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COMMON FIXED POINT AND BEST APPROXIMATION FOR BANACH OPERATOR PAIRS IN NON-STARSHAPED DOMAIN

Abstract. Common fixed point results for Banach operator pair with generalized nonexpansive mappings in non-starshaped domain of metric space have been obtained in the present work. As application, more general best approximation results in normed space have also been determined. These results extend and generalize various existing known results with the aid of Banach operator pair and without starshaped condition of domain.

1. Introduction

Fixed point theorems have been applied in the field of invariant approximation theory since last four decades and several interesting and valuable results have been studied.

Meinardus [17] was the one to employ a fixed-point theorem of Schauder to establish the existence of an invariant approximation. Further, Brosowski [2] obtained a celebrated result and generalized the Meinardus's result. Later, several results [7, 25, 27] have been proved in the direction of Brosowski [2]. In the year 1988, Sahab, Khan and Sessa [22] extended the result of Hicks and Humpheries [7] and Singh [25] by using two mappings, one linear and the other nonexpansive mappings for commuting mappings.

Al-Thagafi [1] extended result of Sahab et al. [22] and proved some results on invariant approximations for commuting mappings. The introduction of non-commuting maps to this area, Shahzad [23, 24] further extended Al-Thagafi's results and obtained some results regarding invariant approximation. Afterwards, numbers of results by changing the nature of mappings for convex domain within various space structures appeared. Main contributors in this direction are Shahzad [23], Hussain et al. [9], Jungck and Hussain

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[12] and O'Regan and Hussain [21] for $\mathcal{R}$-subweakly commuting, compatible and $\mathcal{C}_q$-commuting maps. All the above mentioned results are obtained on starshaped domain and linearity or affinness condition of mappings.

Recently, Chen and Li [3] introduced the notion of Banach operator pair as a new class of noncommuting maps. Using this concept, common fixed-point theorems are obtained without the assumption of linearity or affinity of mappings and which is further applied to prove best approximation results.

In this context, it may be mentioned that Dotson [4] proved the existence of fixed point for nonexpansive mapping. He further extended his result without starshapedness under non-convex condition [5]. In a paper, Khan and Khan [14] extended a fixed point theorem of Dotson [5] and generalized an invariant approximation result of Smoluk [26] in the setting of q-normed space. Further, Khan et al. [14] extended the results of Khan and Khan [14] and generalized the result of Singh [25] by using the concept of nonconvexity of Dotson [5]. Again, Khan et al. [16] proved some results on invariant approximations for commuting mappings in non-starshaped set of q-normed space and extended and generalized the results of Al-Thagafi [1], Habiniak [6], Khan et al. [14], Sahab et al. [22] and Singh [25]. The validity of results of Khan et al. [16] is shown by Nashine [18, 19, 20] for noncommuting nonlinear generalized nonexpansive mappings.

The purpose of this paper is to show the validity of results of Chen and Li [3] for generalized $\mathcal{I}$-nonexpansive maps without starshaped condition of domain. Also, some more general approximation results have been determined as application of common fixed point theorem; incidently, the results of Al-Thagafi [1], Dotson [5], Habiniak [6], Jungck and Hussain [12], Khan and Khan [14], Khan et al. [15], Khan et al. [16], Sahab et al. [22], Nashine [18, 19, 20], O'Regan and Hussain [21] and Shahzad [23, 24] have been also extended.

2. Preliminaries

In the material to be produced here, the following definitions have been used:

Let $\mathcal{M}$ be a subset of a normed space $(\mathcal{X}, \|, \|)$. The set $\mathcal{P}_{\mathcal{M}}(\widehat{x}) = \{x \in \mathcal{M} : \|x - \widehat{x}\| = dist(\widehat{x}, \mathcal{M})\}$ is called the set of best approximants to $\widehat{x} \in \mathcal{X}$ out of $\mathcal{M}$, where $dist(\widehat{x}, \mathcal{M}) = inf\{\|y - \widehat{x}\| : y \in \mathcal{M}\}$. We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(\mathcal{M})$ to denote the closure of a set $\mathcal{M}$ and $wcl(\mathcal{M})$ to denote the weak closure of a set $\mathcal{M}$. Let $\mathcal{I} : \mathcal{M} \to \mathcal{M}$ be a mapping. A mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is called an $\mathcal{I}$-contraction if, for any $x, y \in \mathcal{M}$, there exists $0 \leq k < 1$ such that $\|\mathcal{T}x - \mathcal{T}y\| \leq k\|\mathcal{I}x - \mathcal{I}y\|$. If $k = 1$, then $\mathcal{T}$ is called $\mathcal{I}$-nonexpansive. The set of fixed points of $\mathcal{T}$ (resp. f) is denoted by $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{I})$). A point $x \in \mathcal{M}$ is a coincidence point (common fixed point) of $\mathcal{I}$ and $\mathcal{T}$ if $\mathcal{I}x = \mathcal{T}x$ ($x = \mathcal{I}x = \mathcal{T}x$). The set of coincidence points of $\mathcal{I}$ and $\mathcal{T}$ is denoted by $\mathcal{C}(\mathcal{I}, \mathcal{T})$. The pair $(\mathcal{I}, \mathcal{T})$ is called (1) commuting if $\mathcal{T}\mathcal{I}x = \mathcal{I}\mathcal{T}x$ for all $x \in \mathcal{M}$, (2) $\mathcal{R}$-weakly commuting [23] if for all $x \in \mathcal{M}$, there exists $\mathcal{R} > 0$ such that $\|\mathcal{I}\mathcal{T}x - \mathcal{T}\mathcal{I}x\| \leq \mathcal{R}\|\mathcal{I}x - \mathcal{I}x\|$. If $\mathcal{R} = 1$, then the maps are called weakly commuting; (3) compatible [11] if $\lim_n \|\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n\| = 0$ when $\{x_n\}$ is a sequence such that $\lim_n \mathcal{T}x_n = \lim_n \mathcal{I}x_n = t$ for some t in $\mathcal{M}$; (4) weakly compatible if they commute at their coincidence points, i.e., if $\mathcal{I}\mathcal{T}x = \mathcal{T}\mathcal{I}x$ whenever $\mathcal{I}x = \mathcal{T}x$. The set $\mathcal{M}$ is called p-starshaped with $p \in \mathcal{M}$, if the segment $[p, x] = \{(1-k)p + kx : 0 \leq k \leq 1\}$ joining p to x, is contained in $\mathcal{M}$ for all $x \in \mathcal{M}$. Suppose that $\mathcal{M}$ is p-starshaped with $p \in \mathcal{F}(\mathcal{I})$ and is both $\mathcal{T}$- and $\mathcal{I}$-invariant. Then $\mathcal{T}$ and $\mathcal{I}$ are called (5) $\mathcal{R}$-subweakly commuting on $\mathcal{M}$ (see [23]) if for all $x \in \mathcal{M}$, there exists a real number $\mathcal{R} > 0$ such that $\|\mathcal{I}\mathcal{T}x - \mathcal{T}\mathcal{I}x\| \leq \mathcal{R}dist(\mathcal{I}x, [p, \mathcal{T}x])$. It is well known that $\mathcal{R}$-subweakly commuting maps are $\mathcal{R}$-weakly commuting and $\mathcal{R}$-weakly commuting maps are compatible but not conversely in general (see for examples [23, 24]).

Further, definition providing the notion of Banach operator pair introduced by Chen and Li [3] may be written as:

Definition 2.1. Banach Operator Pair. The ordered pair $(\mathcal{T}, \mathcal{I})$ of two self-maps of a metric space $(\mathcal{X}, d)$ is called a Banach operator pair, if the set $\mathcal{F}(\mathcal{I})$ is $\mathcal{T}$-invariant, namely $\mathcal{T}(\mathcal{F}(\mathcal{I})) \subseteq \mathcal{F}(\mathcal{I})$. Obviously commuting pair $(\mathcal{T}, \mathcal{I})$ is Banach operator pair but not conversely in general, see [3]. If $(\mathcal{T}, \mathcal{I})$ is Banach operator pair then $(\mathcal{I}, \mathcal{T})$ need not be Banach operator pair (see [3, Example 1]).

If the self-maps $\mathcal{T}$ and $\mathcal{I}$ of $\mathcal{X}$ satisfy

$$d(\mathcal{I}\mathcal{T}x, \mathcal{T}x) \leq kd(\mathcal{I}x, x) \tag{2.1}$$

for all $x \in \mathcal{X}$ and $k \geq 0$, then $(\mathcal{T}, \mathcal{I})$ is Banach operator pair. In particular, when $\mathcal{I} = \mathcal{T}$ and $\mathcal{X}$ is a normed space, (2.1) can be rewritten as

$$\|\mathcal{T}^2x - \mathcal{T}x\| \leq k\|\mathcal{T}x - x\| \tag{2.2}$$

for all $x \in \mathcal{X}$. Such $\mathcal{T}$ is called Banach operator of type k in [27].

Further, definition providing the notion of contractive jointly continuous family introduced by Dotson [5] may be written as:

Definition 2.2. [5] Let $\mathcal{M}$ be a subset of metric space $(\mathcal{X}, d)$ and $\Delta = \{f_\alpha\}_{\alpha \in \mathcal{M}}$ a family of functions from $[0, 1]$ into $\mathcal{M}$ such that $f_\alpha(1) = \alpha$ for each $\alpha \in \mathcal{M}$. The family Δ is said to be *contractive* if whenever there exists a function $\phi : (0, 1) \to (0, 1)$ such that for all $\alpha, \beta \in \mathcal{M}$ and all $t \in (0, 1)$ we have

$$d(f_\alpha(t), f_\beta(t)) \leq \phi(t)d(\alpha, \beta).$$

The family is said to be *jointly continuous* if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in $\mathcal{M}$ imply that $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in $\mathcal{M}$.

Definition 2.3. [5] If $\mathcal{X}$ is a normed linear space and Δ is a family as in Definition 2.2, then Δ is said to be *jointly weakly continuous* if $t \to t_0$ in $[0,1]$ and $\alpha \to^w \alpha_0$ in $\mathcal{M}$ imply that $f_\alpha(t) \to^w f_{\alpha_0}(t_0))$ in $\mathcal{M}$.

Hence, *property* (Γ) on contractive jointly continuous family Δ can now be defined as:

Definition 2.4. Let $\mathcal{T}$ be a selfmap of the set $\mathcal{M}$ having a family of functions $\Delta = \{f_x\}_{x\in\mathcal{M}}$ as defined above. Then $\mathcal{T}$ is said to satisfy the property (Γ), if $\mathcal{T}(f_x(t)) = f_{\mathcal{T}x}(t)$, for all $x \in \mathcal{M}$ and $t \in [0,1]$,

The following result would also be used in the sequel.

Theorem 2.5. [21, Corollary 2.2] *Let $\mathcal{M}$ be a nonempty closed subset of a metric space $(\mathcal{X}, d)$, and $\mathcal{T}$ and $\mathcal{I}$ be self-maps of $\mathcal{M}$. Assume that $cl\mathcal{T}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$, $cl\mathcal{T}(\mathcal{M})$ is complete, $\mathcal{T}$ is $\mathcal{I}$-continuous and $\mathcal{T}$ and $\mathcal{I}$ satisfy for all $x, y \in \mathcal{M}$ and $0 \le h < 1$,*

$$d(\mathcal{T}x,\mathcal{T}y) \le h max\{d(\mathcal{I}x,\mathcal{I}y), d(\mathcal{T}x,\mathcal{I}x), d(\mathcal{T}y,\mathcal{I}y), d(\mathcal{T}x,\mathcal{I}y), d(\mathcal{T}y,\mathcal{I}x)\}. \tag{2.3}$$

Then $\mathcal{C}(\mathcal{T},\mathcal{I}) \ne \emptyset$.

3. Main results

Before proving the main results, a lemma is presented below, which extends and improves Lemma 3.1 of [3]:

Lemma 3.1. *Let $\mathcal{M}$ be a nonempty closed subset of a metric space $(\mathcal{X}, d)$, and $(\mathcal{T},\mathcal{I})$ be Banach operator pair on $\mathcal{M}$. Assume that $cl\mathcal{T}(\mathcal{M})$ is complete, and $\mathcal{T}$ and $\mathcal{I}$ satisfy for all $x, y \in \mathcal{M}$ and $0 \le h < 1$,*

$$d(\mathcal{T}x,\mathcal{T}y) \le h max\{d(\mathcal{I}x,\mathcal{I}y), d(\mathcal{T}x,\mathcal{I}x), d(\mathcal{T}y,\mathcal{I}y), d(\mathcal{T}x,\mathcal{I}y), d(\mathcal{T}y,\mathcal{I}x)\}. \tag{3.1}$$

If $\mathcal{I}$ is continuous, $\mathcal{F}(\mathcal{I})$ is nonempty and $\mathcal{T}$ is $\mathcal{I}$-continuous, then there is unique common fixed point of $\mathcal{T}$ and $\mathcal{I}$.

Proof. According to assumptions, $\mathcal{T}(\mathcal{F}(\mathcal{I})) \subseteq \mathcal{F}(\mathcal{I})$ and $\mathcal{F}(\mathcal{I})$ is nonempty closed and $cl\mathcal{T}(\mathcal{F}(\mathcal{I})) \subseteq cl\mathcal{T}(\mathcal{M})$ is complete. Also (3.1) implies that

$$\begin{aligned} d(\mathcal{T}x,\mathcal{T}y) &\le h\max\{d(\mathcal{I}x,\mathcal{I}y), d(\mathcal{I}x,\mathcal{T}x), d(\mathcal{I}y,\mathcal{T}y), d(\mathcal{T}y,\mathcal{I}x), d(\mathcal{T}x,\mathcal{I}y)\} \\ &= h\max\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), d(\mathcal{T}y,x), d(\mathcal{T}x,y)\} \end{aligned}$$

for all $x,\ y \in \mathcal{F}(\mathcal{I})$. Hence $\mathcal{T}$ is generalized contraction on $\mathcal{F}(\mathcal{I})$ and $cl\mathcal{T}(\mathcal{F}(\mathcal{I})) \subseteq cl\mathcal{F}(\mathcal{I}) = \mathcal{F}(\mathcal{I})$. Thus, Theorem 2.5 guarantees that, $\mathcal{T}$ has a unique fixed point w in $\mathcal{F}(\mathcal{I})$ and consequently $\mathcal{F}(\mathcal{T},\mathcal{I})$ is singleton. ∎

Theorem 3.2. *Let $\mathcal{M}$ be a nonempty closed subset of a metric space $(\mathcal{X}, d)$ and $\mathcal{T}$ and $\mathcal{I}$ be self-maps of $\mathcal{M}$. Suppose that $\mathcal{M}$ admits a contractive and jointly continuous family $\Delta = \{f_\alpha\}_{\alpha\in\mathcal{M}}$, $\mathcal{I}$ is continuous and $\mathcal{T}$ is $\mathcal{I}$-continuous. If $(\mathcal{T}, \mathcal{I})$ is Banach operator pair and satisfies, for all $x, y \in \mathcal{M}$,*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \max\{d(\mathcal{I}x, \mathcal{I}y), dist(\mathcal{I}x, f_{\mathcal{T}x}(k)), dist(\mathcal{I}y, f_{\mathcal{T}y}(k)), dist(\mathcal{I}x, f_{\mathcal{T}y}(k)), dist(\mathcal{I}y, f_{\mathcal{T}x}(k))\}, \tag{3.2}$$

where $0 < k < 1$. If $cl\mathcal{T}(\mathcal{M})$ is compact, then $\mathcal{M} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$.

Proof. Choose a sequence $k_n \in (0, 1)$ with $k_n \to 1$ as $n \to \infty$, and define, for each $n \in \mathbb{N}$, the mapping

$$\mathcal{T}_n x = f_{\mathcal{T}x}(k_n).$$

Each $\mathcal{T}_n$ maps $\mathcal{M}$ into itself. Again by (3.2),

$$\begin{aligned} d(\mathcal{T}_n x, \mathcal{T}_n y) &= d(f_{\mathcal{T}x}(k_n), f_{\mathcal{T}y}(k_n)) \leq \phi(k_n) d(\mathcal{T}x, \mathcal{T}y) \\ &\leq \phi(k_n) \max\{d(\mathcal{I}x, \mathcal{I}y), dist(\mathcal{I}x, f_{\mathcal{T}x}(k_n)), dist(\mathcal{I}y, f_{\mathcal{T}y}(k_n)), \\ &\qquad dist(\mathcal{I}x, f_{\mathcal{T}y}(k_n)), dist(\mathcal{I}y, f_{\mathcal{T}x}(k_n))\} \\ &\leq \phi(k_n)\ max\{d(\mathcal{I}x, \mathcal{I}y), d(\mathcal{I}x, \mathcal{T}_n x), d(\mathcal{I}y, \mathcal{T}_n y), d(\mathcal{I}x, \mathcal{T}_n y), \\ &\qquad d(\mathcal{I}y, \mathcal{T}_n x)\}, \end{aligned}$$

for each $x, y \in \mathcal{M}$ and $0 < k_n < 1$. Since $(\mathcal{T}, \mathcal{I})$ is Banach operator pair, for $x \in \mathcal{F}(\mathcal{I})$, we have $\mathcal{T}x \in \mathcal{F}(\mathcal{I})$, and hence $\mathcal{T}_n x = f_{\mathcal{T}x}(k_n) \in \mathcal{F}(\mathcal{I})$. Thus $(\mathcal{T}_n, \mathcal{I})$ is Banach operator pair on $\mathcal{M}$ for each n.

As $cl\mathcal{T}(\mathcal{M})$ is compact, for each $n \in \mathbb{N}$, $cl\mathcal{T}_n(\mathcal{M})$ is compact and hence complete. By Lemma 3.1, for each $n \geq 1$, there exists $y_n \in \mathcal{M}$ such that $y_n = \mathcal{I}y_n = \mathcal{T}_n y_n$. The compactness of $cl(\mathcal{T}(\mathcal{M}))$ implies that there exists a subsequence $\{\mathcal{T}y_m\}$ of $\{\mathcal{T}y_n\}$ such that $\mathcal{T}y_m \to z \in cl(\mathcal{T}(\mathcal{M}))$ as $m \to \infty$. Since $k_m \to 1$, $y_m = \mathcal{T}_m y_m = f_{\mathcal{T}x}(k_m) \to z$. As $\mathcal{I}$ is continuous, then $\mathcal{I}y_m$ converges to y and hence $y = \mathcal{I}y$. The $\mathcal{I}$ continuity of $\mathcal{T}$ implies that $\mathcal{T}y_m$ converges to $\mathcal{T}y$. Consequently, $y = \mathcal{T}y = \mathcal{I}y$. Thus $\mathcal{M} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$. ∎

Theorem 3.3. *Let $\mathcal{M}$ be a nonempty closed subset of a Banach space $\mathcal{X}$ and $\mathcal{T}$ and $\mathcal{I}$ be self-maps of $\mathcal{M}$. Suppose that $\mathcal{M}$ has a contractive family of functions $\Delta = \{f_\alpha\}_{\alpha\in\mathcal{M}}$, $\mathcal{I}$ is continuous and $\mathcal{T}$ is $\mathcal{I}$-continuous. If $(\mathcal{T}, \mathcal{I})$ is Banach operator pair and satisfies* (3.2) *of Theorem 3.2(d is the metric induced on $\mathcal{M}$ from $\mathcal{X}$), for all $x, y \in \mathcal{M}$, then $\mathcal{M} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$, provided one of the following conditions holds:*

(i) *$\mathcal{M}$ is weakly compact, $\mathcal{T}$ and $\mathcal{I}$ are weakly continuous and the family Δ is weakly jointly continuous.*
(ii) *$\mathcal{M}$ is weakly compact, $\mathcal{T}$ is completely continuous, and the family Δ is jointly continuous.*

Proof. (i) As in Theorem 3.2, there exists $x_n \in \mathcal{M}$ such that $x_n = \mathcal{T}_n x_n = \mathcal{I}x_n$. Since $\mathcal{M}$ is weakly compact, $\{x_n\}$ contains a convergent subsequence, say, $\{x_m\}$ such that $x_m \to u \in \mathcal{M}$. Since $\mathcal{T}$ is weakly continuous, $\mathcal{T}x_m \to^w \mathcal{T}u$ and hence $x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}u}(1) = \mathcal{T}u$. Also since $x_m \to u$ and the weak topology is Hausdorff, we have $\mathcal{T}u = u$. From the weakly continuity of $\mathcal{I}$ we have $x_m = \mathcal{I}x_m \to \mathcal{I}u$, so that $\mathcal{I}u = u$. Hence $\mathcal{M} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$.

(ii) As in Theorem 3.2, there exists $x_n \in \mathcal{M}$ such that $x_n = \mathcal{T}_n x_n = \mathcal{I}x_n$. Since $\mathcal{M}$ is weakly compact, $\{x_n\}$ contains a convergent subsequence, say, $\{x_m\}$ such that $x_m \to u \in \mathcal{M}$. Since $\mathcal{T}$ is completely continuous, $\mathcal{T}x_m \to \mathcal{T}y$ as $m \to \infty$. Then we have

$$x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}y}(1) = \mathcal{T}y.$$

Thus $\mathcal{T}x_m \to \mathcal{T}^2 y$ and consequently $\mathcal{T}^2 y = \mathcal{T}y$ implies that $\mathcal{T}z = z$, where $z = \mathcal{T}y$. But, since $\mathcal{I}x_m = x_m \to \mathcal{T}y = z$, using the continuity of $\mathcal{I}$ and the uniqueness of the limit, we have $\mathcal{I}z = z$. Hence $\mathcal{M} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$. ∎

Remark 3.4. Theorem 3.2 and Theorem 3.3 extends and improves Theorem 2.2 of [?], Theorems 3.2-3.3 of [3], Theorem 1 and 2 of Dotson [5], Theorem 4 of Habiniak [6] and Theorem 6 of [13] to non-starshaped domain.

From Theorem 3.2, one obtains the following:

Corollary 3.5. *Let $\mathcal{M}$ be a nonempty closed subset of a metric space $(\mathcal{X}, d)$ and $\mathcal{T}$ and $\mathcal{I}$ be self-maps of $\mathcal{M}$. Suppose that $\mathcal{M}$ admits a contractive and jointly continuous family $\Delta = \{f_\alpha\}_{\alpha \in \mathcal{M}}$ and $\mathcal{I}$-continuous. If $(\mathcal{T}, \mathcal{I})$ is Banach operator pair and $\mathcal{T}$ is $\mathcal{I}$-nonexpansive on $\mathcal{M}$. If $cl\mathcal{T}(\mathcal{M})$ is compact, then $\mathcal{M} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$, provided one of the following conditions holds:*

(i) $clf(\mathcal{M})$ is compact,
(ii) $\mathcal{X}$ is Banach space, $\mathcal{M}$ is weakly compact, $\mathcal{I}$ and $\mathcal{T}$ are weakly continuous, family Δ is weakly jointly continuous,
(iii) $\mathcal{X}$ is Banach space, $\mathcal{M}$ is weakly compact, $\mathcal{T}$ is completely continuous, and family Δ is jointly continuous.

Remark 3.6. In the light of the comment given by Dotson [5] and Khan et al. [16] if $\mathcal{M} \subseteq \mathcal{X}$ is $p-$starshaped and $f_\alpha(t) = (1-t)p + t\alpha$, $(\alpha \in \mathcal{M}, t \in [0,1])$, then $\{f_\alpha\}_{\alpha \in \mathcal{M}}$ is a contractive jointly continuous family with $\phi(t) = t$. Thus the class of subsets of $\mathcal{X}$ with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contain the class of convex sets. If for a subset $\mathcal{M}$ of $\mathcal{X}$, there exists a contractive jointly continuous of family $\Delta = \{f_\alpha\}_{\alpha \in \mathcal{M}}$, then we say that $\mathcal{M}$ has the property of contractiveness and joint continuity.

Corollary 3.7. *Let $\mathcal{M}$ be a nonempty closed subset of a metric space $(\mathcal{X}, d)$ which is starshaped with respect to $p \in \mathcal{M}$, and $\mathcal{T}$ and $\mathcal{I}$ be self-maps of $\mathcal{M}$. Suppose that $\mathcal{I}$ is continuous, $\mathcal{F}(\mathcal{I})$ is p-starshaped with $p \in \mathcal{F}(\mathcal{I})$ and $\mathcal{T}$ is $\mathcal{I}$-continuous. If $(\mathcal{T}, \mathcal{I})$ is Banach operator pair and satisfies, for all $x, y \in \mathcal{M}$,*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \max\{\|\mathcal{I}x - \mathcal{I}y\|, dist(\mathcal{I}x, [\mathcal{T}x, p]), dist(\mathcal{I}y, [\mathcal{T}y, p]), dist(\mathcal{I}x, [\mathcal{T}y, p]), dist(\mathcal{I}y, [\mathcal{T}x, p])\}, \tag{3.3}$$

then $\mathcal{M} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$ under each of the conditions of Corollary 3.5.

As application of Theorem 3.2, the following is a more general result in best approximation theory without convexity of $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ and with the aid of Banach operator pair, a generalized class of noncommuting mappings.

Theorem 3.8. *Let $\mathcal{M}$ be subset of a normed space $\mathcal{X}$ and $\mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$ be mappings such that $\widehat{x} \in \mathcal{F}(\mathcal{T}, \mathcal{I})$ for some $\widehat{x} \in \mathcal{X}$ and $\mathcal{T}(\partial\mathcal{M}) \subset \mathcal{M}$. Suppose that $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(\widehat{x})$ is nonempty and has a contractive family $\Delta = \{f_\alpha\}_{\alpha \in \mathcal{D}}$, $\mathcal{I}$ is continuous on $\mathcal{D}$, $\mathcal{I}(\mathcal{D}) = \mathcal{D}$ and $\mathcal{T}$ is $\mathcal{I}$-continuous. If the pair $(\mathcal{T}, \mathcal{I})$ is a Banach operator pair on $\mathcal{D}$ and satisfies*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \begin{cases} \|\mathcal{I}x - \mathcal{I}\widehat{x}\| \;\; if \;\; y = \widehat{x}, \\ max\{\|\mathcal{I}x - \mathcal{I}y\|, \; dist(\mathcal{I}x, f_{\mathcal{T}x}(k)), \; dist(\mathcal{I}y, f_{\mathcal{T}y}(k)), \\ dist(\mathcal{I}x, f_{\mathcal{T}y}(k)), \; dist(\mathcal{I}y, f_{\mathcal{T}x}(k))\}, \;\; if \; y \in \mathcal{D}, \end{cases} \tag{3.4}$$

where $0 < k < 1$, then $\mathcal{D} \cap \mathcal{F}(\mathcal{T}, \mathcal{I}) \neq \emptyset$, provided one of the following conditions is satisfied;

(i) $cl(f(\mathcal{D}))$ is compact,
(ii) $\mathcal{X}$ is Banach space, $\mathcal{D}$ is weakly compact, $\mathcal{I}$ and $\mathcal{T}$ are weakly continuous, family $\mathcal{F}$ is weakly jointly continuous,
(iii) $\mathcal{X}$ is Banach space, $\mathcal{D}$ is weakly compact, $\mathcal{T}$ is completely continuous, and family $\mathcal{F}$ is jointly continuous.

Proof. First, we show that $\mathcal{T}$ is self-map on $\mathcal{D}$, i.e., $\mathcal{T} : \mathcal{D} \to \mathcal{D}$. Let $y \in \mathcal{D}$, then $\mathcal{I}y \in \mathcal{D}$, since $\mathcal{I}(\mathcal{D}) = \mathcal{D}$. Also, if $y \in \partial\mathcal{M}$, then $\mathcal{I}y \in \mathcal{M}$, since $\mathcal{I}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{I}\widehat{x} = \widehat{x} = \mathcal{T}\widehat{x}$, one may have from (3.4)

$$\|\mathcal{T}y - \widehat{x}\| = \|\mathcal{T}y - \mathcal{T}\widehat{x}\| \leq \|\mathcal{I}y - \mathcal{I}\widehat{x}\| = \|\mathcal{I}y - \widehat{x}\| = dist(\widehat{x}, \mathcal{M}).$$

Thus, $\mathcal{T}y \in \mathcal{D}$. Consequently $\mathcal{T}$ and $\mathcal{I}$ are self-maps on $\mathcal{D}$. The conditions of Theorem 3.2 and Theorem 3.3((i)and(ii)) are satisfied and hence, there exists a $z \in \mathcal{D}$ such that $\mathcal{T}z = z = \mathcal{I}z$. ∎

Defines $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(\widehat{x}) \cap \mathcal{C}^{\mathcal{I}}_{\mathcal{M}}(\widehat{x})$, where $\mathcal{C}^{\mathcal{I}}_{\mathcal{M}}(\widehat{x}) = \{x \in \mathcal{M} : \mathcal{I}x \in \mathcal{P}_{\mathcal{M}}(\widehat{x})\}$.

Theorem 3.9. *Let $\mathcal{M}$ be subset of a normed space $\mathcal{X}$ and $\mathcal{T},\mathcal{I}:\mathcal{X}\to\mathcal{X}$ be mappings such that $\widehat{x}\in\mathcal{F}(\mathcal{T})\cap\mathcal{F}(\mathcal{I})$ for some $\widehat{x}\in\mathcal{X}$ and $\mathcal{T}(\partial\mathcal{M}\cap\mathcal{M})\subset\mathcal{M}$. Suppose that $\mathcal{D}$ has a contractive family $\Delta=\{f_\alpha\}_{\alpha\in\mathcal{D}}$ and $\mathcal{D}\cap\mathcal{F}(\mathcal{I})$ is nonempty closed, $\mathcal{I}(\mathcal{D}(\widehat{x}))=\mathcal{D}$ and $\mathcal{T}$ is $\mathcal{I}$-continuous. If $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(\widehat{x})\cup\{\widehat{x}\}$ and the pair $(\mathcal{T},\mathcal{I})$ is a Banach operator pair on $\mathcal{D}$ and satisfies*

$$(3.5)\quad \|\mathcal{T}x-\mathcal{T}y\| \leq \begin{cases} \|\mathcal{I}x-\mathcal{I}\widehat{x}\| & \text{if } y=\widehat{x},\\ max\{\|\mathcal{I}x-\mathcal{I}y\|,\ dist(\mathcal{I}x,f_{\mathcal{T}x}(k)),\ dist(\mathcal{I}y,f_{\mathcal{T}y}(k)),\\ dist(\mathcal{I}x,f_{\mathcal{T}y}(k)),\ dist(\mathcal{I}y,f_{\mathcal{T}x}(k))\}, & \text{if } y\in\mathcal{D},\end{cases}$$

where $0<k<1$, then $\mathcal{P}_{\mathcal{M}}(\widehat{x})\cap\mathcal{F}(\mathcal{T},\mathcal{I})\neq\emptyset$, provided one of the following conditions is satisfied;

(i) *$cl(\mathcal{T}(\mathcal{D}))$ is compact,*
(ii) *$\mathcal{X}$ is Banach space, $\mathcal{D}$ is weakly compact, $\mathcal{I}$ and $\mathcal{T}$ are weakly continuous, family $\mathcal{F}$ is weakly jointly continuous,*
(iii) *$\mathcal{X}$ is Banach space, $\mathcal{D}$ is weakly compact, $\mathcal{T}$ is completely continuous, and family $\mathcal{F}$ is jointly continuous.*

Proof. Let $x\in\mathcal{D}$. Then, $x\in\mathcal{P}_{\mathcal{M}}(\widehat{x})$ and hence $\|x-\widehat{x}\|=dist(x_0,\mathcal{M})$. Note that for any $t\in(0,1)$,

$$\|t\widehat{x}+(1-t)x-\widehat{x}\|=(1-t)\|x-\widehat{x}\|<dist(\widehat{x},\mathcal{M}).$$

It follows that the line segment $\{t\widehat{x}+(1-t)x:0<t<1\}$ and the set $\mathcal{M}$ are disjoint. Thus x is not in the interior of $\mathcal{M}$ and so $x\in\partial\mathcal{M}\cap\mathcal{M}$. Since $f(\partial\mathcal{M}\cap\mathcal{M})\subset\mathcal{M}$, $\mathcal{T}x$ must be in $\mathcal{M}$. Also, proceeding as in the proof of Theorem 3.8, we have $\mathcal{T}x\in\mathcal{P}_{\mathcal{M}}(\widehat{x})$. As $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(\widehat{x})\cup\{\widehat{x}\}$, we have

$$\|\mathcal{I}\mathcal{T}x-\widehat{x}\|\leq\|\mathcal{T}x-\mathcal{T}\widehat{x}\|\leq\|\mathcal{I}x-\mathcal{I}\widehat{x}\|=\|\mathcal{I}x-\widehat{x}\|=dist(\widehat{x},\mathcal{M}).$$

Thus $\mathcal{I}\mathcal{T}x\in\mathcal{P}_{\mathcal{M}}(\widehat{x})$ and so $\mathcal{T}x\in\mathcal{C}^{\mathcal{I}}_{\mathcal{M}}(\widehat{x})$. Hence $\mathcal{T}x\in\mathcal{D}$. Consequently, $\mathcal{T}(\mathcal{D})\subset\mathcal{D}=\mathcal{I}(\mathcal{D})$. Now Theorem 3.2 and Theorem 3.3((i) and (ii)) guarantee that $\mathcal{P}_{\mathcal{M}}(\widehat{x})\cap\mathcal{F}(\mathcal{T},\mathcal{I})\neq\phi$. ∎

Remark 3.10. It is remark that the Theorem 3.9 is trivial if $\widehat{x}\in\mathcal{M}$, because the statement in the proof that $\mathcal{M}$ and the line segment $t\widehat{x}+(1-t)x$ are disjoint is no longer necessarily true if $\widehat{x}\in\mathcal{M}$.

For $h\geq 0$, let ${\mathcal{D}_{\mathcal{M}}}^{h,\mathcal{I}}(\widehat{x})=\mathcal{P}_{\mathcal{M}}(\widehat{x})\cap{\mathcal{G}_{\mathcal{M}}}^{h,\mathcal{I}}(\widehat{x})$, where ${\mathcal{G}_{\mathcal{M}}}^{h,\mathcal{I}}(\widehat{x})=\{x\in\mathcal{M}:\|\mathcal{I}x-\widehat{x}\|\leq(2h+1)dist(\widehat{x},\mathcal{M})\}$.

Theorem 3.11. *Let $\mathcal{M}$ be subset of a metric space $(\mathcal{X},d)$ and $\mathcal{T},\mathcal{I}:\mathcal{X}\to\mathcal{X}$ be mappings such that $\widehat{x}\in\mathcal{F}(\mathcal{T},\mathcal{I})$ for some $\widehat{x}\in\mathcal{X}$ and $\mathcal{T}(\partial\mathcal{M}\cap\mathcal{M})\subset\mathcal{M}$.*

Suppose that $\mathcal{D}$ has a contractive family $\Delta = \{f_\alpha\}_{\alpha\in\mathcal{D}}$ and ${\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(u)\cap\mathcal{F}(\mathcal{I})$ is nonempty closed, $\mathcal{I}({\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})) = {\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$ and $\mathcal{T}$ is $\mathcal{I}$-continuous. If $\mathcal{I}$ is continuous on ${\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$, the pair $(\mathcal{T},\mathcal{I})$ satisfies

(a) $\|\mathcal{I}\mathcal{T}x - \mathcal{T}x\| \le h\|\mathcal{I}x - x\|$ *for all* $x \in {\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$ *and* $h \ge 0$
(b) *for all* $x \in {\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x}) \cup \{\widehat{x}\}$,

$$(3.6)\quad \|\mathcal{T}x - \mathcal{T}y\| \le \begin{cases} \|\mathcal{I}x - \mathcal{I}\widehat{x}\| \;\; if \;\; y = \widehat{x}, \\ max\{\|\mathcal{I}x - \mathcal{I}y\|, dist(\mathcal{I}x, f_{\mathcal{T}x}(k)), dist(\mathcal{I}y, f_{\mathcal{T}y}(k)), \\ dist(\mathcal{I}x, f_{\mathcal{T}y}(k)), dist(\mathcal{I}y, f_{\mathcal{T}x}(k))\}, \;\; if\; y \in {\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x}), \end{cases}$$

where $0 < k < 1$, then $\mathcal{P}_\mathcal{M}(\widehat{x}) \cap \mathcal{F}(\mathcal{T},\mathcal{I}) \neq \emptyset$, provided one of the following conditions is satisfied;

(i) *$cl(\mathcal{T}({\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})))$ is compact,*
(ii) *$\mathcal{X}$ is Banach space, ${\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$ is weakly compact, $\mathcal{I}$ and $\mathcal{T}$ are weakly continuous, family $\mathcal{F}$ is weakly jointly continuous,*
(iii) *$\mathcal{X}$ is Banach space, ${\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$ is weakly compact, $\mathcal{T}$ is completely continuous, and family $\mathcal{F}$ is jointly continuous.*

Proof. Let $x \in {\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$. Then, along in the line of the proof of Theorem 3.8, we have $\mathcal{T}x \in \mathcal{P}_\mathcal{M}(\widehat{x})$. From inequality in (a) and (3.6), it follow that,

$$\begin{aligned} \|\mathcal{I}\mathcal{T}x - \widehat{x}\| &= \|\mathcal{I}\mathcal{T}x - \mathcal{T}x + \mathcal{T}x - \widehat{x}\| \\ &\le \|\mathcal{I}\mathcal{T}x - \mathcal{T}x\| + \|\mathcal{T}x - \widehat{x}\| \le h\|\mathcal{I}x - x\| + \|\mathcal{T}x - \widehat{x}\| \\ &= h\|\mathcal{I}x - \widehat{x} + \widehat{x} - x\| + \|\mathcal{T}x - \widehat{x}\| \\ &\le h(\|\mathcal{I}x - u\| + \|x - \widehat{x}\|) + \|\mathcal{T}x - \widehat{x}\| \\ &\le h(dist(\widehat{x},\mathcal{M}) + dist(\widehat{x},\mathcal{M})) + dist(\widehat{x},\mathcal{M}) \\ &\le (2h+1)dist(\widehat{x},\mathcal{M}). \end{aligned}$$

Thus $\mathcal{T}x \in {\mathcal{G}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$. Consequently, $\mathcal{T}({\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})) \subset {\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x}) = \mathcal{I}({\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x}))$. Inequality in (a) also implies that $(\mathcal{T},\mathcal{I})$ is a Banach operator pair. Now by Theorem 3.2 and Theorem 3.3 ((i) and (ii)), we obtain, $\mathcal{P}_\mathcal{M}(\widehat{x}) \cap \mathcal{F}(\mathcal{I},\mathcal{I}) \neq \emptyset$ in each of the cases (i) and (ii). ∎

Remark 3.12. If we take ${\mathcal{C}_\mathcal{M}}^{\mathcal{I}}(\widehat{x}) = \{x \in \mathcal{M} : \mathcal{I}x \in \mathcal{P}_\mathcal{M}(\widehat{x})\}$. Then $\mathcal{I}(\mathcal{P}_\mathcal{M}(\widehat{x})) \subset \mathcal{P}_\mathcal{M}(\widehat{x})$ implies $\mathcal{P}_\mathcal{M}(u) \subset {\mathcal{C}_\mathcal{M}}^{\mathcal{I}}(\widehat{x}) \subset {\mathcal{G}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x})$ and hence ${\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x}) = \mathcal{P}_\mathcal{M}(\widehat{x})$. Consequently, Theorem 3.11 remains valid when ${\mathcal{D}_\mathcal{M}}^{h,\mathcal{I}}(\widehat{x}) = \mathcal{P}_\mathcal{M}(\widehat{x})$ and the pair $(\mathcal{T},\mathcal{I})$ is Banach operator on $\mathcal{P}_\mathcal{M}(\widehat{x})$ instead of satisfying (a), which in turn extends many results (see [1, 13, 14, 17, 22, 25, 27]).

Remark 3.13. Theorem 3.11 extends and improve the results in [1, 24, 27].

Remark 3.14. Theorem 3.1–Theorem 3.11 generalize Theorem 3.2–Theorem 4.2 in [3] by relaxing the starshaped condition of domain $\mathcal{M}$ or $\mathcal{D}$ and $\mathcal{F}(\mathcal{I})$, and using more generalized relatively nonexpansive mappings.

Remark 3.15. Theorem 3.2–Theorem 3.11 generalize results of Nashine [18, 19, 20] and hence [14, 15, 16] by using more general noncommuting, called Banach pair operator, weak conditions and relaxing the property Γ of mappings in metric or normed space.

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