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Bothway embedding of circulant network into grid ☆

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ABSTRACT

Graph embedding is an important technique that maps a guest graph into a host graph, usually an interconnection network. In this paper, we compute the dilation and wirelength of embedding circulant network into grid and vice versa.

1. Introduction

A parallel algorithm or a massively parallel computer can each be modeled by a graph, in which the vertices of the graph represent the processes or processing elements, and the edges represent the communications among processes or processors. Thus, the problem of efficiently executing a parallel algorithm A on a parallel computer M can be often reduced to the problem of mapping the graph G, representing A, on the graph H, representing M, so that the mapping satisfies some predefined constraints. This is called a graph embedding [3].

An embedding of a guest graph G into a host graph H is a one-to-one mapping of the vertex set of G into that of H. The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *dilation*. The dilation of an embedding is defined as the maximum distance between pairs of vertices of H that are images of adjacent vertices of G. It is a measure for the communication time needed when simulating one network on another [12].

Another important cost criterion is the *wirelength*. The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [22,39].

The circulant graph is a natural generalization of the double loop network [38] and has been used for decades in the design of computer and telecommunication networks due to its optimal fault-tolerance and routing capabilities [7]. It is also used in VLSI design and distributed computation [1,2,37]. Circulant graphs have been employed for designing binary codes [20]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [2]. Every circulant graph is a vertex transitive graph and a Cayley graph [39]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [2,7].

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Graph embeddings have been well studied for hypercubes into grids [24], meshes into crossed cubes [14], meshes into locally twisted cubes [17], meshes into faulty crossed cubes [40], generalized ladders into hypercubes [8], rectangular grids into hypercubes [10], rectangular grids into hypercubes [13], grids into grids [34], binary trees into grids [28], meshes into Möbius cubes [36], circulant into arbitrary trees, cycles, certain multicyclic graphs and ladders [33], tori and grids into twisted cubes [21].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [3,9]. But the Congestion Lemma and the Partition Lemma [24] have enabled to obtain exact wirelength of embeddings for various architectures [23–25,29,31–33]. This technique focuses on specific partitioning of the edge set of the host graph. It is interesting to note that not all host graphs can be partitioned to apply the Partition Lemma. In this paper, we overcome this difficulty by partially retaining a set of edges on which the wirelength is computed using Partition Lemma and compute minimum congestion on the rest of the edges using certain other procedure.

The paper is organized as follows. Section 2 gives definitions and other preliminaries. Sections 3 and 4 establish the main results. Finally, concluding remarks and future work are given in Section 5.

2. Basic concepts

In this section we give the basic definitions and preliminaries related to embedding problems.

Definition 2.1. (See [3].) Let G and H be finite graphs. An *embedding* $\phi = (f, P_f)$ of G into H is defined as follows:

1. f is a one-to-one map from $V(G) \to V(H)$.
2. P_f is a one-to-one map from $E(G)$ to $\{P_f(u, v) : P_f(u, v)$ is a path in H between $f(u)$ and $f(v)$ for $(u, v) \in E(G)\}$.

For brevity, we denote the pair (f, P_f) as f.

Definition 2.2. (See [3].) If $e = (u, v) \in E(G)$, then the length of $P_f(u, v)$ in H is called the *dilation* of the edge e. The maximal dilation over all edges of G is called the dilation of the embedding f. The dilation of embedding G into H denoted by $d(G, H)$ is the minimum dilation taken over all embeddings f of G into H. The *expansion* of an embedding f is the ratio of the number of vertices of H to the number of vertices of G.

In this paper, we consider embeddings with expansion one.

The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(e)$ denote the number of edges (u, v) of G such that e is in the path $P_f(u, v)$ between $f(u)$ and $f(v)$ in H.

In other words,

$$EC_f(e) = \left|\left\{(u, v) \in E(G) : e \in P_f(u, v)\right\}\right|$$

where $P_f(u, v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f. On the other hand, if S is any subset of $E(H)$, then $EC_f(S) = \sum_{e \in S} EC_f(e)$.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum, over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [4]. See Fig. 1.

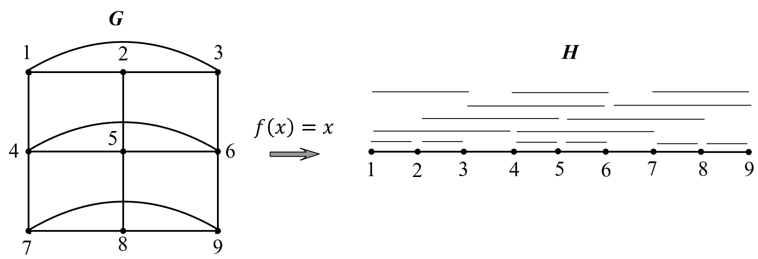

Fig. 1. Wiring diagram of a cylinder G into path H with $WL_f(G, H) = 30$.

Definition 2.3. (See [24].) The wirelength of an embedding f of G into H is given by

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} d_H\big(f(u), f(v)\big) = \sum_{e\in E(H)} EC_f(e)$$

where $d_H(f(u), f(v))$ denotes the distance of the path $P_f(u,v)$ in H.

The *wirelength* of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

The *wirelength problem* [3,4,9,24,28,29] of a graph G into H is to find an embedding of G into H that induces the wirelength $WL(G,H)$. It is interesting to note that the embedding parameters wirelength, dilation sum and congestion sum are all equal [29]. The following two versions of the edge isoperimetric problem of a graph $G(V,E)$ have been considered in the literature [5], and are NP-complete [15].

Problem 1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. More formally, for a given m, if $\theta_G(m) = \min_{A\subseteq V, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u,v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

Problem 2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximum among all induced subgraphs with the same number of vertices. More formally, for a given m, if $I_G(m) = \max_{A\subseteq V, |A|=m} |I_G(A)|$, where $I_G(A) = \{(u,v) \in E : u, v \in A\}$; then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$.

For a given m, where $m = 1, 2, \ldots, n$, we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called *optimal* [5,18].

Further, if a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. But it is not true for Problem 2 in general. However for regular graphs a subset of vertices S is optimal with respect to Problem 1 if and only if S is optimal for Problem 2 [5]. In the literature, Problem 2 is defined as the *maximum subgraph* problem [15].

Lemma 2.4 *(Congestion Lemma). (See [24,25].) Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S splits H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Let S also satisfy the following conditions:*

(i) *For every edge $(a,b) \in G_i$, $i = 1, 2$, $P_f(a,b)$ has no edges in S.*
(ii) *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum in the sense that $EC_f(S) \le EC_g(S)$ for an arbitrary embedding g of G into H and $EC_f(S) = r|V(G_1)| - 2|E(G_1)|$.

Lemma 2.5 *(Partition Lemma). (See [24,25].) Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

Definition 2.6. (See [2,32].) The undirected circulant graph $G(n; \pm S)$, $S \subseteq \{1, 2, \ldots, j\}$, $1 \le j \le \lfloor n/2 \rfloor$ is a graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and the edge set $E = \{(i,k) : |k-i| \equiv s \pmod n, s \in S\}$.

The circulant graph shown in Fig. 2 is $G(8; \pm\{1,3,4\})$. It is clear that $G(n; \pm 1)$ is the undirected cycle C_n and $G(n; \pm\{1, 2, \ldots, \lfloor n/2 \rfloor\})$ is the complete graph K_n. The cycle $G(n; \pm 1) \simeq C_n$ contained in $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j \le \lfloor n/2 \rfloor$ is sometimes referred to as the outer cycle C of G.

Theorem 2.7. *(See [33].) A set of k consecutive vertices of $G(n; \pm 1)$, $1 \le k \le n$ induces a maximum subgraph of $G(n; \pm S)$, where $S = \{1, 2, \ldots, j\}$, $1 \le j < \lfloor n/2 \rfloor$, $n \ge 3$.* $\square$

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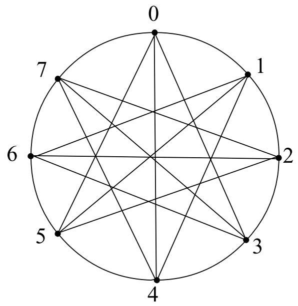

Fig. 2. Circulant graph $G(8; \pm\{1, 3, 4\})$.

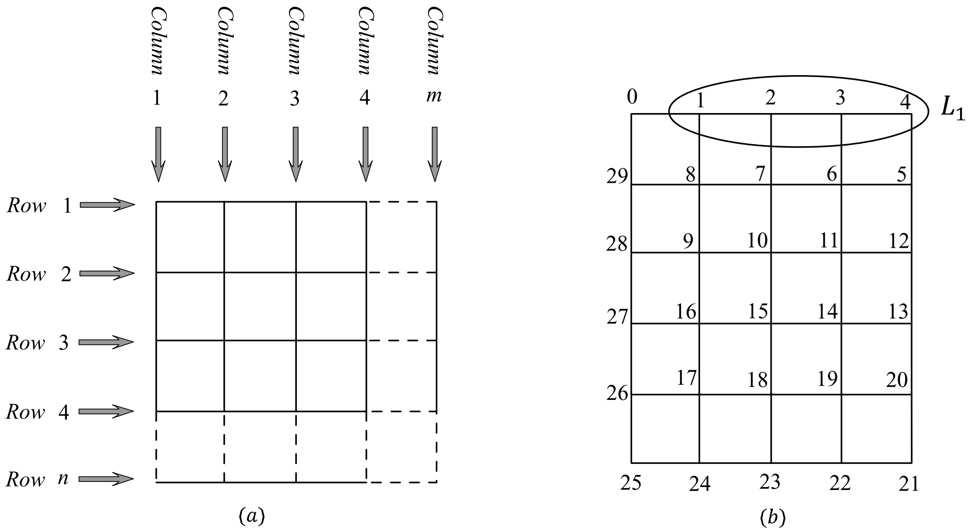

Fig. 3. (a) Grid $M[n \times m]$, (b) snake-wise labeling of $M[6 \times 5]$.

3. Embedding of circulant network into grid

In this section, we compute the dilation and wirelength of embedding circulant network into grid.

Definition 3.1. An $n \times m$ *mesh* denoted by $M[n \times m]$ is the Cartesian product $P_n \times P_m$, where P_n denotes a path on n vertices. A mesh is also referred to as a grid.

Remark 3.2. Grid $M[n \times m]$ has nm vertices and $2nm - (n + m)$ edges. The diameter of $M[n \times m]$ is $n + m - 2$, see Fig. 3(a).

Notation. For a vertex u in G, let $N_k(u)$ denote the kth neighborhood of u in G. Let L_i denote the path induced by the vertices in row i of $M[n \times m]$ except the first node from left, see Fig. 3(b).

3.1. Dilation

As dilation problem is NP-complete, Havel [19] conjectured that a binary tree can be embedded into a k-dimensional hypercube Q^k with dilation 1 if and only if each of its partite sets contains at most 2^{k-1} vertices. In 1985, Bhatt and Ipsen [6] conjectured that a binary tree can be embedded into its optimal hypercube with dilation at most 2 as well as into its next-to-optimal hypercube with dilation 1. Monien and Sudborough [27] proved that every binary tree can be embedded into a hypercube with dilation 3 and $O(1)$ expansion. Chen and Stallmann [11] proved that a simple linear-time heuristic embeds an arbitrary binary tree into a hypercube with expansion 1 and average dilation no more than 2.

Sunitha [35] constructed an embedding of some hierarchical caterpillars into their optimal hypercube with dilation 2. Gupta et al. [16] present efficient graph embeddings for complete k-ary trees into Boolean hypercubes. They describe an efficient embedding of a complete ternary tree ($k = 3$) of height h into a hypercube, which achieves dilation 3. In the recent years Rajasingh et al. [26,30,32] proved the existence of an embedding m-sequential k-ary trees into hypercubes with dilation 2, embedding of variants of hypercubes with dilation 2, circulant networks into hypercubes with dilation 2,

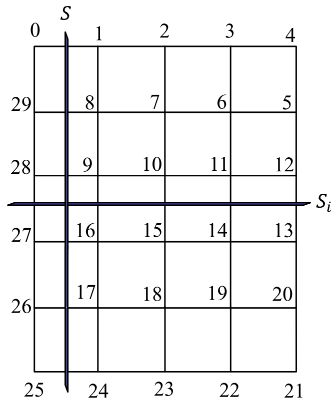

Fig. 4. Edge cut of $M[5 \times 5]$.

circulant networks into generalized Petersen graphs with dilation 2 and vice versa. We now embed circulant network $G(nm; \pm\{1, 2, 3\})$ into grid $M[n \times m]$ with dilation 3, where $n, m \geq 3$ and n is even.

Theorem 3.3. *Let G be the circulant graph $G(nm; \pm\{1, 2, 3\})$ and H be the grid $M[n \times m]$, $n, m \geq 3$ and n is even. Then the dilation of embedding G into H satisfies*

$$d(G, H) \geq 3.$$

Proof. Since G is vertex-transitive, without loss of generality, let u be an arbitrarily chosen vertex in G mapped to a vertex of degree 2 in H, say $f(u)$. By definition of grid, the neighborhood $N_1(f(u)) \cup N_2(f(u))$ contains only 5 vertices. Since G is 6-regular, $d(G, H) \geq 3$. □

In this section, sequentially labeling L_i, $1 \leq i \leq n$ of $M[n \times m]$ alternately from left to right and from right to left, beginning with L_1 and continuing the labeling of the first column from bottom to top is termed the *snake-wise labeling* of $M[n \times m]$, see Fig. 3(*b*), when n is even, the hamiltonian cycle traced by $0, 1, \ldots, nm - 1$ in H is called *snake-wise cycle*.

Dilation Algorithm A.

Input: The circulant graph $G(nm; \pm\{1, 2, 3\})$ and the grid $M[n \times m]$, $n, m \geq 3$ and n is even.

Algorithm: Label the consecutive vertices of $G(nm; \pm\{1\})$ in $G(nm; \pm\{1, 2, 3\})$ as $0, 1, 2, \ldots, nm-1$ in the clockwise sense. Label the vertices of $M[n \times m]$ using snake-wise labeling, see Fig. 4. Let $f(x) = x$ for all $x \in V(G)$ and for $(a, b) \in E(G)$, let $P_f(a, b)$ be a shortest path between $f(a)$ and $f(b)$ in H.

Output: The snake-wise embedding f of $G(nm; \pm\{1, 2, 3\})$ into $M[n \times m]$ with dilation 3.

Theorem 3.4. *Let G be the circulant graph $G(nm; \pm\{1, 2, 3\})$ and H be the grid $M[n \times m]$, $n, m \geq 3$ and n is even. Then the dilation of embedding G into H is given by*

$$d(G, H) = 3.$$

Proof. Label the vertices of G and H using Dilation Algorithm A. We assume that the labels represent the vertices to which they are assigned. The labeling of H generates a hamiltonian cycle $C = (0, 1, \ldots, nm - 1, 0)$ in H. Therefore, the maximum stretch of any edge of G in H is at most 3. Hence $d(G, H) = 3$. □

3.2. Wirelength

Grid embedding plays an important role in computer architecture. VLSI Layout Problem, Crossing Number Problem, Graph Drawing, and Edge Embedding Problem are all a part of grid embedding. There are very few papers in the literature which provide the exact wirelength of grid embedding [24].

Now, we compute the exact wirelength of embedding circulant network $G(nm; \pm\{1, 2\})$ into grid $M[n \times m]$. For proving the main result, we need the following results.

Lemma 3.5. *A subset A of $G(n; \pm\{1, 2\})$ induces a maximum subgraph of $G(n; \pm\{1, 2\})$ if and only if A is a set of consecutive vertices of $G(n; \pm\{1\})$, where $|A| \geq 3$.*

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Proof. Suppose $|A| = k$. If $k = n$ or $n - 1$, labels of A are consecutive. Therefore it is enough to consider the case when $3 \leq k \leq n - 2$. Any k consecutive vertices of $G(n; \pm\{1\})$ induce $2k - 3$ edges in $G(n; \pm\{1, 2\})$, for $3 \leq k \leq n - 2$. Let A be a maximum subgraph on k vertices of $G(n; \pm\{1, 2\})$. Suppose the labels of A are not consecutively labeled. Without loss of generality let A contain 2 arcs of $G(n; \pm\{1\})$ labeled $1, 2, \ldots, j$ and $j + 2, j + 3, \ldots, k + 1$. Then $|E(A)| = 2k - 5 < 2k - 3$, which is a contradiction. Converse follows from Theorem 2.7. □

Lemma 3.6. *Let G be the circulant graph $G(nm; \pm\{1, 2\})$ and H be the grid $M[n \times m]$, $n, m \geq 3$. Let $f : G \to H$ be an embedding. If $EC_f(\theta_H(L_i))$ is minimum, then $f^{-1}(L_i)$ is a maximum subgraph of G for all i, $1 \leq i \leq n$.*

Proof. Suppose $EC_f(\theta_H(L_i))$ is minimum where $|L_i| = m - 1$. We claim that $A = f^{-1}(L_i)$ is a maximum subgraph of G on $m - 1$ vertices. Suppose not, there exists $B \subseteq V(G)$ such that $|I_G(A)| < |I_G(B)|$. Since G is 4-regular,

$$\begin{aligned} EC_f\big(\theta_H(L_i)\big) &= 4(m-1) - 2\big|I_G(A)\big| \\ &> 4(m-1) - 2\big|I_G(B)\big| \\ &= EC_f\big(\theta_H\big(f(B)\big)\big) \end{aligned}$$

which is a contradiction to our assumption that $EC_f(\theta_H(L_i))$ is minimum. Therefore $f^{-1}(L_i)$ is a maximum subgraph of G. □

Wirelength Algorithm.

Input: The circulant graph $G(nm; \pm\{1, 2\})$ and the grid $M[n \times m]$, $n, m \geq 3$ and n is even.

Algorithm: Same as Dilation Algorithm A, see Fig. 4.

Output: An embedding f of $G(nm; \pm\{1, 2\})$ into $M[n \times m]$ with wirelength $= 3mn$.

Lemma 3.7. *Let G be the circulant graph $G(nm; \pm\{1, 2\})$ and H be the grid $M[n \times m]$, $n, m \geq 3$ and n is even. Let f be the snake-wise embedding of G into H. Then*

$$WL(G, H) = WL_f(G, H).$$

Proof. Let vertices of $G(nm; \pm\{1\})$ be labeled $0, 1, \ldots, nm - 1$ in the clockwise sense. Suppose $g : G \to H$ be an embedding such that $g(x) = x$ and the snake-wise cycle in H is not consecutively labeled. Without loss of generality let the sequence of labeling the snake-wise cycle be $0, 1, 2, \ldots, i - 1, i, k, i + 2, \ldots, nm - 1, 0$. We assume that $i + 1$ lies to the right of k, where $k \neq i + 1$. The contribution to the wirelength by virtue of the position of $i + 1$ is at least 5 in contrast to the contribution of at least 4, when $i - 1, i, i + 1$ are consecutive. Thus $WL(G, H) \geq WL_f(G, H)$. Clearly $WL(G, H) \leq WL_f(G, H)$. Hence the theorem. □

Theorem 3.8. *Let G be the circulant graph $G(nm; \pm\{1, 2\})$ and H be the grid $M[n \times m]$, $n, m \geq 3$ and n is even. Let f be the snake-wise embedding of G into H. Then the wirelength is given by*

$$WL_f(G, H) = 3nm.$$

Proof. Label the vertices of G and H using Wirelength Algorithm. We assume that the labels represent the vertices to which they are assigned.

For $1 \leq i \leq n - 1$, let S_i be an edge cut of H consisting of edges between the rows i and $i + 1$ of H such that S_i disconnects H into two components H_{i1} and H_{i2} where $V(H_{i1})$ is consecutively labeled [29], see Fig. 4. Let G_{i1} and G_{i2} be the inverse images of H_{i1} and H_{i2} under f respectively. By Theorem 2.7, G_{i1} is a maximum subgraph of G. By Lemma 2.4, $EC_f(S_i)$ is minimum and is equal to $4im - 2(2im - 3) = 6$, $1 \leq i \leq n - 1$.

Let S be an edge cut of H consisting of edges between the columns 1 and 2 of H such that S disconnects H into two components H_1 and H_2 where $V(H_1)$ is consecutively labeled [29], see Fig. 4. Let G_1 and G_2 be the inverse images of H_1 and H_2 under f respectively. By Theorem 2.7, G_1 is a maximum subgraph of G. By Lemma 2.4, $EC_f(S)$ is minimum and is equal to 6. Thus

$$\sum_{i=1}^{n-1} EC_f(S_i) + EC_f(S) = \sum_{i=1}^{n-1} 6 + 6 = 6n \tag{3.1}$$

Next we claim that the sum of $EC_f(e)$ is $3m - 6$ over all the edges e of L_i, for all i, $1 \leq i \leq n$. By Lemma 3.6, $f^{-1}(L_i)$ is a maximum subgraph with $2m - 5$ edges. We know that L_i is a path induced by $m - 1$ vertices and $m - 2$ edges of $M[n \times m]$, $1 \leq i \leq n$.

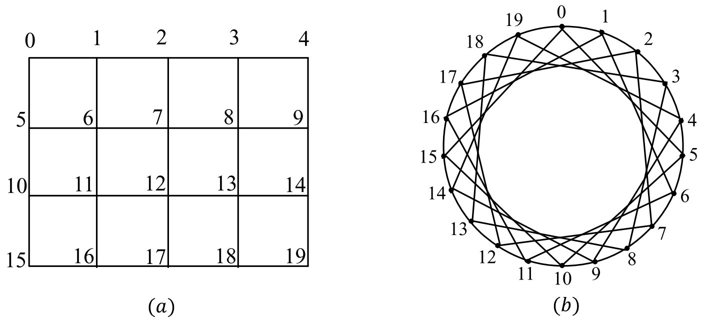

Fig. 5. (a) Labeling of $M[4 \times 5]$, (b) labeling of $G(20; \pm\{1, 5\})$.

Let $u_{i2} \ldots, u_{im}$ and $u_{(i+1)2} \ldots, u_{(i+1)m}$ be the vertices of L_i and L_{i+1} respectively. By Lemma 3.5, the labels of L_i and L_{i+1} are consecutively labeled, say $x_{i2}, x_{i3} \ldots, x_{im}$ and $x_{(i+1)2}, x_{(i+1)3}, \ldots, x_{(i+1)m}$. Then by definition of grid there is no vertex in L_i which is adjacent to both y and z, where $y, z \in \{u_{(i+1)2}, u_{(i+1)3}, \ldots, u_{(i+1)m}\}$. Thus $\sum_{e \in \cup L_i} EC_f(e)$ is increased by $2n$. Hence

$$\sum_{e \in \cup L_i} EC_f(e) = \big(m - 2 + 2(m-3)\big)n + 2n = (3m-6)n \tag{3.2}$$

By Lemma 2.5,

$$\begin{aligned} WL_f(G, H) &= \sum_{i=1}^{n-1} EC_f(S_i) + EC_f(S) + \sum_{e \in \cup L_i} EC_f(e) \\ &= 6n + (3m-6)n \quad \big[\text{by Eqs. (3.1) and (3.2)}\big] \\ &= 3nm. \qquad \square \end{aligned}$$

4. Embedding of grid into circulant network

In this section, we compute the dilation and wirelength of embedding grid into circulant network.

Dilation Algorithm B.

Input: The grid $M[n \times m]$ and the circulant graph $G(nm; \pm\{1, m\})$, $n, m \geq 3$.

Algorithm: Label the vertices in the ith row of $M[n \times m]$ as $(i-1)m, (i-1)m+1, \ldots, im-1$, $1 \leq i \leq n$ and label the consecutive vertices of $G(nm; \pm\{1\})$ in $G(nm; \pm\{1, m\})$ as $0, 1, 2, \ldots, nm-1$ in the clockwise sense, see Figs. 5(a) and 5(b). Let $f(x) = x$ for all $x \in V(G)$ and for $(a, b) \in E(G)$, let $P_f(a, b)$ be a shortest path between $f(a)$ and $f(b)$ in H.

Output: An embedding f of $M[n \times m]$ into $G(nm; \pm\{1, m\})$ with dilation 1.

Theorem 4.1. *Let G be the grid $M[n \times m]$ and H be the circulant graph $G(nm; \pm\{1, m\})$, $n, m \geq 3$. Let $f : G \to H$ be an embedding. Then G is a subgraph of H.*

Proof. Label the vertices of G and H using Dilation Algorithm B. We assume that the labels represent the vertices to which they are assigned. Let x and y be the labeling of vertices u and v in G respectively. Without loss of generality let $x > y$.

If $e = (u, v)$ is an edge in G, then $x - y = 1$ or m. If $x - y = 1$, then $(f(x), f(y))$ is an edge in $G(nm; \pm\{1\})$. Otherwise $(f(x), f(y))$ is an edge in $G(nm; \pm\{m\})$. Hence the proof. $\square$

Corollary 4.2. *Let G be the grid $M[n \times m]$ and H be the circulant graph $G(nm; \pm\{1, m\})$, $n, m \geq 3$. Then the wirelength of embedding G into H is given by $WL(G, H) = |E(G)| = nm - (n + m)$.*

5. Concluding remarks

In this paper, we compute the dilation and wirelength of embedding circulant network into grid and vice versa. Finding the wirelength of embedding circulant networks into other interconnection networks such as cylinder, torus, n-dimensional grid, hypercube, crossed cube, twisted cube, Möbius cube, Fibonacci cube and generalized books are under investigation.

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