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BUCKLING OF ELASTIC CIRCULAR PLATES WITH AN ELASTICALLY RESTRAINED EDGES AGAINST ROTATION AND INTERNAL ELASTIC RING SUPPORT

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The buckling of elastic circular plates with an internal elastic ring support and elastically restrained edges against rotation and simply supported is concerned. The classical plate theory is used to derive the governing differential equation. This work presents the existence of buckling mode switching with respect to the radius of internal elastic ring support. The plate may buckle in an axisymmetric mode in general, but when the radius of the ring support becomes small, the plate may buckle in an asymmetric mode. The cross-over ring support radius varies from 0.09891 to 0.1545 times the plate radius, depending on the rotational stiffness of the elastic restraint at the edges and elastic restraint of the ring. The optimum radius of the internal elastic ring support for maximum buckling load is also determined. Extensive data is tabulated so that pertinent conclusions can be arrived at on the influence of rotational restraint, translational restraint of internal elastic ring support, Poisson's ratio, and other boundary conditions on the buckling of uniform isotropic circular plates. The numerical results obtained are in good agreement with the previously published data

Keywords: buckling; circular plate; elastic ring support; rotational spring stiffness; mode switching

I. Introduction. Buckling of plates is an important topic in structural engineering. The prediction of buckling of structural members restrained laterally is important in the design of various engineering components. In particular, circular plates with an internal elastic ring support find applications in aeronautical (instrument mounting bases for space vehicles), rocket launching pads, aircrafts (instrument mounting bases for aircraft vehicles) and naval vessels (instrument mounting bases). Based on Kirchhoff's theory, the elastic buckling of thin circular plates has been extensively studied by many authors after the pioneering work published by Bryan [1]. Since then, there have been extensive studies on the subject covering various aspects such as different materials, boundary and loading conditions. Also the buckling of circular plates was studied by different authors Wolkowisky [2] and Brushes [3]. However, these sources only considered axisymmetric case, which may not lead to the correct buckling load. Introducing an internal elastic ring supports may increase the elastic buckling capacity of in-plane loaded circular plates significantly. Laura et al. [4] investigated the elastic buckling problem of the aforesaid type of circular plates, who modeled the plate using the classical thin plate theory. In their study only axisymmetric modes are considered.

Kunukkasseril and Swamidas [5] are probably the first to consider elastic ring supports. They formulated the equations in general, but presented only the case of circular plate with a free edge. Wang and Wang [6] studied the fundamental frequency of the circular plate with internal elastic ring support. They have considered the four basic boundary conditions.

Although the circular symmetry of the problem allows for its significant simplification, many difficulties very often arise due to complexity and uncertainty of boundary conditions. This uncertainty could be due to practical engineering applications where the edge of the plate does not fall into the classical boundary conditions. It is accepted fact that the condition on a periphery often tends to be part way between the classical boundary conditions (free, clamped and simply supported) and

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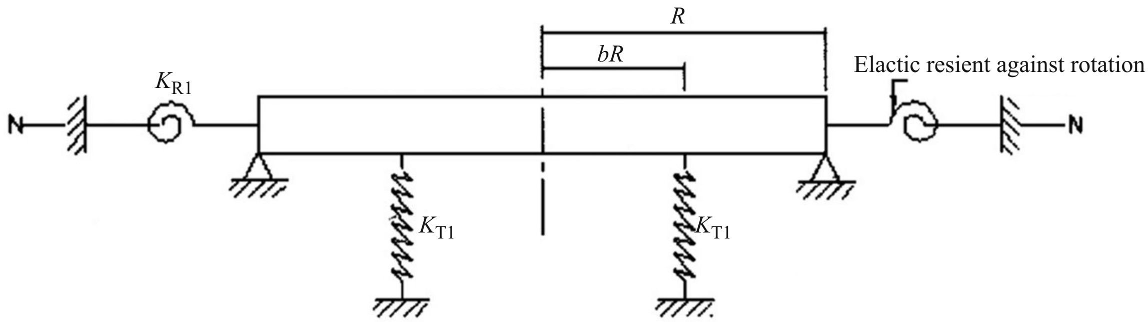

Fig. 1. Buckling of a circular plate with an internal elastic ring support and elastically restrained edge against rotation and simply supported.

may correspond more closely to some form of elastic restraints, i.e., rotational and translational restraints Kim and Dickinson [7], Wang et al. [8], Wang and Wang [9], Ashour [10], Rdzanek et al. [11], and Zagrai and Donskoy [12]. In a recent study, Wang et al. [8] showed that when the ring support has a small radius, the buckling mode takes the asymmetric mode. Wang and Wang [9] showed that the axisymmetric mode assumed by the previous authors might not yield the correct buckling load. In certain cases, an asymmetric mode would yield a lower buckling load. But they have studied only the circular plate with rigid ring support and elastically restrained edge against rotation. Recently, Wang [13] studied the buckling of a circular plate with internal elastic ring support by considering only the classical boundary conditions. The purpose of the present work is to complete the results of the buckling of circular plates with an internal elastic ring support and elastically restrained edge against rotation and simply supported by including the asymmetric buckling modes, thus correctly determining the buckling loads.

II. Definition of the Problem. Consider a thin circular plate of radius R, uniform thickness h, Young's modulus E and Poisson's ratio ν and subjected to a uniform in-plane load, N along its boundary, as shown in Fig. 1. The circular plate is also assumed to be made of linearly elastic, homogeneous and isotropic material. The edge of the circular plate is elastically restrained against rotation and simply supported and supported by an internal elastic ring support, as shown in Fig. 1. The problem at hand is to determine the elastic critical buckling load of a circular plate with an internal elastic ring support and elastically restrained edge against rotation and simply supported.

III. Mathematical Formulation of the Problem. The plate is elastically restrained against rotation and simply supported at the edge of radius R and supported on an internal elastic ring of smaller radius bR as shown in Fig. 1. Let subscript I denote the outer region $b \le \bar{r} \le 1$ and the subscript II denote the inner region $0 \le \bar{r} \le b$. Here, all lengths are normalized by R. Using the classical Kirchhoff's plate theory, we get the following fourth-order differential equation for buckling in polar coordinates (r, θ):

$$D\nabla^4 w + N\nabla^2 w = 0, \tag{1}$$

where w is the lateral displacement, N is the uniform compressive load at the edge. After normalizing the lengths by the radius of the plate R, Eq. (1) can be written as

$$D\nabla^4 \overline{w} + k^2 \nabla^2 \overline{w} = 0, \tag{2}$$

where $\nabla^2 = \frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplace operator in the polar coordinates r and θ, $\bar{r}$ is the radial distance normalized by R, $\overline{D} = Eh^3 / 12(1-\nu^2)$ is the flexural rigidity, $\overline{w} = w / R$ is the normalized transverse displacement of the plate, $k^2 = R^2 N / \overline{D}$ is the non-dimensional load parameter. Suppose there are n nodal diameters. In polar coordinates (r, θ) set

$$\overline{w}(\bar{r}, \theta) = \bar{u}(\bar{r})\cos(n\theta). \tag{3}$$

General solutions (Yamaki [14]) for the two regions are

$$\bar{u}_I(r) = C_1 J_n(k\bar{r}) + C_2 Y_n(k\bar{r}) + C_3 \bar{r}^n + C_4 \begin{Bmatrix} \log \bar{r} \\ \bar{r}^{-n} \end{Bmatrix}, \tag{4}$$

$$\bar{u}_{II}(r) = C_5 J_n(k\bar{r}) + C_6 \bar{r}^n, \tag{5}$$

where the top form of Eq. (4) is used for $n = 0$ and the bottom form is used for $n \neq 0$, $C_1, C_2, C_3, C_4, C_5, C_6$ are constants, $J_n(.)$ and $Y_n(.)$ are the Bessel functions of the first and second order n, respectively. Substituting Eqs. (4) and (5) into Eq. (3) yields the following:

$$\bar{w}_I(\bar{r},\theta) = \left[C_1 J_n(k\bar{r}) + C_2 Y_n(k\bar{r}) + C_3 \bar{r}^n + C_4 \begin{Bmatrix} \log \bar{r} \\ \bar{r}^{-n} \end{Bmatrix} \right] \cos(n\theta), \tag{6}$$

$$\bar{w}_{II}(\bar{r},\theta) = [C_5 J_n(k\bar{r}) + C_6 \bar{r}^n] \cos(n\theta). \tag{7}$$

The boundary conditions at the outer region of the circular plate in terms of rotational stiffness (K_{R1}) is given by the following expressions:

$$M_r(\bar{r}) = K_{R1} \bar{u}_I'(\bar{r}), \tag{8}$$

$$\bar{u}_I(\bar{r}) = 0. \tag{9}$$

The radial moment at the outer edge is defined as follows:

$$M_r(\bar{r}) = -\frac{D}{R^3}\left[\bar{u}_I''(\bar{r}) + \nu(\bar{u}_I'(\bar{r}) - n^2 \bar{u}_I(\bar{r})) \right]. \tag{10}$$

Equations (8) and (10) yield the following:

$$\left[\bar{u}_I''(\bar{r}) + \nu(\bar{u}_I'(\bar{r}) - n^2 \bar{u}_I(\bar{r})) \right] = -\frac{K_{R1}R^2}{D} \bar{u}_I'(\bar{r}). \tag{11}$$

Therefore, the boundary conditions are as follows:

$$\left[\bar{u}_I''(\bar{r}) + \nu(\bar{u}_I'(\bar{r}) - n^2 \bar{u}_I(\bar{r})) \right] = -R_{11} \bar{u}_I'(\bar{r}), \tag{12}$$

$$\bar{u}_I(\bar{r}) = 0, \tag{13}$$

where $R_{11} = \dfrac{K_{R1}R^2}{D}$.

Apart from the elastically restrained edge against rotation and simply supported edge, there is an internal elastic ring support constraint and the continuity requirements of slope and curvature at the support, i.e., at $\bar{r} = b$,

$$\bar{u}_I(b) = \bar{u}_{II}(b), \tag{14}$$

$$\bar{u}_I'(b) = \bar{u}_{II}'(b), \tag{15}$$

$$\bar{u}_I''(b) = \bar{u}_{II}''(b), \tag{16}$$

$$\bar{u}_I'''(b) = \bar{u}_{II}'''(b) - T_{22} \bar{u}_{II}(b), \tag{17}$$

where $T_{22} = \dfrac{K_{T2}R}{D}$. The prime (') denotes the differentiation with respect to $\bar{r}$. Non-trivial solutions to Eqs. (12), (13), (14)–(17) are sought. The lowest value of k is the square root of the normalized buckling load. From Eqs. (4), (5), (12), (13) and (14)–(17), we get the following:

$$\left[\frac{k^2}{4} P_2 + \frac{k}{2}(\nu + R_{11}) P_1 - \left(\frac{k^2}{2} + \nu n^2 \right) J_n(k) \right] C_1$$

$$+\left[\frac{k^2}{4}Q_2+\frac{k}{2}(\nu+R_{11})Q_1-\left(\frac{k^2}{2}+\nu n^2\right)Y_n(k)\right]C_2+\left[n((n-1)(1-\nu)+R_{11})\right]C_3$$

$$+\begin{Bmatrix}(\nu+R_{11})-1\\ n\big((n+1)(1-\nu)-R_{11}\big)\end{Bmatrix}C_4=0, \tag{18}$$

$$\left[J_n(k)\right]C_1+Y_n(k)C_2+1C_3+\begin{Bmatrix}0\\1\end{Bmatrix}C_4=0, \tag{19}$$

$$J_n(kb)C_1+Y_n(kb)C_2+b^nC_3+\begin{Bmatrix}\log b\\ b^{-n}\end{Bmatrix}C_4-J_n(kb)C_5-b^nC_6=0, \tag{20}$$

$$\frac{k}{2}P_1'C_1+\frac{k}{2}Q_1'C_2+nb^{n-1}C_3+\begin{Bmatrix}\frac{1}{b}\\ -nb^{-n-1}\end{Bmatrix}C_4-\frac{k}{2}P_1'C_5-nb^{n-1}C_6=0, \tag{21}$$

$$\frac{k^2}{4}(P_2'-2J_n(kb))C_1+\frac{k^2}{4}(Q_2'-2Y_n(kb))C_2+n(n-1)b^{n-2}C_3-\begin{Bmatrix}\frac{1}{b^2}\\ n(n+1)b^{-n-2}\end{Bmatrix}C_4, \tag{22}$$

$$-\frac{k^2}{4}(P_2'-2J_n(kb))C_5-n(n-1)b^{n-2}C_6=0,$$

$$+\frac{k^2}{8}(P_3'-3P_1')C_1+\frac{k^2}{8}(Q_3'-Q_1')C_2+n(n-1)(n-2)b^{n-3}C_3+\begin{Bmatrix}\frac{2}{b^3}\\ -n(n+1)(n+2)b^{-n-3}\end{Bmatrix}C_4$$

$$-\left[\frac{k^2}{8}(P_3'-3P_1')-T_{22}J_n(kb)\right]C_5-[n(n-1)(n-2)b^{n-3}-T_{22}b^n]C_6=0, \tag{23}$$

$$P_1=J_{n-1}(k)-J_{n+1}(k),\quad P_2=J_{n-2}(k)+J_{n+2}(k),\quad P_3=J_{n-3}(k)-J_{n+3}(k),$$

$$Q_1=Y_{n-1}(k)-Y_{n+1}(k),\quad Q_2=Y_{n-2}(k)+Y_{n+2}(k),\quad Q_3=Y_{n-3}(k)-Y_{n+3}(k),$$

$$P_1'=J_{n-1}(kb)-J_{n+1}(kb),\quad P_2'=J_{n-2}(kb)+J_{n+2}(kb),\quad P_3'=J_{n-3}(kb)-J_{n+3}(kb),$$

$$Q_1'=Y_{n-1}(kb)-Y_{n+1}(kb),\quad Q_2'=Y_{n-2}(kb)+Y_{n+2}(kb),\quad Q_3'=Y_{n-3}(kb)-Y_{n+3}(kb).$$

The top forms of Eqs. (17)–(23) are used for $n=0$ (axisymmetric buckling) and the bottom forms are used for $n\neq 0$ (asymmetric buckling).

IV. Solution. For the given values of $n, \nu, R_{11}, T_{22}, b$ the above set of equations gives the exact characteristic equation for non-trivial solutions of the coefficients $C_1, C_2, C_3, C_4, C_5, C_6$. For non-trivial solution, the determinant of $[C]_{6x6}$ must be removed. The value of *k* calculated from the characteristic equation by a simple root search method. Using Mathematica, computer software with symbolic capabilities, we solve this problem.

V. Results and Discussions. The influence of the rotational spring stiffness parameter on the buckling load for the given translational spring stiffness parameters of an elastic ring support is shown in Figs. 2–5. Figures 2–5 show the variations of the buckling load parameter *k*, with respect to the internal elastic ring support radius *b*, for various values of rotational spring stiffness parameters ($R_{11}=0, 0.5, 10, 100, \infty$) by keeping the translational spring stiffness parameter of an internal elastic ring support constant ($T_{22}=100000$). It is observed from Figs. 2–5 that for a given value of R_{11} and constant T_{22}, the curve is composed of two segments. This is due to the switching of buckling modes. For a smaller internal elastic ring support radius *b*, the plate buckles in an asymmetric mode (i.e., $n=1$). In this segment (as shown by the dotted lines in Figs. 2–5) the buckling load

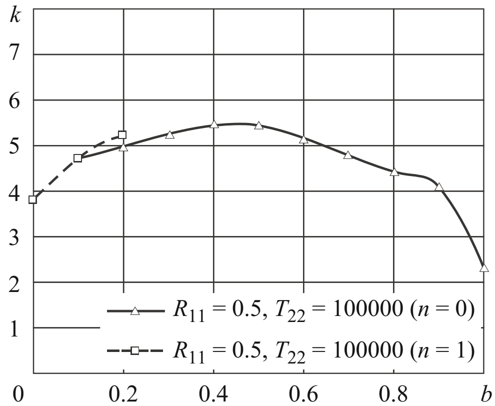

Fig. 2. Buckling load parameter *k* versus internal elastic ring support radius *b* for $R_{11} = 0.5, T_{22} = 100000$.

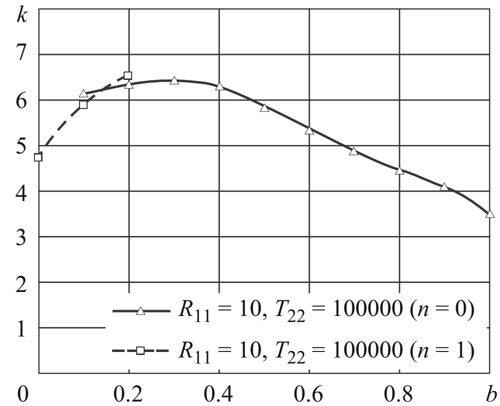

Fig. 3. Buckling load parameter *k* versus internal elastic ring support radius *b* for $R_{11} = 10, T_{22} = 100000$.

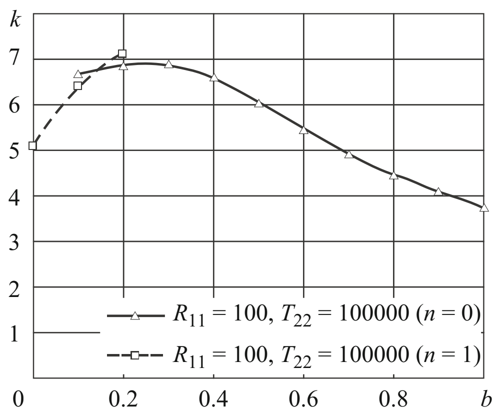

Fig. 4. Buckling load parameter *k* versus internal elastic ring support radius *b* for $R_{11} = 100, T_{22} = 100000$.

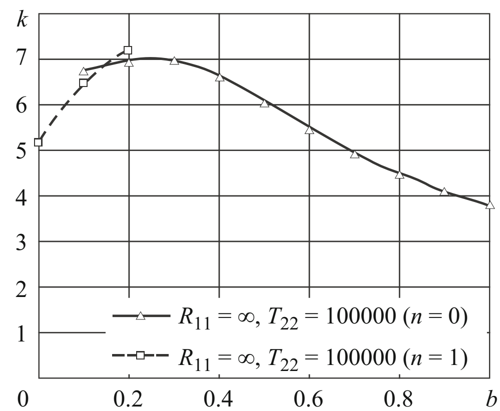

Fig. 5. Buckling load parameter *k* versus internal elastic ring support radius *b* for $R_{11} = \infty, T_{22} = 100000$.

TABLE 1. Optimal location of internal elastic ring support b_{opt}, corresponding buckling load parameter k_{opt}, and percentage increase in buckling load parameter

Parameter	$T_{22} = 100000$				
R_{11}	0	0.5	10	100	∞
b_{opt}	0.4998	0.4010	0.3001	0.2982	0.2966
k_{opt}	5.3669	5.4571	6.4333	6.9313	6.9989
%	161.95	135.52	84.39	82.71	82.66

decreases with *b*. For larger internal elastic ring support radius *b*, the plate buckles in an axisymmetric mode (i.e., $n = 0$). In this segment (as shown by the continuous lines in Figs. 2–5) the buckling load increases as *b* decreases up to a peak point corresponding to the maximum buckling load and thereafter decrease with *b*.

The cross over radius varies from $b = 0.09891$ for $R_{11} = 0$ and $T_{22} = 100000$ to $b = 0.1545$ for $R_{11} = \infty$ and $T_{22} = 100000$ as shown in Figs. 2 and 5, respectively. The major interest in the design of supported circular plates is the optimal location of the internal elastic ring support for the maximum buckling load. The optimal solutions for this case are presented in Table 1. It is observed that the optimal ring support radius parameter decreases with increase in the rotational spring stiffness parameter and

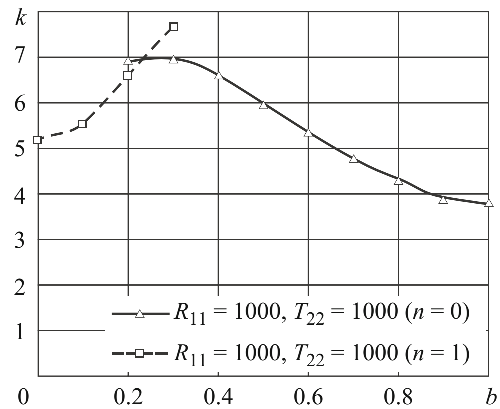

Fig. 6. Buckling load parameter k versus internal elastic ring support radius b for $T_{22} = 1000$, $R_{11} = 1000$.

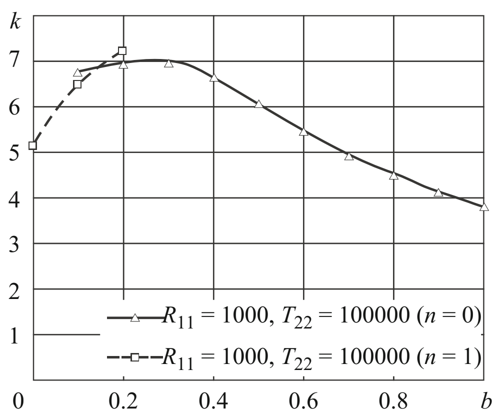

Fig. 7. Buckling load parameter k versus internal elastic ring support radius b for $T_{22} = 100000$, $R_{11} = 1000$.

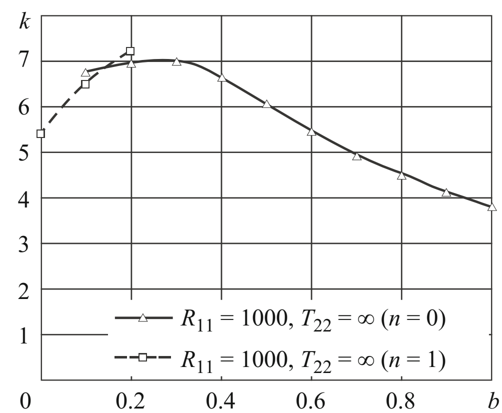

Fig. 8. Buckling load parameter k versus internal elastic ring support radius b for $T_{22} = \infty$, $R_{11} = 1000$.

also the optimal buckling load capacity increases with the rotational spring stiffness parameter. Introducing internal elastic ring support in the optimal position increases the elastic buckling capacity significantly, and the percentage of increase in buckling loads is presented in Table 1. It is observed that the percentage increase in the buckling load parameter decreases with increase in R_{11}. This because the amount of increase in the buckling load without elastic ring support with R_{11} is more than that of increase in the buckling load with elastic ring support with R_{11}.

The influence of translational spring stiffness parameter of an elastic ring support on the buckling load for a given rotational spring stiffness parameter is shown in Figs. 6–8. Figures 6–8 show the variations of the buckling load parameter k with respect to the internal elastic ring support radius b for various values of the translational spring stiffness parameter of the internal elastic ring support ($T_{22} = 1000, 100000, \infty$) by keeping rotational spring stiffness parameters constant ($R_{11} = 1000$). It is observed from Figs. 6–8 that for a given value of T_{22} and constant R_{11}, the curve is composed of two segments. This is due to the switching of buckling modes. For a smaller internal elastic ring support radius b, the plate buckles in an asymmetric mode (i.e., $n = 1$). In this segment (as shown by the dotted lines in Figs. 6–8) the buckling load decreases with b. For larger internal elastic ring support radius b, the plate buckles in an axisymmetric mode (i.e., $n = 0$). In this segment (as shown by the continuous lines in Figs. 6–8) the buckling load increases as b decreases up to a peak point corresponding to the maximum buckling load and thereafter decrease with b.

The cross over radius varies from $b = 0.2333$ for $T_{22} = 1000$ and $R_{11} = 1000$ to $b = 0.1518$ for $T_{22} = \infty$ and $R_{11} = 1000$ as shown in Figs. 6 and 8, respectively. The optimal solutions for this case are presented in Table 2.

Introducing internal elastic ring support in the optimal position increases the elastic buckling capacity significantly, and the percentage of increase in the buckling loads is presented in Table 2.

TABLE 2. Optimal locations of internal elastic ring support b_{opt}, corresponding buckling load parameter k_{opt}, and percentage increase in buckling load parameter

Parameter	$R_{11} = 1000$		
T_{22}	1000	100000	∞
b_{opt}	0.2999	0.2987	0.2984
k_{opt}	6.9857	6.9898	6.9901
%	82.49	82.60	82.61

TABLE 3. Comparison of buckling load parameter b_{opt} with Wang et al. [17] for various rotational stiffness parameters R_{11} and Poisson's ratio = 0.3

R_{11}	0	0.1	5	10	100	∞
Wang et al.	4.198	4.449	10.462	12.173	14.392	14.682
Present	4.19766	4.44864	10.46134	12.17242	14.39200	14.6814

TABLE 4. Comparison of buckling load parameter k with Laura et al. [4], Wang et al. [17], and Bhaskara Rao and Kameswara Rao [16] for rotational stiffness parameter $R_{11} = 0$ and $\nu = 0.3$

Ring support radius, b	Laura et al. [4]	Wang et al. [17]	Bhaskara Rao and Kameswara Rao [16]	Present
0.1	4.5244	4.5235	4.52341	4.52341
0.2	4.7718	4.7702	4.77018	4.77018
0.3	5.0725	5.071	5.07091	5.07091
0.4	5.3301	5.3296	5.32964	5.32964
0.5	5.3666	5.3666	5.36659	5.36659
0.6	5.1284	5.1261	5.12606	5.12606
0.7	4.7789	4.7727	4.77266	4.77266
0.8	4.4249	4.4215	4.42141	4.42141
0.9	4.1122	4.1063	4.10629	4.10629

TABLE 5. Comparison of buckling load parameter k with Laura et al. [4] and Bhaskara Rao and Kameswara Rao [16] for rotational stiffness parameter $R_{11} = \infty$ and $\nu = 0.3$

Ring support radius, b	Laura et al. [4]	Bhaskara Rao and Kameswara Rao [16]	Present
0.1	6.772	6.50105	6.50105
0.2	6.9649	6.95592	6.95592
0.3	6.9964	6.99485	6.99485
0.4	6.6693	6.66257	6.66257
0.5	6.0852	6.07454	6.07454
0.6	5.4845	5.4755	5.4755
0.7	4.9588	4.95263	4.95263
0.8	4.5277	4.51266	4.51266
0.9	4.1509	4.14357	4.14357

The results of this kind were scarce in the literature. However, the results are compared with the following cases:

(i) for any value of R_{11} and as $T_{22} \to \infty$ and $b \to 1$, all the curves converge to $k = 3.83165$, which is of the clamped plate, and this is in agreement with Wang et al. [9];

(ii) as $R_{11} \to \infty$ and $T_{22} = 10$, or clamped support with internal elastic ring support, the optimum location is at radius $b = 0.290$, with buckling load $k = 4.20875$, and also as $b \to 1$, the buckling load $k = 3.83163$, which is in good agreement with Wang [6];

(iii) as $R_{11} \to 0$ and $T_{22} = 10$, or simply supported edge plate with internal elastic ring support, the optimum location is at radius $b = 0.417$, with buckling load $k = 2.69104$, and also as $b \to 1$, the buckling load $k = 2.04882$, which is in good agreement with Wang [6];

(iv) Table 3 presents the buckling load parameters k for a circular plate with simply supported edge and rotational restraint with $T_{22} = 0$ (i.e., with no ring support), against those obtained by Wang et. al. [15];

(v) as $R_{11} \to \infty$ and $T_{22} \to \infty$, or rotationally restrained and simply supported circular plate with internal rigid support, the optimum location is at radius $b = 0.265$, with buckling load $k = 7.01554$, which is in agreement with Wang et al [15];

(vi) Tables 4 and 5 present the buckling load parameters k for a circular plate with an internal ring support ($T_{22} \to \infty$, i.e., rigid ring support) and elastically restrained edge against rotation and simply supported, against those obtained by Laura et al. [4], Wang et al. [17] and Bhaskara Rao and Kameswara Rao [16].

VI. Conclusions. The buckling problem for thin circular plates with an internal elastic ring support and elastically restrained edge against rotation and simply supported has been solved. The buckling loads are given for various rotational restraints [R_{11}] and translational restraints of internal ring support [T_{22}]. It is observed that the buckling mode switches from an asymmetric mode to an axisymmetric mode at a particular ring support radius. The cross-over radius is determined for different values of rotational restraints and translational restraints of elastic ring support. The optimal ring support is affected by the rotational stiffness parameters and translational spring stiffness parameters of the internal elastic ring support. The optimum location increases with decreasing T_{22}, whereas the bucking load decreases with T_{22}. The optimum location increases with decreasing R_{11}, whereas the bucking load decreases with R_{11}. However, it is observed that the influence of the rotational restrains on the buckling load is more predominant than that of the translational restraints of the internal elastic ring support. In this paper, the characteristic equations are exact; therefore the results can be calculated to any accuracy. These exact solutions can be used to check numerical or approximate results. The tabulated buckling results are useful to designers in structural design and vibration control.

Nomenclature:

$w(r, \theta)$ — Transverse deflection of the plate;
h — Thickness of the plate;
R — Radius of the plate;
b — Non-dimensional radius of the ring support;
ν — Poisson's ratio;
E — Young's modulus of the material;
D — Flexural rigidity of the material;
K_{T2} — Translational spring stiffness of internal elastic ring;
K_{R1} — Rotational spring stiffness;
R_{11} — Non-dimensional rotational spring stiffness parameter;
T_{22} — Non-dimensional translational spring stiffness parameter of internal elastic ring;
N — Uniform in-plane compressive load;
k — Non-dimensional buckling load parameter.

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