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Research Article

Coefficient Estimate of Biunivalent Functions of Complex Order Associated with the Hohlov Operator

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We introduce and investigate a new subclass of the function class Σ of biunivalent functions of complex order defined in the open unit disk, which are associated with the Hohlov operator, satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass. Several, known or new, consequences of the results are also pointed out.

1. Introduction, Definitions, and Preliminaries

Let $\mathscr{A}$ denote the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}. \quad (2)$$

By $\mathscr{S}$ we denote the class of all functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the class $\mathscr{S}$ include, for example, the class $\mathscr{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathscr{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. It is well known that every function $f \in \mathscr{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$
$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \geqq \frac{1}{4}\right), \quad (3)$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \quad (4)$$

A function $f \in \mathscr{A}$ is said to be biunivalent in $\mathbb{U}$, if $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of biunivalent functions in $\mathbb{U}$ given by (1).

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided that there is an analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = g(\omega(z))$. Ma and Minda [1] unified various subclasses of starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1+(zf''(z)/f'(z))$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\mathbb{U}$, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathscr{A}$ satisfying the subordination $zf'(z)/f(z) \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathscr{A}$ satisfying the subordination $1 + (zf''(z)/f'(z)) \prec \phi(z)$.

A function f is bi-starlike of Ma-Minda type or biconvex of Ma-Minda type, if both f and f^{-1} are, respectively, Ma-Minda starlike or convex. These classes are denoted, respectively, by $\mathscr{S}^*_{\Sigma}(\phi)$ and $\mathscr{K}_{\Sigma}(\phi)$. In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0) = 1$ and $\phi'(0) > 0$, and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{5}$$

The convolution or Hadamard product of two functions f and $h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{6}$$

where $f(z)$ is given by (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Here, in our present investigation, we recall a convolution operator $\mathcal{I}_{a,b,c}$ due to Hohlov [2, 3], which indeed is a special case of the Dziok-Srivastava operator [4, 5].

For the complex parameters a, b, and $c(c \neq 0, -1, -2, -3, \ldots)$, the Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ is defined as

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\ &= 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!} \quad (z \in \mathbb{U}), \end{aligned} \tag{7}$$

where $(\alpha)_n$ is the Pochhammer symbol (or the shifted factorial) defined as follows:

$$\begin{aligned} (\alpha)_n &= \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \\ &= \begin{cases} 1 & (n = 0), \\ \alpha(\alpha + 1)(\alpha + 2), \ldots, (\alpha + n - 1) & (n = 1, 2, 3, \ldots). \end{cases} \end{aligned} \tag{8}$$

For the positive real values a, b, and $c(c \neq 0, -1, -2, -3, \ldots)$, by using the Gaussian hypergeometric function given by (7), Hohlov [2, 3] introduced the familiar convolution operator $\mathcal{I}_{a,b,c}$ as follows:

$$\begin{aligned} \mathcal{I}_{a,b;c} f(z) &= z\, {}_2F_1(a, b, c; z) * f(z), \\ &= z + \sum_{n=2}^{\infty} \varphi_n a_n z^n \quad (z \in \mathbb{U}), \end{aligned} \tag{9}$$

where

$$\varphi_n = \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!}. \tag{10}$$

Hohlov [2, 3] discussed some interesting geometrical properties exhibited by the operator $\mathcal{I}_{a,b;c}$. The three-parameter family of operators $\mathcal{I}_{a,b;c}$ contains, as its special cases, most of the known linear integral or differential operators. In particular, if $b = 1$ in (9), then $\mathcal{I}_{a,b;c}$ reduces to the Carlson-Shaffer operator. Similarly, it is easily seen that the Hohlov operator $\mathcal{I}_{a,b;c}$ is also a generalization of the Ruscheweyh derivative operator as well as the Bernardi-Libera-Livingston operator.

Recently, there has been triggering interest to study biunivalent function class Σ and obtained nonsharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1). But the coefficient problem for each of the Taylor-Maclaurin coefficients,

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{11}$$

is still an open problem (see [6–11]). Many researchers (see [12–17]) have recently introduced and investigated several interesting subclasses of the biunivalent function class Σ and they have found nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Motivated by the earlier work of Deniz [18] (see [19–21]) and Peng and Han [22], in the present paper, we introduce new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C} \setminus \{0\}$, involving Hohlov operator $\mathcal{I}_{a,b;c}$, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ. Several related classes are also considered, and connection to earlier known results are made.

Definition 1. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma, \lambda, \phi)$, if the following conditions are satisfied:

$$\begin{gathered} 1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}_{a,b;c} f(z))'}{(1-\lambda) z + \lambda \mathcal{I}_{a,b;c} f(z)} - 1\right) \prec \phi(z) \\ (\gamma \in \mathbb{C} \setminus \{0\}; 0 \leqq \lambda \leqq 1;\ z \in \mathbb{U}), \\ 1 + \frac{1}{\gamma}\left(\frac{w(\mathcal{I}_{a,b;c} g(w))'}{(1-\lambda) w + \lambda \mathcal{I}_{a,b;c} g(w)} - 1\right) \prec \phi(w) \\ (\gamma \in \mathbb{C} \setminus \{0\}; 0 \leqq \lambda \leqq 1;\ w \in \mathbb{U}), \end{gathered} \tag{12}$$

where the function g is given by (4).

On specializing the parameters λ and a, b, and c, one can state the various new subclasses of Σ as illustrated in the following examples.

Example 2. For $\lambda = 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1), is said to be in the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma, \phi)$, if the following conditions are satisfied:

$$\begin{gathered} 1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}_{a,b;c} f(z))'}{\mathcal{I}_{a,b;c} f(z)} - 1\right) \prec \phi(z), \\ 1 + \frac{1}{\gamma}\left(\frac{w(\mathcal{I}_{a,b;c} g(w))'}{\mathcal{I}_{a,b;c} g(w)} - 1\right) \prec \phi(w), \end{gathered} \tag{13}$$

where $z, w \in \mathbb{U}$ and the function g is given by (4).

Example 3. For $\lambda = 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1), is said to be in the class $\mathcal{G}_{\Sigma}^{a,b;c}(\gamma, \phi)$, if the following conditions are satisfied:

$$\begin{gathered} 1 + \frac{1}{\gamma}\left((\mathcal{I}_{a,b;c} f(z))' - 1\right) \prec \phi(z), \\ 1 + \frac{1}{\gamma}\left((\mathcal{I}_{a,b;c} g(w))' - 1\right) \prec \phi(w), \end{gathered} \tag{14}$$

where $z, w \in \mathbb{U}$ and the function g is given by (4).

It is of interest to note that, for $a = c$ and $b = 1$, the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\lambda,\phi)$ reduces to the following new subclasses.

Example 4. For $\lambda = 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1), is said to be in the class $\mathcal{S}_{\Sigma}^{*}(\gamma,\phi)$, if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\phi(z),\qquad 1+\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}-1\right)\prec\phi(w), \tag{15}$$

where $z, w \in \mathbb{U}$ and the function g is given by (4).

Example 5. For $\lambda = 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1), is said to be in the class $\mathcal{H}_{\Sigma}^{*}(\gamma,\phi)$, if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(f'(z)-1\right)\prec\phi(z),\qquad 1+\frac{1}{\gamma}\left(g'(w)-1\right)\prec\phi(w), \tag{16}$$

where $z, w \in \mathbb{U}$ and the function g is given by (4).

In the following section, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\lambda,\phi)$ of the function class Σ by employing the technique which is different from that used by earlier authors. Earlier authors investigated the coefficients of biunivalent functions mainly by using the following lemma.

Lemma 6 (see [23]). *If $h \in \mathcal{P}$, then $|c_k| \leqq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\{h(z)\}>0\quad(z\in\mathbb{U}), \tag{17}$$

where

$$h(z)=1+c_1z+c_2z^2+\cdots\quad(z\in\mathbb{U}). \tag{18}$$

2. Coefficient Bounds for the Function Class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\lambda,\phi)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\lambda,\phi)$.

Suppose that $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 0 = q(0)$, $|p(z)| < 1$, and $|q(z)| < 1$ and suppose that

$$p(z)=p_1z+p_2z^2+\cdots\quad(|z|<1),\qquad q(z)=q_1z+q_2z^2+\cdots\quad(|z|<1). \tag{19}$$

It is well known that

$$|p_1|\le 1,\quad |p_2|\le 1-|p_1|^2,\qquad |q_1|\le 1,\quad |q_2|\le 1-|q_1|^2. \tag{20}$$

Thus, from (5), it follows that

$$\phi(p(z))=1+B_1p_1z+\left(B_1p_2+B_2p_1^2\right)z^2+\cdots, \tag{21}$$

$$\phi(q(w))=1+B_1q_1w+\left(B_1q_2+B_2q_1^2\right)w^2+\cdots. \tag{22}$$

Theorem 7. *Let a function $f(z)$, given by (1), be in the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\lambda,\phi)$. Then*

$$|a_2|\le\frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{\left|\left[\gamma(\lambda^2-2\lambda)B_1^2-(2-\lambda)^2B_2\right]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3\right|+(2-\lambda)^2B_1\varphi_2^2}},$$

$$|a_3|\le\begin{cases}\dfrac{|\gamma|B_1}{(3-\lambda)\varphi_3}, & |\gamma|\le\dfrac{(2-\lambda)^2\varphi_2^2}{(3-\lambda)\varphi_3B_1},\\[2ex] \dfrac{|\gamma|B_1\left|\left[\gamma(\lambda^2-2\lambda)B_1^2-(2-\lambda)^2B_2\right]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3\right|+(3-\lambda)\varphi_3B_1^3|\gamma|^2}{(3-\lambda)\varphi_3\left\{\left|\left[\gamma(\lambda^2-2\lambda)B_1^2-(2-\lambda)^2B_2\right]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3\right|+(2-\lambda)^2B_1\varphi_2^2\right\}}, & |\gamma|>\dfrac{(2-\lambda)^2\varphi_2^2}{(3-\lambda)\varphi_3B_1},\end{cases} \tag{23}$$

where φ_2 and φ_3 are given by (10).

Proof. It follows from (12) that

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{I}_{a,b;c}f(z))'}{(1-\lambda)z+\lambda\mathcal{I}_{a,b;c}f(z)}-1\right)=\phi(p(z)),\qquad 1+\frac{1}{\gamma}\left(\frac{w(\mathcal{I}_{a,b;c}g(w))'}{(1-\lambda)w+\lambda\mathcal{I}_{a,b;c}g(w)}-1\right)=\phi(q(w)), \tag{24}$$

where $\phi(p(z))$ and $\phi(q(w))$ are given by (21) and (22), respectively.

Now, by equating the coefficients in (24), we get

$$\frac{(2-\lambda)}{\gamma}\varphi_2a_2=B_1p_1, \tag{25}$$

$$\frac{\left(\lambda^2-2\lambda\right)}{\gamma}\varphi_2^2a_2^2+\frac{(3-\lambda)}{\gamma}\varphi_3a_3=B_1p_2+B_2p_1^2, \tag{26}$$

$$-\frac{(2-\lambda)}{\gamma}\varphi_2 a_2 = B_1 q_1, \tag{27}$$

$$\frac{(\lambda^2-2\lambda)}{\gamma}\varphi_2^2 a_2^2 + \frac{(3-\lambda)}{\gamma}\varphi_3\left(2a_2^2-a_3\right) = B_1 q_2 + B_2 q_1^2. \tag{28}$$

From (25) and (27), we find that

$$a_2 = \frac{\gamma B_1 p_1}{(2-\lambda)\varphi_2} = \frac{-\gamma B_1 q_1}{(2-\lambda)\varphi_2}, \tag{29}$$

which implies

$$p_1 = -q_1, \tag{30}$$

$$(2-\lambda)^2\varphi_2^2 a_2^2 = \gamma^2 B_1^2 p_1^2. \tag{31}$$

By adding (26) and (28) and by using (29) and (30), we obtain

$$\left\{\left[2\gamma\left(\lambda^2-2\lambda\right)B_1^2 - 2(2-\lambda)^2 B_2\right]\varphi_2^2 + 2\gamma(3-\lambda)B_1^2\varphi_3\right\}a_2^2 = B_1^3\gamma^2\left(p_2+q_2\right). \tag{32}$$

Now, by using (20) and (31), we get

$$\left\{\left|\left[\gamma\left(\lambda^2-2\lambda\right)B_1^2 - (2-\lambda)^2 B_2\right]\varphi_2^2 + \gamma(3-\lambda)B_1^2\varphi_3\right| + (2-\lambda)^2 B_1\varphi_2^2\right\}\left|a_2\right|^2 \le \left|\gamma^2\right| B_1^3. \tag{33}$$

Hence,

$$\left|a_2\right| \le \left(\left|\gamma\right| B_1\sqrt{B_1}\right) \times \left(\left|\left[\gamma\left(\lambda^2-2\lambda\right)B_1^2 - (2-\lambda)^2 B_2\right] \times \varphi_2^2 + \gamma(3-\lambda)B_1^2\varphi_3\right| + (2-\lambda)^2 B_1\varphi_2^2\right)^{-1/2}. \tag{34}$$

This gives the bound on $|a_2|$ as asserted in (23).

Next, in order to find the bound on $|a_3|$, by subtracting (28) from (26), we get

$$\frac{2(3-\lambda)}{\gamma}\varphi_3 a_3 = B_1\left(p_2-q_2\right) + \frac{2(3-\lambda)}{\gamma}\varphi_3 a_2^2. \tag{35}$$

It follows from (20), (30), and (35) that

$$\left|a_3\right| \le \frac{|\gamma| B_1}{(3-\lambda)\varphi_3} + \frac{(3-\lambda)\varphi_3|\gamma| B_1 - (2-\lambda)^2\varphi_2^2}{(3-\lambda)\varphi_3|\gamma| B_1}\left|a_2\right|^2. \tag{36}$$

By using (34), we obtain

$$\left|a_3\right| \le \begin{cases} \dfrac{|\gamma| B_1}{(3-\lambda)\varphi_3}, & |\gamma| \le \dfrac{(2-\lambda)^2\varphi_2^2}{(3-\lambda)\varphi_3 B_1}, \\ \dfrac{|\gamma| B_1\left|\left[\gamma\left(\lambda^2-2\lambda\right)B_1^2-(2-\lambda)^2B_2\right]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3\right| + (3-\lambda)\varphi_3 B_1^3|\gamma|^2}{(3-\lambda)\varphi_3\left\{\left|\left[\gamma\left(\lambda^2-2\lambda\right)B_1^2-(2-\lambda)^2B_2\right]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3\right| + (2-\lambda)^2B_1\varphi_2^2\right\}}, & |\gamma| > \dfrac{(2-\lambda)^2\varphi_2^2}{(3-\lambda)\varphi_3 B_1}. \end{cases} \tag{37}$$

This completes the proof of Theorem 7. □

By putting $\lambda = 1$ in Theorem 7, we have the following corollary.

Corollary 8. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\phi)$. Then*

$$\left|a_2\right| \le \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{\left|2\gamma B_1^2\varphi_3 - \left(\gamma B_1^2+B_2\right)\varphi_2^2\right| + B_1\varphi_2^2}},$$

$$\left|a_3\right| \le \begin{cases} \dfrac{|\gamma| B_1}{2\varphi_3}, \\ \qquad |\gamma| \le \dfrac{\varphi_2^2}{2\varphi_3 B_1} \\ \dfrac{|\gamma| B_1\left|2\gamma B_1^2\varphi_3-\left(\gamma B_1^2+B_2\right)\varphi_2^2\right| + 2\varphi_3 B_1^3|\gamma|^2}{2\varphi_3\left\{\left|2\gamma B_1^2\varphi_3-\left(\gamma B_1^2+B_2\right)\varphi_2^2\right| + B_1\varphi_2^2\right\}}, \\ \qquad |\gamma| > \dfrac{\varphi_2^2}{2\varphi_3 B_1}. \end{cases} \tag{38}$$

By taking $a = c$ and $b = 1$, in Corollary 8, we get the following corollary.

Corollary 9. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{S}_{\Sigma}^{*}(\gamma,\phi)$. Then*

$$\left|a_2\right| \le \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{\left|\gamma B_1^2 - B_2\right| + B_1}}, \tag{39}$$

$$\left|a_3\right| \le \begin{cases} \dfrac{|\gamma| B_1}{2}, & |\gamma| \le \dfrac{1}{2B_1}, \\ \dfrac{|\gamma| B_1\left|\gamma B_1^2 - B_2\right| + 2B_1^3|\gamma|^2}{2\left(\left|\gamma B_1^2-B_2\right| + B_1\right)}, & |\gamma| > \dfrac{1}{2B_1}. \end{cases} \tag{40}$$

By putting $\lambda = 0$ in Theorem 7, we have the following corollary.

Corollary 10. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{G}_{\Sigma}^{a,b;c}(\gamma,\phi)$. Then*

$$|a_2| \le \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|3\gamma B_1^2\varphi_3 - 4B_2\varphi_2^2| + 4B_1\varphi_2^2}},$$

$$|a_3| \le \begin{cases} \dfrac{|\gamma| B_1}{3\varphi_3}, & |\gamma| \le \dfrac{4\varphi_2^2}{3\varphi_3 B_1}, \\ \dfrac{|\gamma| B_1 |3\gamma B_1^2\varphi_3 - 4B_2\varphi_2^2| + 3\varphi_3 B_1^3 |\gamma|^2}{3\varphi_3 (|3\gamma B_1^2\varphi_3 - 4B_2\varphi_2^2| + 4B_1\varphi_2^2)}, & |\gamma| > \dfrac{4\varphi_2^2}{3\varphi_3 B_1}. \end{cases} \quad (41)$$

By taking $a = c$ and $b = 1$, in Corollary 10, we get the following corollary.

Corollary 11. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{H}_{\Sigma}^{*}(\gamma,\phi)$. Then*

$$|a_2| \le \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|3\gamma B_1^2 - 4B_2| + 4B_1}},$$

$$|a_3| \le \begin{cases} \dfrac{|\gamma| B_1}{3}, & |\gamma| \le \dfrac{4}{3B_1}, \\ \dfrac{|\gamma| B_1 |3\gamma B_1^2 - 4B_2| + 3B_1^3 |\gamma|^2}{3 (|3\gamma B_1^2 - 4B_2| + 4B_1)}, & |\gamma| > \dfrac{4}{3B_1}. \end{cases} \quad (42)$$

3. Concluding Remarks

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \le 1), \quad (43)$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$.

Remark 12. From Theorem 7, when $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$ for the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\lambda,\phi)$ [8], we get

$$|a_2| \le \frac{|2\gamma| \alpha}{\sqrt{|(\lambda-2)(2\gamma\lambda-\lambda+2)\alpha\varphi_2^2 + 2(3-\lambda)\gamma\alpha\varphi_3| + (2-\lambda)^2\varphi_2^2}},$$

$$|a_3| \le \begin{cases} \dfrac{|2\gamma| \alpha}{(3-\lambda)\varphi_3}, & |\gamma| \le \dfrac{(2-\lambda)^2\varphi_2^2}{2(3-\lambda)\varphi_3\alpha}, \\ \dfrac{|2(\lambda-2)(2\gamma\lambda-\lambda+2)\gamma\alpha^2\varphi_2^2 + 4\gamma^2(3-\lambda)\alpha^2\varphi_3| + 4(3-\lambda)\alpha^2\varphi_3|\gamma|^2}{(3-\lambda)\varphi_3 \{|(\lambda-2)(2\gamma\lambda-\lambda+2)\alpha\varphi_2^2 + 2\gamma(3-\lambda)\alpha\varphi_3| + (2-\lambda)^2\varphi_2^2\}}, & |\gamma| > \dfrac{(2-\lambda)^2\varphi_2^2}{2(3-\lambda)\varphi_3\alpha}. \end{cases} \quad (44)$$

On the other hand, if we take

$$\phi(z) = \frac{1 + (1-2\beta) z}{1-z} = 1 + 2(1-\beta) z + 2(1-\beta) z^2 + \cdots \quad (0 \le \beta < 1), \quad (45)$$

then $B_1 = B_2 = 2(1-\beta)$.

Remark 13. From Theorem 7, when $B_1 = B_2 = 2(1-\beta)$ for the class $\mathcal{S}_{\Sigma}^{a,b;c}(\gamma,\lambda,\phi)$, we get

$$|a_2| \le \frac{2(1-\beta)|\gamma|}{\sqrt{|[2(1-\beta)\lambda\gamma-\lambda+2](\lambda-2)\varphi_2^2 + 2(1-\beta)(3-\lambda)\gamma\varphi_3| + (2-\lambda)^2\varphi_2^2}},$$

$$|a_3| \le \begin{cases} \dfrac{2(1-\beta)|\gamma|}{(3-\lambda)\varphi_3}, \\ \qquad |\gamma| \le \dfrac{(2-\lambda)^2\varphi_2^2}{2(1-\beta)(3-\lambda)\varphi_3} \\ \dfrac{2(1-\beta)|(\lambda-2)[2(1-\beta)\lambda\gamma-\lambda+2]\gamma\varphi_2^2 + 2(1-\beta)(3-\lambda)\gamma^2\varphi_3| + 4(1-\beta)^2(3-\lambda)|\gamma|^2\varphi_3}{(3-\lambda)\varphi_3 \{|(\lambda-2)[2(1-\beta)\gamma\lambda-\lambda+2]\varphi_2^2 + 2(1-\beta)(3-\lambda)\gamma\varphi_3| + (2-\lambda)^2\varphi_2^2\}}, \\ \qquad |\gamma| > \dfrac{(2-\lambda)^2\varphi_2^2}{2(1-\beta)(3-\lambda)\varphi_3}. \end{cases} \quad (46)$$

Remark 14. By putting $\gamma = 1$ in Corollary 11 we obtain more accurate results corresponding to the results obtained in [19]. Further, by taking $\gamma = 1$ and $\phi(z)$ is given by (43) (or by (45), the results obtained in Theorem 7 and Corollary 11 yield more accurate results than the results obtained in [15, 21].

Remark 15. If $a = 1$, $b = 1 + \delta$, and $c = 2 + \delta$ with $\Re(\delta) > -1$, then the operator $I_{a,b,c} f$ turns into well-known Bernardi operator:

$$B_f(z) = \left[\mathcal{I}_{a,b,c}(f)\right](z) = \frac{1+\delta}{z^{\delta}} \int_0^1 t^{\delta-1} f(t)\, dt. \quad (47)$$

$\mathcal{I}_{1,1,2} f$ and $\mathcal{I}_{1,2,3} f$ are the well-known Alexander and Libera operators, respectively. Further, if $b = 1$ in (9), then $\mathcal{I}_{a,b;c}$ immediately yields the Carlson-Shaffer operator $L(a,c)(f) := \mathcal{I}_{a,1,c} f$. So, various other interesting corollaries and consequences of our main results (which are asserted by Theorem 7 above) can be derived similarly. The details involved may be left as an exercise for the interested reader.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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