in \mathcal{P}$ , then  $|p_k| \leq 2$  for each  $k$ , where  $\mathcal{P}$  is the family of all functions  $p$  analytic in  $\Delta$  for which  $\Re(p(z)) > 0$ , where  $p(z) = 1 + p_1z + p_2z^2 + \dots$  for  $z \in \Delta$ .

THEOREM 2.1

Let the function  $f(z)$  given by (1) be in the class  $\mathcal{S}_{\Sigma}^q(\gamma, \lambda, \delta; \phi)$ . Then

$$|a_2| \leq \sqrt{\frac{N}{D}}, \quad (11)$$

where

$$\begin{aligned} N &= 2|\gamma|^2 B_1^3 (1+q)^2 (1+q+q^2), \\ D &= |\gamma B_1^2 [2(1+q)^2 (\lambda+q+q^2) \Theta_3 + (\lambda-1)(\lambda+2q)(1+q+q^2) \Theta_2^2] \\ &\quad - 2(B_2 - B_1)(\lambda+q)^2 (1+q+q^2) \Theta_2^2| \end{aligned}$$

and

$$|a_3| \leq \left( \frac{|\gamma| B_1 (1+q)}{(\lambda+q) \Theta_2} \right)^2 + \frac{|\gamma| B_1 (1+q+q^2)}{(\lambda+q+q^2) \Theta_3}. \quad (12)$$

*Proof.* Let  $f \in \mathcal{S}_{\Sigma}^q(\gamma, \lambda, \delta; \phi)$  and  $g = f^{-1}$  be given by (3). Then there are analytic functions  $u, v: \Delta \rightarrow \Delta$  with  $u(0) = 0 = v(0)$ , satisfying

$$1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{(\mathcal{R}_q^\delta f(z))^{1-\lambda}} - 1 \right) = \phi(u(z)) \quad (13)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{(\mathcal{R}_q^\delta g(w))^{1-\lambda}} - 1 \right) = \phi(v(w)). \quad (14)$$

Define the functions  $p(z)$  and  $q(z)$  by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1z + p_2z^2 + \dots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1z + q_2z^2 + \dots$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2} \left[ p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \dots \right] \quad (15)$$

and

$$v(z) := \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \dots \right]. \quad (16)$$

Then  $p(z)$  and  $q(z)$  are analytic in  $\Delta$  with  $p(0) = 1 = q(0)$ . Since  $u, v: \Delta \rightarrow \Delta$ , the functions  $p(z)$  and  $q(z)$  have a positive real part in  $\Delta$ , and  $|p_k| \leq 2$  and  $|q_k| \leq 2$  for each  $k$ . Using (15) and (16) in (13) and (14) respectively, we have

$$1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{(\mathcal{R}_q^\delta f(z))^{1-\lambda}} - 1 \right) = \phi \left( \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \dots \right] \right) \quad (17)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{z^{1-\lambda} \mathcal{D}_q(\mathcal{R}_q^\delta g(w))}{(\mathcal{R}_q^\delta g(w))^{1-\lambda}} - 1 \right) = \phi \left( \frac{1}{2} \left[ q_1 w + \left( q_2 - \frac{q_1^2}{2} \right) w^2 + \dots \right] \right). \quad (18)$$

In light of (8)–(10), from (17) and (18), it is evident that

$$\begin{aligned} 1 + \frac{1}{\gamma} \frac{\lambda + q}{1 + q} \Theta_2 a_2 z + \frac{1}{\gamma} \left[ \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 + \frac{(\lambda - 1)(\lambda + 2q)}{2(1 + q)^2} \Theta_2^2 a_2^2 \right] z^2 + \dots \\ = 1 + \frac{1}{2} B_1 p_1 z + \left[ \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \right] z^2 + \dots \end{aligned}$$

and

$$\begin{aligned} 1 - \frac{1}{\gamma} \frac{\lambda + q}{1 + q} \Theta_2 a_2 w + \frac{1}{\gamma} \left[ - \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 \right. \\ \left. + \left( 2 \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 + \frac{(\lambda - 1)(\lambda + 2q)}{2(1 + q)^2} \Theta_2^2 \right) a_2^2 \right] w^2 + \dots \\ = 1 + \frac{1}{2} B_1 q_1 w + \left[ \frac{1}{2} B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} B_2 q_1^2 \right] w^2 + \dots \end{aligned}$$

which yields the following relations:

$$\frac{\lambda + q}{1 + q} \Theta_2 a_2 = \frac{\gamma}{2} B_1 p_1, \quad (19)$$

$$\frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 + \frac{(\lambda - 1)(\lambda + 2q)}{2(1 + q)^2} \Theta_2^2 a_2^2 = \frac{\gamma}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{\gamma}{4} B_2 p_1^2, \quad (20)$$

$$- \frac{\lambda + q}{1 + q} \Theta_2 a_2 = \frac{\gamma}{2} B_1 q_1 \quad (21)$$

and

$$\begin{aligned} - \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 + \left( 2 \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 + \frac{(\lambda - 1)(\lambda + 2q)}{2(1 + q)^2} \Theta_2^2 \right) a_2^2 \\ = \frac{\gamma}{2} B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{\gamma}{4} B_2 q_1^2. \end{aligned} \quad (22)$$

From (19) and (21), it follows that

$$p_1 = -q_1 \quad (23)$$

and

$$8 \frac{(\lambda + q)^2}{(1 + q)^2} \Theta_2^2 a_2^2 = \gamma^2 B_1^2 (p_1^2 + q_1^2). \quad (24)$$

Adding (20) and (22), we obtain

$$\begin{aligned} & \left( 2 \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 + \frac{(\lambda - 1)(\lambda + 2q)}{(1 + q)^2} \Theta_2^2 \right) a_2^2 \\ &= \frac{\gamma B_1}{2} (p_2 + q_2) + \frac{\gamma}{4} (B_2 - B_1) (p_1^2 + q_1^2). \end{aligned} \quad (25)$$

Using (24) in (25), we get

$$a_2^2 = \frac{N_0}{D_0},$$

where

$$\begin{aligned} N_0 &= \gamma^2 B_1^3 (1 + q)^2 (1 + q + q^2) (p_2 + q_2), \\ D_0 &= 2\gamma B_1^2 [2(1 + q)^2 (\lambda + q + q^2) \Theta_3 + (\lambda - 1)(\lambda + 2q)(1 + q + q^2) \Theta_2^2] \\ &\quad - 4(B_2 - B_1)(\lambda + q)^2 (1 + q + q^2) \Theta_2^2. \end{aligned}$$

Applying Lemma 2.1 for the coefficients  $p_2$  and  $q_2$ , we immediately have

$$|a_2|^2 \leq \frac{N}{D},$$

where

$$\begin{aligned} N &= 2|\gamma|^2 B_1^3 (1 + q)^2 (1 + q + q^2), \\ D &= |\gamma B_1^2 [2(1 + q)^2 (\lambda + q + q^2) \Theta_3 + (\lambda - 1)(\lambda + 2q)(1 + q + q^2) \Theta_2^2] \\ &\quad - 2(B_2 - B_1)(\lambda + q)^2 (1 + q + q^2) \Theta_2^2|. \end{aligned}$$

This gives the bound on  $|a_2|$  as asserted in (11).

Next, in order to find the bound on  $|a_3|$ , by subtracting (22) from (20), we get

$$\begin{aligned} & 2 \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_3 - 2 \frac{\lambda + q + q^2}{1 + q + q^2} \Theta_3 a_2^2 \\ &= \frac{\gamma B_1}{2} \left[ (p_2 - q_2) - \frac{1}{2} (p_1^2 - q_1^2) \right] + \frac{\gamma B_2}{4} (p_1^2 - q_1^2). \end{aligned} \quad (26)$$

Using (23) and (24) in (26), we get

$$a_3 = \frac{\gamma^2 B_1^2 (1 + q)^2 (p_1^2 + q_1^2)}{8(\lambda + q)^2 \Theta_2^2} + \frac{\gamma B_1 (1 + q + q^2) (p_2 - q_2)}{4(\lambda + q + q^2) \Theta_3}. \quad (27)$$

Applying Lemma 2.1 once again for the coefficients  $p_1, q_1, p_2$  and  $q_2$ , we readily get (12). This completes the proof of Theorem 2.1.



### 3. Corollaries and Consequences

By setting  $\lambda = 0$  in Theorem 2.1, we have the following Theorem.

#### THEOREM 3.1

Let the function  $f(z)$  given by (1) be in the class  $\mathcal{S}_\Sigma^q(\gamma, \delta; \phi)$ . Then

$$|a_2| \leq \sqrt{\frac{|\gamma|^2 B_1^3 (1+q)^2 (1+q+q^2)}{q|\gamma B_1^2 [(1+q)^3 \Theta_3 - (1+q+q^2) \Theta_2^2] - (B_2 - B_1) q (1+q+q^2) \Theta_2^2}}$$

and

$$|a_3| \leq \left( \frac{|\gamma| B_1 (1+q)}{q \Theta_2} \right)^2 + \frac{|\gamma| B_1 (1+q+q^2)}{q (1+q) \Theta_3}.$$

By setting  $\lambda = 1$  in Theorem 2.1, we have the following result.

#### THEOREM 3.2

Let the function  $f(z)$  given by (1) be in the class  $\mathcal{H}_\Sigma^q(\gamma, \delta; \phi)$ . Then

$$|a_2| \leq \sqrt{\frac{2|\gamma|^2 B_1^3 (1+q)^2}{|\gamma B_1^2 [2(1+q)^2 \Theta_3 + (1+2q) \Theta_2^2] - 2(B_2 - B_1) (1+q)^2 \Theta_2^2}}$$

and

$$|a_3| \leq \left( \frac{|\gamma| B_1}{\Theta_2} \right)^2 + \frac{|\gamma| B_1}{\Theta_3}.$$

By setting  $\phi(z) = \frac{1+Az}{1+Bz}$ ,  $-1 \leq B < A \leq 1$ , in Theorem 2.1, we state the following Theorem.

#### THEOREM 3.3

Let the function  $f(z)$  given by (1) be in the class  $\mathcal{S}_\Sigma^q(\gamma, \lambda, \delta; A, B)$ . Then

$$|a_2| \leq \sqrt{\frac{N}{D}},$$

where

$$\begin{aligned} N &= 2|\gamma|^2 (A-B)^2 (1+q)^2 (1+q+q^2), \\ D &= |\gamma(A-B)[2(1+q)^2 (\lambda+q+q^2) \Theta_3 + (\lambda-1)(\lambda+2q)(1+q+q^2) \Theta_2^2] \\ &\quad + 2(B+1)(\lambda+q)^2 (1+q+q^2) \Theta_2^2| \end{aligned}$$

and

$$|a_3| \leq \left( \frac{|\gamma|(A-B)(1+q)}{(\lambda+q) \Theta_2} \right)^2 + \frac{|\gamma|(A-B)(1+q+q^2)}{(\lambda+q+q^2) \Theta_3}.$$

Further, by setting  $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$ ,  $0 \leq \beta < 1$  in Theorem 2.1 we get the following result.

## THEOREM 3.4

Let the function  $f(z)$  given by (1) be in the class  $S_{\Sigma}^q(\gamma, \lambda, \delta; \beta)$ . Then

$$|a_2| \leq \sqrt{\frac{4|\gamma|(1-\beta)(1+q)^2(1+q+q^2)}{|2(1+q)^2(\lambda+q+q^2)\Theta_3 + (\lambda-1)(\lambda+2q)(1+q+q^2)\Theta_2^2|}}$$

and

$$|a_3| \leq \left( \frac{2|\gamma|(1-\beta)(1+q)}{(\lambda+q)\Theta_2} \right)^2 + \frac{2|\gamma|(1-\beta)(1+q+q^2)}{(\lambda+q+q^2)\Theta_3}.$$

**Concluding Remarks.** By taking  $\delta = 0$  and specializing the parameters  $\lambda$  and  $\gamma$ , various other interesting corollaries and consequences of our main results (which are asserted by Theorem 2.1 above) can be derived easily hence we omit the details.

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