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COEFFICIENT ESTIMATE OF CERTAIN SUBCLASSES OF CONVEX $p-$VALENT FUNCTIONS WITH A BOUNDED POSITIVE REAL PART

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Abstract: We estimate the bounds of coefficients and solve Fekete-Szegö problem for $p-$valent Mocanu-convex and Pascu-type functions in the open unit disk $\triangle$ which maps $\triangle$ onto the strip domain w with $p\alpha < \Re\, w < p\beta$.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of all functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and $p-$valent in the open unit disk $\triangle = \{z : z \in \mathbb{C} : |z| < 1\}$. Note that $\mathcal{A}_1 := \mathcal{A}$the class of analytic functions further $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\triangle$. A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ in $\triangle$ if it satisfies $\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [4]. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$. Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f \in \mathcal{A}$ which satisfy $\Re \left(\frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in \triangle)$ for some real number β with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi et. al [6].

Let $P(z)$ and $Q(z)$ be analytic in $\triangle$. Then the function $P(z)$ is said to subordinate to $Q(z)$ in $\triangle$ written by

$$P(z) \prec Q(z) \qquad (z \in \triangle), \tag{1}$$

if there exists a function $w(z)$ which is analytic in $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \triangle)$, and such that $P(z) = Q(w(z))$ $(z \in \triangle)$. From the definition of the subordinations, it is easy to show that the subordination (1) implies that

$$P(0) = Q(0) \quad \text{and} \quad P(\triangle) \subset Q(\triangle). \tag{2}$$

In particular, if $Q(z)$ is univalent in $\triangle$, then the subordination (1) is equivalent to the condition (2).

Motivated by the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{M}(\beta)$, we define a new class for certain $p-$valent functions.

Definition 1. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{S}^p(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < \Re \left(\frac{zf\ (z)}{pf(z)} \right) < \beta \qquad (z \in \triangle, p \in \mathbb{N}). \tag{3}$$

Definition 2. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{C}^p_\lambda(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < \Re \left\{ \frac{1}{p} \left(1 + \frac{zf\ \ (z)}{f\ (z)} \right) \right\} < \beta \qquad (z \in \triangle, \lambda \geq 0). \tag{4}$$

Definition 3. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{M}_\lambda^p(\alpha,\beta)$ if f satisfies the following inequality

$$\alpha < \Re\left\{\frac{(1-\lambda)}{p}\frac{zf'(z)}{f(z)} + \frac{\lambda}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} < \beta \quad (z \in \triangle, \lambda \geq 0, p \in \mathbb{N}). \tag{5}$$

Remark 4. When $p = 1, \mathcal{M}_\lambda^p(\alpha,\beta)$ reduces to $\mathcal{M}_\lambda(\alpha,\beta)$, the class of Mocanu-convex functions with bounded positive real part. Further we note that $\mathcal{M}_0^p(\alpha,\beta) = \mathcal{S}^p(\alpha,\beta)$ [1], $\mathcal{M}_0^1(\alpha,\beta) = \mathcal{S}(\alpha,\beta)$ [3], $\mathcal{M}_1^p(\alpha,\beta) = \mathcal{C}^p(\alpha,\beta)$, the class of $p-$valent convex functions with bounded positive real part and $\mathcal{M}_1^1(\alpha,\beta) = \mathcal{C}(\alpha,\beta)$, the class of convex functions with bounded positive real part.

Definition 5. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{N}_\lambda^p(\alpha,\beta)$ if f satisfies the following inequality

$$\alpha < \Re\left\{\frac{(1-\lambda)\frac{1}{p}zf'(z) + \lambda\frac{1}{p^2}z\left(zf'(z)\right)'}{(1-\lambda)f(z)+\lambda\frac{1}{p}zf'(z)}\right\} < \beta \quad (z \in \triangle, \lambda \geq 0, p \in \mathbb{N}). \tag{6}$$

Remark 6. When $p = 1, \mathcal{N}_\lambda^p(\alpha,\beta)$ reduces to $\mathcal{N}_\lambda(\alpha,\beta)$, the class of Pascu-type functions with positive real part. Further we note that $\mathcal{N}_0^p(\alpha,\beta) = \mathcal{S}^p(\alpha,\beta)$ [1], $\mathcal{N}_0^1(\alpha,\beta) = \mathcal{S}(\alpha,\beta)$ [3], $\mathcal{N}_1^p(\alpha,\beta) = \mathcal{C}^p(\alpha,\beta)$ and $\mathcal{N}_1^1(\alpha,\beta) = \mathcal{C}(\alpha,\beta)$.

Now, we define an analytic function $\mathcal{S}_{\alpha,\beta}(z) : \triangle \longrightarrow \mathbb{C}$ by

$$\mathcal{S}_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi}\, i \log\left(\frac{1-e^{\frac{i\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha}}z}\right) \tag{7}$$

due to Kuroki and Owa [3] and they proved $\mathcal{S}_{\alpha,\beta}(z)$ maps $\triangle$ onto a convex domain w with $\alpha < \Re(w) < \beta$, conformaly.

2. Coefficient Estimates for $f \in \mathcal{M}_\lambda^p(\alpha,\beta)$

Applying the function $\mathcal{S}_{\alpha,\beta}(z)$ defined by (7), we give a necessary and sufficient condition for $f(z) \in \mathcal{A}_p$ to belong to the class $\mathcal{M}_\lambda^p(\alpha,\beta)$.

Lemma 7. *Let $f(z) \in \mathcal{A}_p$ and $0 \leq \alpha < 1 < \beta$. Then $f(z) \in \mathcal{M}_\lambda^p(\alpha, \beta)$ if and only if*

$$\frac{(1-\lambda)}{p}\frac{zf\ (z)}{f(z)} + \frac{\lambda}{p}\left(1 + \frac{zf\ \ (z)}{f\ (z)}\right) \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right) \tag{8}$$

in $\triangle$.

By taking $\lambda = 0$ and $\lambda = 1$ we state the following lemmas respectively:

Lemma 8. *[1] Let $f \in \mathcal{A}_p$. Then $f(z) \in \mathcal{S}^p(\alpha, \beta)$ if and only if*

$$\frac{zf\ (z)}{pf(z)} \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right) \qquad (z \in \triangle) \tag{9}$$

where $\alpha < 1$ and $\beta > 1$.

Lemma 9. *Let $f \in \mathcal{A}_p$. Then $f(z) \in \mathcal{C}^p(\alpha, \beta)$ if and only if*

$$\frac{1}{p}\left(1 + \frac{zf\ \ (z)}{f\ (z)}\right) \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right) \qquad (z \in \triangle) \tag{10}$$

where $\alpha < 1$ and $\beta > 1$.

We note that

$$\mathcal{S}_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi}\, i \log\left(\frac{1 - e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}{1 - e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}} z}\right) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{2(\beta-\alpha)}{n\pi} \sin \frac{n\pi(1-\alpha)}{\beta-\alpha} \qquad (n = 1, 2, 3, \ldots). \tag{11}$$

Using the subordination (8), we find sharp bounds on the second and third coefficients for $f(z) \in \mathcal{M}_\lambda^p(\alpha, \beta)$, by applying the following lemma due to Rogosinki [5].

Lemma 10. *Let $P(z) = \sum\limits_{n=1}^{\infty} A_n z^n$ and $Q(z) = \sum\limits_{n=1}^{\infty} B_n z^n$ be analytic in $\triangle$. If $P(z) \prec Q(z)$ $(z \in \triangle)$, then*

$$\sum_{k=1}^{m} |A_k|^2 \leq \sum_{k=1}^{m} |B_k|^2 \qquad (m = 1, 2, \ldots).$$

Theorem 11. *If the function* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{M}_{\lambda}^{p}(\alpha, \beta)$, *then*

$$|a_{p+1}| \leq \frac{2p^2(\beta-\alpha)}{\pi(p+\lambda)} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}$$

and

$$|a_{p+2}| \leq \frac{2p^2(\beta-\alpha)}{\pi(p+\lambda)} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \times \left(\cos \frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p(p^2+2p\lambda+\lambda)(\beta-\alpha)}{\pi(p+\lambda)^2} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \right).$$

Proof. Let

$$\begin{aligned} P(z) &= \frac{(1-\lambda)}{p} \frac{zf'(z)}{f(z)} + \frac{\lambda}{p} \left(1 + \frac{zf''(z)}{f'(z)} \right) \\ &= 1 + \frac{(1+p\lambda)}{p} a_{p+1} z + \left[\frac{2(p+2\lambda)}{p^2} a_{p+2} - \frac{(p^2+2p\lambda+\lambda)}{p^3} a_{p+1}^2 \right] z^2 + \cdots \end{aligned}$$

and

$$Q(z) = \mathcal{S}_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{12}$$

where B_n is as in (11). Applying Lemma 10 we can get the results as asserted. □

When $\lambda = 0$ and $\lambda = 1$ we state the following corollaries respectively:

Corollary 12. *[1] If the function* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{S}^p(\alpha, \beta)$, *then*

$$|a_{p+1}| \leq \frac{2p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}$$

and

$$|a_{p+2}| \leq \frac{p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \left(\cos \frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \right).$$

Corollary 13. *If the function* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{C}^p(\alpha, \beta)$, *then*

$$|a_{p+1}| \leq \frac{2p^2(\beta-\alpha)}{\pi(p+1)} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}$$

and

$$|a_{p+2}| \leq \frac{p^2(\beta-\alpha)}{\pi(p+2)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left[\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right].$$

When $p = 1$, from Theorem 11, we state the following corollary:

Corollary 14. *If the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{M}_\lambda(\alpha, \beta)$, *then*

$$|a_2| \leq \frac{2(\beta-\alpha)}{\pi(1+\lambda)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

and

$$|a_3| \leq \frac{(\beta-\alpha)}{\pi(1+2\lambda)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left(\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2(1+3\lambda)(\beta-\alpha)}{\pi(1+\lambda)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right).$$

Making use of the following lemma we shall solve the Fekete-Szegö problem for $f(z) \in \mathcal{M}_\lambda^p(\alpha, \beta)$.

Lemma 15. *[2]Let* $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$ *be a function with positive real part in* $\triangle$. *Then for any complex number* ν,

$$|h_2 - \nu h_1^2| \leq 2\max\{1, |1-2\nu|\}.$$

Theorem 16. *Let* $0 \leq \alpha < 1 < \beta$ *and let the function* f *given by* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ *be in the class* $\mathcal{M}_\lambda^p(\alpha, \beta)$. *Then for any complex number* μ,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^2(\beta-\alpha)}{\pi(p+2\lambda)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$
$$\times\max\left\{1, \left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p[(p^2+2p\lambda+\lambda) - 2\mu p(p+2\lambda)](\beta-\alpha)}{\pi(p+\lambda)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

Proof. Let $P(z) = \frac{(1-\lambda)}{p}\frac{zf'(z)}{f(z)} + \frac{\lambda}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)$. Then, since $f \in \mathcal{M}_\lambda^p(\alpha, \beta)$, we have $P(z) \prec Q(z)$, where $Q(z)$ is given by (12).

Let

$$h(z) = \frac{1 + Q^{-1}(P(z))}{1 - Q^{-1}(P(z))} = 1 + h_1 z + h_2 z^2 + \cdots \qquad (z \in \triangle).$$

Then h is analytic and has positive real part in the open disk $\triangle$. We also have

$$P(z) = Q\left(\frac{h(z)-1}{h(z)+1}\right) \qquad (z \in \triangle). \tag{13}$$

We find from the equation (13) that

$$a_{p+1} = \frac{p^2 B_1 h_1}{(p+\lambda)}$$
$$a_{p+2} = \frac{p^2}{2(p+2\lambda)}\left[\frac{B_2 h_1^2}{4} - \frac{B_1 h_1^2}{4} + \frac{B_1 h_2}{2} + \frac{p(p^2+2p\lambda+\lambda)}{(p+\lambda)^2} B_1^2 h_1^2\right]$$

which imply that

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p^2 B_1}{4(p+2\lambda)}(h_2 - \nu h_1^2),$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{p(p^2+2p\lambda+\lambda)B_1}{(p+\lambda)^2} + \frac{2\mu p^2(p+2\lambda)B_1}{(p+\lambda)^2}\right).$$

Applying Lemma 15, we obtain

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &= \frac{p^2}{4(p+2\lambda)}|B_1||h_2 - \nu h_1^2| \\ &\le \frac{p^2}{2(p+2\lambda)} B_1 \max\{1, |1-2\nu|\}. \end{aligned} \tag{14}$$

Substituting $B_1 = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$ and $B_2 = \frac{(\beta-\alpha)}{\pi}\sin\frac{2\pi(1-\alpha)}{\beta-\alpha}$ in (14), we can obtain the results as asserted. □

By taking $\lambda = 0$ and $\lambda = 1$ we state the following corollaries respectively:

Corollary 17. *[1] Let $0 \le \alpha < 1 < \beta$ and let the function f given by $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ be in the class $\mathcal{S}^p(\alpha, \beta)$. Then for any complex number μ,*

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha} \times \max\left\{1, \left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p(1-2\mu)(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

Corollary 18. *Let $0 \le \alpha < 1 < \beta$ and let the function f given by $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ be in the class $\mathcal{C}^p(\alpha, \beta)$. Then for any complex number μ,*

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{p^2(\beta-\alpha)}{\pi(p+2)}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$
$$\times \max\left\{1, \left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha}+\frac{2p[(p+1)^2-2\mu p(p+2)](\beta-\alpha)}{\pi(p+1)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

Putting $p=1$ in Theorem 16, we get the following corollary.

Corollary 19. *Let $0 \le \alpha < 1 < \beta$ and let the function f given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{M}_\lambda(\alpha,\beta)$. Then for any complex number μ,*

$$|a_3-\mu a_2^2| \le \frac{(\beta-\alpha)}{\pi(1+2\lambda)}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$
$$\times \max\left\{1, \left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha}+\frac{2[(1+3\lambda)-2\mu(1+2\lambda)](\beta-\alpha)}{\pi(1+\lambda)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

3. Coefficient Estimates for $f \in \mathcal{N}_\lambda^p(\alpha,\beta)$

First, by applying the function $\mathcal{S}_{\alpha,\beta}(z)$ defined by (7), we give a necessary and sufficient condition for $f(z) \in \mathcal{A}_p$ to belong to the class $\mathcal{N}_\lambda^p(\alpha,\beta)$.

Lemma 20. *Let $f(z) \in \mathcal{A}_p$ and $0 \le \alpha < 1 < \beta$. Then $f(z) \in \mathcal{N}_\lambda^p(\alpha,\beta)$ if and only if*

$$\frac{(1-\lambda)\frac{1}{p}zf\ (z)+\lambda\frac{1}{p^2}\,z\,(zf\ (z))}{(1-\lambda)f(z)+\lambda\frac{1}{p}zf\ (z)} \prec 1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}{1-e^{-i\frac{\pi(1-\alpha)}{\beta-\alpha}}z}\right) \qquad (15)$$

in $\triangle$.

Remark 21. For $\lambda=0$ and $\lambda=1$, we get Lemma 8 and Lemma 9 respectively.

Using the subordination (15), we find sharp bounds on the second and third coefficients for $f(z) \in \mathcal{N}_\lambda^p(\alpha,\beta)$, by applying Lemma 10.

Theorem 22. *If the function $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{N}_\lambda^p(\alpha,\beta)$, then*

$$|a_{p+1}| \le \frac{2p^2(\beta-\alpha)}{\pi(p+\lambda)}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

and

$$|a_{p+2}| \leq \frac{p^2(\beta-\alpha)}{\pi(p+2\lambda)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left(\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2p(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right).$$

Proof. Let

$$\begin{aligned} P(z) &= \frac{(1-\lambda)\frac{1}{p}zf\ (z)+\lambda\frac{1}{p^2}\,z\,(zf\ (z))}{(1-\lambda)f(z)+\lambda\frac{1}{p}zf\ (z)} \\ &= 1+\frac{(p+\lambda)}{p}a_{p+1}z+\left[\frac{2(p+2\lambda)}{p^2}a_{p+2}-\frac{(p+\lambda)^2}{p^3}a_{p+1}^2\right]z^2+\cdots \end{aligned} \tag{16}$$

and

$$Q(z) = \mathcal{S}_{\alpha,\beta}(z) = 1+\sum_{n=1}^{\infty} B_n z^n, \tag{17}$$

where B_n is as in (11). Applying Lemma 10 we can get the results as asserted. □

Remark 23. For $\lambda = 0$ and $\lambda = 1$, we get Corollary 12 and Corollary 13 respectively.

When $p = 1$, from Theorem 22, we state the following corollary:

Corollary 24. *If the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{N}_\lambda(\alpha,\beta)$, *then*

$$|a_2| \leq \frac{2(\beta-\alpha)}{\pi(1+\lambda)}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

and

$$|a_3| \leq \frac{(\beta-\alpha)}{\pi(1+2\lambda)} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}\left(\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2(\beta-\lambda)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right).$$

Again using Lemma 15 we shall solve the Fekete-Szegö problem for $f(z) \in \mathcal{N}_\lambda^p(\alpha,\beta)$.

Theorem 25. *Let* $0 \leq \alpha < 1 < \beta$ *and let the function* f *given by* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ *be in the class* $\mathcal{N}_\lambda^p(\alpha,\beta)$. *Then for any complex number* μ,

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{p^2(\beta-\alpha)}{\pi(p+2\lambda)}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

$$\times\max\left\{1,\left|\cos\frac{\pi(1-\alpha)}{\beta-\alpha}+\frac{2p[(p+\lambda)^2-2\mu p(p+2\lambda)](\beta-\alpha)}{\pi(p+\lambda)^2}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right|\right\}.$$

Proof. Let

$$P(z) = \frac{(1-\lambda)\frac{1}{p}zf\,(z) + \lambda\frac{1}{p^2}\,z\,(zf\,(z))}{(1-\lambda)f(z) + \lambda\frac{1}{p}zf\,(z)}.$$

Then, since $f \in \mathcal{N}_\lambda^p(\alpha,\beta)$, we have $P(z) \prec Q(z)$, where $Q(z)$ is given by (17).

Let

$$h(z) = \frac{1+Q^{-1}(P(z))}{1-Q^{-1}(P(z))} = 1 + h_1 z + h_2 z^2 + \cdots \qquad (z \in \triangle).$$

Then h is analytic and has positive real part in the open disk $\triangle$. We also have

$$P(z) = Q\left(\frac{h(z)-1}{h(z)+1}\right) \qquad (z \in \triangle). \tag{18}$$

We find from the equation (18) that

$$\begin{aligned} a_{p+1} &= \frac{p^2 B_1 h_1}{2(p+\lambda)} \\ a_{p+2} &= \frac{p^2}{2(p+2\lambda)}\left[\frac{B_2 h_1^2}{4} - \frac{B_1 h_1^2}{4} + \frac{B_1 h_2}{2} + \frac{pB_1^2 h_1^2}{4}\right] \end{aligned}$$

which imply that

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p^2 B_1}{4(p+2\lambda)}(h_2 - \nu h_1^2),$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{p[(p+\lambda)^2 - 2\mu p(p+2\lambda)]B_1}{(p+\lambda)^2}\right).$$

Applying Lemma 15, we obtain

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &= \frac{p^2}{4(p+2\lambda)}|B_1||h_2 - \nu h_1^2| \\ &\le \frac{p^2}{2(p+2\lambda)} B_1 \max\{1, |1-2\nu|\}. \end{aligned} \tag{19}$$

Substituting $B_1 = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$ and $B_2 = \frac{(\beta-\alpha)}{\pi}\sin\frac{2\pi(1-\alpha)}{\beta-\alpha}$ in (19), we can obtain the results as asserted. □

Remark 26. By taking $\lambda = 0$ and $\lambda = 1$ we get Corollary 17 and Corollary 18 respectively.

Putting $p = 1$ in Theorem 25, we get the following corollary.

Corollary 27. *Let $0 \leq \alpha < 1 < \beta$ and let the function f given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{N}_\lambda(\alpha, \beta)$. Then for any complex number μ,*

$$|a_3 - \mu a_2^2| \leq \frac{(\beta - \alpha)}{\pi(1 + 2\lambda)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$
$$\times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} + \frac{2[(1 + \lambda)^2 - 2\mu(1 + 2\lambda)](\beta - \alpha)}{\pi(1 + \lambda)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.$$

Remark 28. By taking $\lambda = 0$ and $\lambda = 1$ one can deduce the results for functions $f \in \mathcal{S}_\lambda^p(\alpha, \beta)$ [1] and $f \in \mathcal{C}^p(\alpha, \beta)$ respectively. In particular $\lambda = 0, p = 1$ leads to the results obtained in [3].

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