

CONTROLLABILITY FOR FRACTIONAL EVOLUTION INCLUSIONS WITHOUT COMPACTNESS

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ABSTRACT. In this paper, we study the existence and controllability for fractional evolution inclusions in Banach spaces. We use a new approach to obtain the existence of mild solutions and controllability results, avoiding hypotheses of compactness on the semigroup generated by the linear part and any conditions on the multivalued nonlinearity expressed in terms of measures of noncompactness. Finally, two examples are given to illustrate our theoretical results.

1. Introduction. Fractional differential equation is concerned with the notion and methods to solve differential equations involving fractional derivatives of the unknown function. It can be also considered as an alternative model to nonlinear differential equations. As a result, differential equation with fractional derivative can be considered as an excellent instrument for the description of memory and hereditary properties of various materials and processes. The fractional order models of real systems are always more adequate than the classical integer order models, since the description of some systems is more accurate when the fractional derivative is used. The advantages of fractional derivatives becomes evident in modeling mechanical and electrical properties of real materials, description of rheological

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properties of rocks and in various other fields. There has been a great deal of interest in the solutions of fractional differential equations in analytical and numerical senses. One can see the monographs of Kilbas et al. [14], Podlubny [20], Diethelm [6], Tarasov [23], Zhou [30, 31] and the papers [25, 21, 15, 8, 29, 32].

In this paper, we are interested in the fractional semilinear differential inclusions in Banach spaces of the type

$$\begin{cases} {}^C D_t^q x(t) \in Ax(t) + F(t, x(t)), & \text{a.e. } t \in [0, b], \quad 0 < q \leq 1, \\ x(0) = x_0, \end{cases} \quad (1)$$

where ${}^C D_t^q$ is the Caputo fractional derivative of order q , $b > 0$ is a finite number, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in X , the state $x(\cdot)$ takes values in a Banach space X , x_0 is an element of the Banach space X , $F : [0, b] \times X \rightarrow X$ is a multivalued map.

Further, we investigate the following fractional control system

$$\begin{cases} {}^C D_t^q x(t) \in Ax(t) + Bu(t) + F(t, x(t)), & \text{a.e. } t \in [0, b], \quad 0 < q \leq 1, \\ x(0) = x_0, \end{cases} \quad (2)$$

where the control function $u(\cdot)$ takes its value in $L^{\frac{1}{q_1}}([a, b]; U)$ for $q_1 \in (0, q)$, a Banach space of admissible control functions and U is a Banach space. $B : U \rightarrow X$ is a bounded linear operator.

In recent years, the existence of mild solutions and controllability problems for various types of nonlinear fractional evolution inclusions in infinite dimensional spaces by using different kinds of approaches have been considered in many recent publications (see, e.g., [24, 11, 16, 26, 27, 28, 1, 17] and the references therein). In most of the existing articles, various fixed point theorems and measure of noncompactness are employed to obtain the fixed points of the solution operator of the Cauchy problems under the restrictive hypotheses of compactness on the semigroup generated by the linear part and on the nonlinear term. But, in infinite dimensional Banach spaces the compactness of the associated evolution operator is in contradiction with the controllability of a linear system while using locally L^p -controls, for $p > 1$. As it was pointed out by [2], it is meaningful to introduce conditions assuring controllability for semilinear equations without requiring the compactness of the semigroup or evolution operator generated by the linear part. In this paper another approach is considered, it exploits the weak topology of the state space. This new tool was introduced to study semilinear differential inclusions associated to boundary value conditions, see [3]. We prove the existence of mild solutions of (1) and the controllability results of (2) by means of weak topology, avoiding hypotheses of compactness on the semigroup generated by the linear part and any conditions on the multivalued nonlinearity expressed in terms of measures of noncompactness.

The paper is organized as follows. In Section 2 we recall some notions and results that we use in the main part of the paper. In section 3 we study the existence of mild solutions for (1). In Section 4 we prove the controllability for the fractional control system (2) and in Section 5 two examples are given to illustrate the obtained theory.

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $A : D(A) \rightarrow X$ be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$. There exists a constant $M_1 > 0$ such that $\sup_{t \in J} \|T(t)\| \leq M_1$. Let $(X, \|\cdot\|)$ be a reflexive

Banach space and X_w denote the space X endowed with the weak topology. For a set $D \subset X$, the symbol \overline{D}^w denotes the weak closure of D . We recall that a bounded subset D of a reflexive Banach space X is weakly relatively compact. In the whole paper, without generating misunderstanding, we denote by $\|\cdot\|_p$ both the $L^p([0, b]; X)$ -norm and $L^p([0, b]; \mathbb{R})$ -norm and by $\|\cdot\|_0$ the $C([0, b]; X)$ -norm. We recall (see [4, Theorem 4.3]) that a sequence $\{x_n\} \subset C([0, b]; X)$ weakly converges to an element $x \in C([0, b]; X)$ if and only if

- (i) there exists $N > 0$ such that, for every $n \in \mathbb{N}$ and $t \in [0, b]$, $\|x_n(t)\| \leq N$;
- (ii) for every $t \in [0, b]$, $x_n(t) \rightharpoonup x(t)$.

For sake of completeness, we recall some results that we will need in the main section. Firstly we state the fixed point theorem by Donal O'Regan.

Theorem 2.1. [18] *Let E be a metrizable locally convex linear topological space and let Q be a weakly compact, convex subset of E . Suppose $G : Q \rightarrow C(Q)$ has weakly sequentially closed graph. Then G has a fixed point; here $C(Q)$ denoted the family of nonempty closed, convex subsets of Q .*

We mention also two results that are contained in the so called Eberlein-Smulian theory.

Theorem 2.2. [13, Theorem 1, p. 219] *Let Ω be a subset of a Banach space E . The following statements are equivalent:*

- (i) Ω is relatively weakly compact;
- (ii) Ω is relatively weakly sequentially compact.

Corollary 2.1. [13, p. 219] *Let Ω be a subset of a Banach space X . The following statements are equivalent:*

- (i) Ω is weakly compact;
- (ii) Ω is weakly sequentially compact.

We recall the Krein-Smulian theorem.

Theorem 2.3. [7, p. 434] *The convex hull of a weakly compact set in a Banach space E is weakly compact.*

In conclusion we recall the Pettis measurability theorem.

Theorem 2.4. [19, p. 278] *Let (E, Σ) be a measure space, X be a separable Banach space. Then a function $f : E \rightarrow X$ is measurable if and only if for every $e \in X'$ the function $e \circ f : E \rightarrow \mathbb{R}$ is measurable with respect to Σ and the Borel σ -algebra in \mathbb{R} .*

Let us recall the following known definitions. For more details, see [14].

Definition 2.5. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.6. The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^L D^\gamma f(t) = \frac{1}{\Gamma(n - \gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t - s)^{\gamma+1-n}} ds, \quad t > 0, \quad n - 1 < \gamma < n.$$

Definition 2.7. The Caputo derivative of order γ for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^C D^\gamma f(t) = {}^L D^\gamma \left(f(t) - \sum_{k=1}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n - 1 < \gamma < n.$$

Remark 2.1. (i) If $f(t) \in C^n[0, \infty)$, then

$$\begin{aligned} {}^C D^\gamma f(t) &= \frac{1}{\Gamma(n - \gamma)} \int_0^t \frac{f^{(n)}(s)}{(t - s)^{\gamma+1-n}} ds \\ &= I^{n-\gamma} f^{(n)}(t), \quad t > 0, \quad n - 1 < \gamma < n; \end{aligned}$$

- (ii) the Caputo derivative of a constant is equal to zero;
- (iii) if f is an abstract function with values in X , then integrals which appear in Definitions 2.5, 2.6 and 2.7 are taken in Bochner’s sense.

3. Existence of mild solutions. We study the fractional semilinear differential inclusion (1) under the following assumptions:

(H_A) the operator A generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in X , and there exists a constant $M_1 \geq 1$ such that $\sup_{t \in J} \|T(t)\| \leq M_1$.

We assume that the multivalued nonlinearity $F : [0, b] \times X \rightrightarrows X$ has nonempty convex and weakly compact values and:

- (H₁) the multifunction $F(\cdot, x) : [0, b] \rightrightarrows X$ has a measurable selection for every $x \in X$;
- (H₂) the multimap $F(t, \cdot) : X \rightrightarrows X$ is weakly sequentially closed for a.e. $t \in [0, b]$, i.e., it has a weakly sequentially closed graph;
- (H₃) there exists a constant $q_1 \in (0, q)$ and for every $r > 0$, there exists a function $\mu_r \in L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ such that for each $c \in X$, $\|c\| \leq r$:

$$\|F(t, c)\| = \sup\{\|x\| : x \in F(t, c)\} \leq \mu_r(t) \text{ for a.e. } t \in [0, b];$$

Now, we define the mild solution of fractional evolution inclusion (1).

Definition 3.1. [30] A continuous function $x : [0, b] \rightarrow X$ is said to be a mild solution of fractional differential system (1) if $x(0) = x_0$ and there exists $f \in L^{\frac{1}{q_1}}([0, b]; X)$ such that $f(t) \in F(t, x(t))$ on $t \in [0, b]$ and x satisfies the following integral equation

$$x(t) = \mathcal{F}(t)(x_0) + \int_0^t (t - s)^{q-1} \mathcal{S}(t - s) f(s) ds,$$

where

$$\begin{aligned} \mathcal{F}(t) &= \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta, \quad \mathcal{S}(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q \theta) d\theta, \\ \xi_q(\theta) &= \frac{1}{q} \theta^{-1-\frac{1}{q}} \varpi_q \left(\theta^{-\frac{1}{q}} \right) \geq 0, \\ \varpi_q(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq + 1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty), \end{aligned}$$

and ξ_q is a probability density function defined on $(0, \infty)$, that is,

$$\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty) \quad \text{and} \quad \int_0^\infty \xi_q(\theta)d\theta = 1.$$

Remark 3.1. It is not difficult to verify that for $v \in [0, 1]$,

$$\int_0^\infty \theta^v \xi_q(\theta)d\theta = \int_0^\infty \theta^{-qv} \varpi_q(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}.$$

The following results will be used in the proof of our main results.

Lemma 3.2. [30, 33] *The operators \mathcal{T} and \mathcal{S} have the following properties:*

- (i) *for any fixed $t \geq 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators, i.e., for any $x \in X$,*

$$\|\mathcal{T}(t)x\| \leq M_1\|x\| \quad \text{and} \quad \|\mathcal{S}(t)x\| \leq \frac{qM_1}{\Gamma(1+q)}\|x\|;$$

- (ii) *$\{\mathcal{T}(t), t \geq 0\}$ and $\{\mathcal{S}(t), t \geq 0\}$ are strongly continuous.*

Given $p \in C([0, b]; X)$, let us denote

$$\Lambda_p = \{f \in L^{\frac{1}{q_1}}([0, b]; X) : f(t) \in F(t, p(t)) \text{ for a.e. } t \in [0, b]\}.$$

The set Λ_p is always nonempty as Proposition 3.1 below shows.

Proposition 3.1. *Assume that a multimap $F : [0, b] \times X \rightrightarrows X$ satisfies (H_1) , (H_2) and (H_3) , the set Λ_p is nonempty for any $p \in C([0, b]; X)$.*

Proof. Let $p \in C([0, b]; X)$. By the uniform continuity of p there exists a sequence $\{p_n\}$ of step functions, $p_n : [0, b] \rightarrow X$ such that

$$\sup_{t \in [0, b]} \|p_n(t) - p(t)\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3}$$

Hence, by (H_1) , there exists a sequence of functions $\{f_n\}$ such that $f_n(t) \in F(t, p_n(t))$ for a.e. $t \in [0, b]$ and $f_n : [0, b] \rightarrow X$ is measurable for any $n \in \mathbb{N}$. From (3) there exists a bounded set $E \subset X$ such that $p_n(t), p(t) \in E$, for any $t \in [0, b]$ and $n \in \mathbb{N}$ and by (H_3) there exists $\eta_n \in L^{\frac{1}{q_1}}([0, b]; \mathbb{R})$ such that

$$\|f_n(t)\| \leq \|F(t, p_n(t))\| \leq \eta_n(t), \quad \forall n \in \mathbb{N}, \quad \text{and a.e. } t \in [0, b].$$

Hence, $\{f_n\} \subset L^{\frac{1}{q_1}}([0, b]; X)$ is bounded and uniformly integrable and $\{f_n(t)\}$ is bounded in X for a.e. $t \in [0, b]$. According to the reflexivity of the space X and by the Dunford-Pettis theorem (see [7, p. 294]), we have the existence of a subsequence, denoted as the sequence, such that

$$f_n \rightharpoonup g \in L^{\frac{1}{q_1}}([0, b]; X).$$

By Mazur's convexity theorem we obtain a sequence

$$\tilde{f}_n = \sum_{i=0}^{k_n} \lambda_{n,i} w_{n+i}, \quad \lambda_{n,i} \geq 0, \quad \sum_{i=0}^{k_n} \lambda_{n,i} = 1$$

such that $\tilde{f}_n \rightarrow g$ in $L^{\frac{1}{q_1}}([0, b]; X)$ and, up to subsequence, $\tilde{f}_n(t) \rightarrow g(t)$ for all $t \in [0, b]$.

By (H_3) , the multimap $F(t, \cdot)$ is locally weakly compact for a.e. $t \in [0, b]$, i.e., for a.e. t and every $x \in X$ there is a neighbourhood V of x such that the restriction of $F(t, \cdot)$ to V is weakly compact. Hence by (H_2) and the locally weak compactness, we

easily get that $F(t, \cdot) : X_w \rightarrow X_w$ is u.s.c. for a.e. $t \in [0, b]$. Thus, $F(t, \cdot) : X \rightarrow X_w$ is u.s.c. for a.e. $t \in [0, b]$.

To conclude we have only to prove that $g(t) \in F(t, p(t))$ for a.e. $t \in [0, b]$. Indeed, let N_0 with Lebesgue measure zero be such that $F(t, \cdot) : X \rightarrow X_w$ is u.s.c. $f_n(t) \in F(t, p_n(t))$ and $\tilde{f}_n(t) \rightarrow g(t)$ for all $t \in [0, b] \setminus N_0$ and $n \in \mathbb{N}$.

Fix $t_0 \notin N_0$ and assume, by contradiction, that $g(t_0) \notin F(t_0, p(t_0))$. Since $F(t_0, p(t_0))$ is closed and convex, from the Hahn-Banach theorem there is a weakly open convex set $V \supset F(t_0, p(t_0))$ satisfying $g(t_0) \notin \bar{V}$. Since $F(t_0, \cdot) : X \rightarrow X_w$ is u.s.c., we can find a neighbourhood U of $p(t_0)$ such that $F(t_0, x) \subset V$ for all $x \in U$. The convergence $p_n(t_0) \rightarrow p(t_0)$ as $m \rightarrow \infty$ then implies the existence of $n_0 \in \mathbb{N}$ such that $p_n(t_0) \in U$ for all $n > n_0$. Therefore $g_0(t_0) \in F(t_0, p_n(t_0)) \subset V$ for all $n > n_0$. Since V is convex we also have that $\tilde{f}_n(t_0) \in V$ for all $n > n_0$ and, by the convergence, we arrive to the contradictory conclusion that $g(t_0) \in \bar{V}$. We obtain that $g(t) \in F(t, p(t))$ for a.e. $t \in [0, b]$. \square

We define the solution multioperator $\Gamma : C([0, b]; X) \rightarrow C([0, b]; X)$ as

$$\Gamma(p) = \{x \in C([0, b]; X) : x(t) = \mathcal{I}(t)x_0 + S(f)(t), f \in \Lambda_p\}, \quad (4)$$

where $S(f)(t) = \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s) ds$.

We first prove that the operator S is continuous.

For any $f_n, f \in L^{\frac{1}{q_1}}([0, b]; X)$ and $f_n \rightarrow f$ ($n \rightarrow \infty$), using (H₃), we get for each $t \in [0, b]$

$$(t-s)^{q-1} \|f_n(s) - f(s)\| \leq 2(t-s)^{q-1} \mu_r(s), \text{ a.e. } s \in [0, t].$$

On the other hand, the function $\int_0^t (t-s)^{q-1} \mu_r(s) ds = \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|\mu_r\|_{\frac{1}{q_1}}$ is integrable for $t \in [0, b]$. By Lebesgue dominated convergence theorem, we have

$$\int_0^t (t-s)^{q-1} \|f_n(s) - f(s)\| ds \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus

$$\begin{aligned} \|S(f_n) - S(f)\| &\leq \left\| \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) (f_n(s) - f(s)) ds \right\| \\ &\leq \frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|f_n(s) - f(s)\| ds \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

So the operator S is continuous.

It is easy to verify that the fixed points of the multioperator Γ are mild solutions of fractional differential system (1).

Fix $n \in \mathbb{N}$, consider Q_n the closed ball of radius n in $C([0, b]; X)$ centered at the origin and denote by $\Gamma_n = \Gamma|_{Q_n} : Q_n \rightarrow C([0, b]; X)$ the restriction of the multioperator Γ on the set Q_n . We describe some properties of Γ_n .

Proposition 3.2. *The multioperator Γ_n has a weakly sequentially closed graph.*

Proof. Let $\{p_m\} \subset Q_n$ and $\{x_m\} \subset C([0, b]; X)$ satisfying $x_m \in \Gamma_n(p_m)$ for all m and $p_m \rightarrow p$, $x_m \rightarrow x$ in $C([0, b]; X)$; we will prove that $x \in \Gamma_n(p)$.

Since $p_m \in Q_n$ for all m and $p_m(t) \rightarrow p(t)$ for every $t \in [0, b]$, it follows that $\|p(t)\| \leq \liminf_{m \rightarrow \infty} \|p_m(t)\| \leq n$ for all t (see [5, Proposition III.5]). The fact that

$x_m \in \Gamma_n(p_m)$ means that there exists a sequence $\{f_m\}$, $f_m \in \Lambda_{p_m}$ such that for every $t \in [0, b]$,

$$x_m(t) = \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f(s)ds.$$

We observe that, according to (H_3) , $\|f_m(t)\| \leq \mu_n(t)$ for a.e. t and every m , i.e., $\{f_m\}$ is bounded and uniformly integrable and $\{f_m(t)\}$ is bounded in X for a.e. $t \in [0, b]$. Hence, by the reflexivity of the space X and by the Dunford-Pettis theorem (see [7, p. 294]), we have the existence of a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^{\frac{1}{q_1}}([0, b]; X)$.

Therefore, we have $Sf_m \rightharpoonup Sg$. Indeed, let $e' : X \rightarrow \mathbb{R}$ be a linear continuous operator. By the linearity and continuity of the operator S , we have that the operator

$$g \rightarrow e' \left(\int_0^t (t-s)^{q-1} \mathcal{S}(t-s)g(s)ds \right)$$

is a linear and continuous operator from $L^{\frac{1}{q_1}}([0, b]; X)$ to \mathbb{R} for all $t \in [0, b]$. Then, from the definition of the weak convergence, we have for every $t \in [0, b]$

$$e' \left(\int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f_m(s)ds \right) \rightarrow e' \left(\int_0^t (t-s)^{q-1} \mathcal{S}(t-s)g(s)ds \right).$$

Thus

$$x_m(t) \rightharpoonup \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)g(s)ds = x_0(t), \quad \forall t \in [0, b].$$

implying, for the uniqueness of the weak limit in X , that $x_0(t) = x(t)$ for all $t \in [0, b]$.

Finally, as the reason for the fourth part of Proposition 3.1, it is possible to show that $g(t) \in F(t, p(t))$ for a.e. $t \in [a, b]$. □

Proposition 3.3. *The multioperator Γ_n is weakly compact.*

Proof. We first prove that $\Gamma_n(Q_n)$ is relatively weakly sequentially compact.

Let $\{p_m\} \subset Q_n$ and $\{x_m\} \subset C([0, b]; X)$ satisfying $x_m \in \Gamma_n(p_m)$ for all m . By the definition of the multioperator Γ_n , there exists a sequence $\{f_m\}$, $f_m \in \Lambda_{p_m}$ such that

$$x_m(t) = \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f_m(s)ds, \quad \forall t \in [0, b].$$

Further, as the reason for Proposition 3.2, we have that there exists a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^{\frac{1}{q_1}}([0, b]; X)$. Therefore,

$$x_m(t) \rightharpoonup l(t) = \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)g(s)ds, \quad \forall t \in [0, b].$$

Furthermore, by the weak convergence of $\{f_m\}$, by (H_A) , we have

$$\|x_m(t)\| \leq M_1 \|x_0\| + \frac{M_1 q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|\mu_n\|_{\frac{1}{q_1}}$$

for all $m \in \mathbb{N}$ and $t \in [0, b]$. As the reason for Proposition 3.2, it is then easy to prove that $x_m \rightharpoonup l$ in $C([0, b]; X)$. Thus $\Gamma_n(Q_n)$ is relatively weakly compact by Theorem 2.2. □

Proposition 3.4. *The multioperator Γ_n has convex and weakly compact values.*

Proof. Fix $p \in Q_n$, since F is convex valued, from the linearity of the integral and of the operators $\mathcal{T}(t)$ and $\mathcal{S}(t)$, it follows that the set $\Gamma_n(p)$ is convex. The weak compactness of $\Gamma_n(p)$ follows by Propositions 3.2 and 3.3. \square

We are able now to state the main results of this section.

Theorem 3.3. *Assume that (H_A) , (H_1) and (H_2) hold. In addition, suppose that (H_4) there exists a sequence of functions $\{\omega_n\} \subset L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ such that*

$$\sup_{\|c\| \leq n} \|F(t, c)\| \leq \omega_n(t)$$

for a.e. $t \in [0, b]$, $n \in \mathbb{N}$ with

$$\liminf_{n \rightarrow \infty} \frac{\|w_n\|_{\frac{1}{q_1}}}{n} = 0. \tag{5}$$

Then inclusion (1) has at least a mild solution.

Proof. We show that there exists $n \in \mathbb{N}$ such that the operator Γ_n maps the ball Q_n into itself.

Assume, to the contrary, that there exist sequences $\{z_n\}, \{y_n\}$ such that $z_n \in Q_n$, $y_n \in \Gamma_n(z_n)$ and $y_n \notin Q_n$, $\forall n \in \mathbb{N}$. Then there exists a sequence $\{f_n\} \subset L^{\frac{1}{q_1}}([0, b]; X)$, $f_n(s) \in F(s, z_n(s))$ such that

$$y_n(t) = \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)y_n(s)ds, \quad \forall t \in [0, b].$$

As the reason for Proposition 3.3, we have

$$n < \|y_n\|_0 \leq M_1 \|x_0\| + \frac{M_1 q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|w_n\|_{\frac{1}{q_1}}.$$

Then

$$1 < \frac{\|y\|_0}{n} \leq \frac{M_1 \|x_0\|}{n} + \frac{M_1 q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \frac{\|w_n\|_{\frac{1}{q_1}}}{n}, \quad n \in \mathbb{N},$$

which contracts (5).

Fix $n \in \mathbb{N}$ such that $\Gamma_n(Q_n) \subseteq Q_n$. By Proposition 3.3 the set $V_n = \overline{\Gamma_n(Q_n)}^w$ is weakly compact. Let now $W_n = \overline{\text{co}}(V_n)$, where $\overline{\text{co}}(V_n)$ denotes the closed convex hull of V_n . By Theorem 2.3, W_n is a weakly compact set. Moreover, from the fact that $\Gamma_n(Q_n) \subset Q_n$ and that Q_n is a convex closed set, we have that $W_n \subset Q_n$ and hence

$$\Gamma_n(W_n) = \Gamma_n(\overline{\text{co}}(\Gamma_n(Q_n))) \subseteq \Gamma_n(Q_n) \subseteq \overline{\Gamma_n(Q_n)}^w = V_n \subset W_n.$$

In view of Proposition 3.2, Γ_n has a weakly sequentially closed graph. Thus from Theorem 2.1, inclusion (1) has a solution. The proof is now completed. \square

Remark 3.2. Suppose, for example, that there exists $\alpha \in L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ and a nondecreasing function $\beta : [0, +\infty) \rightarrow [0, +\infty)$ such that $\|F(t, c)\| \leq \alpha(t)\beta(\|c\|)$ for a.e. $t \in [0, b]$ and every $c \in X$. Then condition (5) is equivalent to

$$\liminf_{n \rightarrow \infty} \frac{\beta(n)}{n} = 0.$$

Theorem 3.4. Assume that (H_A) , (H_1) and (H_2) hold. If

(H_5) there exists $\alpha \in L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ such that

$$\|F(t, c)\| \leq \alpha(t)(1 + \|c\|), \text{ for a.e. } t \in [0, b], \forall c \in X$$

and

$$\frac{M_1 q}{\Gamma(1 + q)} \left[\left(\frac{1 - q_1}{q - q_1} \right) b^{\frac{q - q_1}{1 - q_1}} \right]^{1 - q_1} \|\alpha\|_{\frac{1}{q_1}} < 1, \tag{6}$$

then inclusion (1) has at least a mild solution.

Proof. As the reason for Theorem 3.3. Assume that there exist $\{z_n\}$, $\{y_n\}$ such that $z_n \in Q_n$, $y_n \in \Gamma_n(z_n)$ and $y_n \notin Q_n$, $\forall n \in \mathbb{N}$, we get

$$\begin{aligned} n &< \|y_n\|_0 \\ &\leq M_1 \|x_0\| + \frac{M_1 q}{\Gamma(1 + q)} \left[\left(\frac{1 - q_1}{q - q_1} \right) b^{\frac{q - q_1}{1 - q_1}} \right]^{1 - q_1} \left(\int_0^b |\alpha(\eta)|^{\frac{1}{q_1}} (1 + \|z_n(\eta)\|)^{\frac{1}{q_1}} d\eta \right)^{q_1} \\ &\leq M_1 \|x_0\| + \frac{M_1 q}{\Gamma(1 + q)} \left[\left(\frac{1 - q_1}{q - q_1} \right) b^{\frac{q - q_1}{1 - q_1}} \right]^{1 - q_1} (1 + n) \|\alpha\|_{\frac{1}{q_1}}, \quad n \in \mathbb{N}, \end{aligned}$$

which contracts (6).

The conclusion then follows by Theorem 2.1, like Theorem 3.3. □

Furthermore we are able to consider also superlinear growth condition, as next theorem shows.

Theorem 3.5. Assume that (H_A) , (H_1) and (H_2) hold. If

(H_6) there exists $\alpha \in L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ and a nondecreasing function $\beta : [0, +\infty) \rightarrow [0, +\infty)$ such that

$$\|F(t, c)\| \leq \alpha(t)\beta(\|c\|), \text{ for a.e. } t \in [0, b], \forall c \in X$$

and $L > 0$ such that

$$\frac{L}{M_1 \|x_0\| + \frac{M_1 q}{\Gamma(1 + q)} \left[\left(\frac{1 - q_1}{q - q_1} \right) b^{\frac{q - q_1}{1 - q_1}} \right]^{1 - q_1} \|\alpha\|_{\frac{1}{q_1}} \beta(L)} > 1, \tag{7}$$

then inclusion (1) has at least a mild solution.

Proof. It is sufficient to prove that the operator Γ maps the ball Q_L into itself. In fact, given any $z \in Q_L$ and $y \in \Gamma(z)$, we have

$$\begin{aligned} \|y_n\|_0 &\leq M_1 \|x_0\| + \frac{M_1 q}{\Gamma(1 + q)} \left[\left(\frac{1 - q_1}{q - q_1} \right) b^{\frac{q - q_1}{1 - q_1}} \right]^{1 - q_1} \left(\int_0^b |\alpha(\eta)|^{\frac{1}{q_1}} (\beta(\|z(\eta)\|))^{\frac{1}{q_1}} d\eta \right)^{q_1} \\ &\leq M_1 \|x_0\| + \frac{M_1 q}{\Gamma(1 + q)} \left[\left(\frac{1 - q_1}{q - q_1} \right) b^{\frac{q - q_1}{1 - q_1}} \right]^{1 - q_1} \|\alpha\|_{\frac{1}{q_1}} \beta(L) < L, \end{aligned}$$

The conclusion then follows by Theorem 2.1, like Theorem 3.3. □

4. Controllability results. In this section, we deals with the controllability for fractional semilinear differential inclusions (2) in a reflexive Banach space. We assume that

(H_B) The control function $u(\cdot)$ takes its value in \bar{U} , a Banach space of admissible control functions, where $\bar{U} = L^{\frac{1}{q_1}}([a, b]; U)$ for $q_1 \in (0, q)$ and U is a Banach space. $B : U \rightarrow X$ is a bounded linear operator, with

$$\|B\| \leq M_2. \quad (8)$$

Definition 4.1. A continuous function $x : [0, b] \rightarrow X$ is said to be a mild solution of system (2) if $x(0) = x_0$ and there exists $f \in L^{\frac{1}{q_1}}([0, b]; X)$ such that $f(t) \in F(t, x(t))$ on $t \in [0, b]$ and x satisfies the following integral equation

$$x(t) = \mathcal{F}(t)(x_0) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)Bu(s)ds + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f(s)ds.$$

We will consider the controllability problem for system (2), i.e., we will study conditions which guarantee the existence of a mild solution to problem (2) satisfying

$$x(b) = x_1, \quad (9)$$

where $x_1 \in X$ is a given point. A pair (x, u) consisting of a mild solution $x(\cdot)$ to (2) satisfying (9) and of the corresponding control $u(\cdot) \in L^{\frac{1}{q_1}}([0, b]; U)$ is called a *solution of the controllability problem*.

We assume the standard assumption that the corresponding linear problem (i.e., when $F(t, c) \equiv 0$) has a solution. More precisely, we suppose that

(H_W) The controllability operator $W : \bar{U} \rightarrow X$ given by

$$Wu = \int_0^b (b-s)^{q-1} \mathcal{S}(b-s)Bu(s)ds$$

has a bounded inverse which takes value in $\bar{U}/\ker(W)$ and there exists a positive constant $M_3 > 0$ such that

$$\|W^{-1}\| \leq M_3. \quad (10)$$

Let $q_1 \in (0, q)$. We denote with $S_1 : L^{\frac{1}{q_1}}([0, b]; X) \rightarrow C([0, b]; X)$ and $S_2 : L^{\frac{1}{q_1}}([0, b]; X) \rightarrow C([0, b]; X)$ the following integral operators

$$S_1 f(t) = \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f(s)ds, \quad \forall t \in [0, b], \quad (11)$$

$$S_2 f(t) = \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)BW^{-1} \left(- \int_0^b (t-s)^{q-1} \mathcal{S}(b-s)f(\eta)d\eta \right) (s)ds, \quad \forall t \in [0, b], \quad (12)$$

and we define the solution multioperator $\Pi : C([0, b]; X) \rightarrow C([0, b]; X)$ as

$$\Pi(p) = \left\{ x \in C([0, b]; X) : x(t) = \mathcal{F}(t)x_0 + S_1(f)(t) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)BW^{-1}(x_1 - \mathcal{F}(b)x_0)(s)ds + S_2 f(t), f \in \Lambda_p \right\}.$$

It is easy to verify that the fixed points of the multioperator Π are mild solutions of fractional differential system (2) and (9).

Proposition 4.1. *The operators S_1 and S_2 defined in (11) and (12) are linear and continuous.*

Proof. The linearity follows from the linearity of the integral operator and of the operators B, W^{-1} , we have

$$\begin{aligned} \|S_1 f(t)\| &= \left\| \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) f(s) ds \right\| \\ &\leq \frac{M_1 q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|f\|_{\frac{1}{q_1}}, \quad \forall t \in [0, b]. \end{aligned}$$

Moreover, by (8) and (10), we obtain

$$\begin{aligned} \|S_2 f(t)\| &= \left\| \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) B W^{-1} \left(- \int_0^b (t-s)^{q-1} \mathcal{S}(b-s) f(\eta) d\eta \right) (s) ds \right\| \\ &\leq \frac{q M_1 M_2}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left\| W^{-1} \left(- \int_0^b (b-s)^{q-1} \mathcal{S}(b-s) f(\eta) d\eta \right) (s) \right\| ds \\ &\leq \frac{q M_1 M_2}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \\ &\quad \times \left\| W^{-1} \left(- \int_0^b (b-s)^{q-1} \mathcal{S}(b-s) f(\eta) d\eta \right) \right\|_{\frac{1}{q_1}} \\ &= \frac{q M_1 M_2 M_3}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \left\| \frac{M_1 q}{\Gamma(1+q)} \int_0^b (b-s)^{q-1} f(\eta) d\eta \right\| \\ &= \frac{q^2 M_1^2 M_2 M_3}{\Gamma^2(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{2(1-q_1)} \|f\|_{\frac{1}{q_1}}, \end{aligned}$$

for $t \in [0, b]$. □

Fix $n \in \mathbb{N}$, we denote by $\Pi_n = \Pi|_{Q_n} : Q_n \rightarrow C([0, b]; X)$ the restriction of the multioperator Π on the set Q_n . We describe some properties of Π_n .

Proposition 4.2. *The multioperator Π_n has a weakly sequentially closed graph.*

Proof. Let $\{p_m\} \subset Q_n$ and $\{x_m\} \subset C([0, b]; X)$ satisfying $x_m \in \Pi_n(p_m)$ for all m and $p_m \rightarrow p, x_m \rightarrow x$ in $C([0, b]; X)$; we will prove that $x \in \Pi_n(p)$.

Since $p_m \in Q_n$ for all m and $p_m(t) \rightarrow p(t)$ for every $t \in [0, b]$, it follows that $\|p(t)\| \leq \liminf_{m \rightarrow \infty} \|p_m(t)\| \leq n$ for all t (see [5, Proposition III.5]). The fact that $x_m \in \Pi_n(p_m)$ means that there exists a sequence $\{f_m\}, f_m \in \Lambda_{p_m}$ such that for every $t \in [0, b]$,

$$\begin{aligned} x_m(t) &= \mathcal{S}(t)x_0 + S_1 f_m(t) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) B W^{-1} (x_1 - \mathcal{S}(b)x_0)(s) ds \\ &\quad + S_2 f_m(t). \end{aligned}$$

We observe that, according to (H₃), $\|f_m(t)\| \leq \mu_n(t)$ for a.e. t and every m , i.e., $\{f_m\}$ is bounded and uniformly integrable and $\{f_m(t)\}$ is bounded in X for a.e. $t \in [0, b]$. Hence, by the reflexivity of the space X and by the Dunford-Pettis theorem (see [7, p. 294]), we have the existence of a subsequence, denoted as the sequence, and a function g such that $f_m \rightarrow g$ in $L^{\frac{1}{q_1}}([0, b]; X)$.

In view of the linearity and continuity for S_i , we have $S_i f_m \rightarrow S_i g$ for $i = 1, 2$. Thus

$$\begin{aligned} x_m(t) &\rightarrow \mathcal{S}(t)x_0 + S_1 g(t) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) B W^{-1} (x_1 - \mathcal{S}(b)x_0)(s) ds \\ &\quad + S_2 g(t) = x_0(t), \quad \forall t \in [0, b]. \end{aligned}$$

implying, for the uniqueness of the weak limit in X , that $x_0(t) = x(t)$ for all $t \in [0, b]$.

Similar to the proof of Proposition 3.2, we can prove that $g(t) \in F(t, p(t))$ for a.e. $t \in [0, b]$. □

Proposition 4.3. *The multioperator Π_n is weakly compact.*

Proof. We first prove that $\Pi_n(Q_n)$ is weakly relatively sequentially compact.

Let $\{p_m\} \subset Q_n$ and $\{x_m\} \subset C([0, b]; X)$ satisfying $x_m \in \Pi_n(p_m)$ for all m . By the definition of the multioperator Π_n , there exist a sequence $\{f_m\}$, $f_m \in \Lambda_{p_m}$ such that

$$x_m(t) = \mathcal{T}(t)x_0 + S_1f_m(t) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)BW^{-1}(x_1 - \mathcal{T}(b)x_0)(s)ds + S_2f_m(t), \quad \forall t \in [0, b].$$

Further, as the reason for Proposition 4.2, we have that there exists a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^{\frac{1}{q_1}}([0, b]; X)$. Therefore

$$x_m(t) \rightharpoonup l(t) = \mathcal{T}(t)x_0 + S_1g(t) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)BW^{-1}(x_1 - \mathcal{T}(b)x_0)(s)ds + S_2g(t), \quad \forall t \in [0, b].$$

Furthermore, by the weak convergence of $\{f_m\}$, by (H_A), (8), (10), and the continuity of the operators S_1 and S_2 we have

$$\begin{aligned} \|x_m(t)\| \leq & M_1\|x_0\| + \frac{M_1q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|\mu_n\|_{\frac{1}{q_1}} \\ & + \frac{qM_1M_2M_3}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \\ & \times \left(\|x_1\| + M_1\|x_0\| + \frac{M_1q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \|\mu_n\|_{\frac{1}{q_1}} \right), \end{aligned}$$

for all $m \in N$ and $t \in [0, b]$. As the reason for Proposition 4.2, it is then easy to prove that $x_m \rightharpoonup l$ in $C([0, b]; X)$. Thus $\Pi_n(Q_n)$ is relatively weakly compact by Theorem 2.2. □

Proposition 4.4. *The multioperator Π_n has convex and weakly compact values.*

Proof. Fix $p \in Q_n$, since F is convex valued, from the linearity of the integral and of the operators $\mathcal{T}(t)$, $\mathcal{S}(t)$, B and W^{-1} , it follows that the set $\Pi_n(p)$ is convex. The weak compactness of $\Pi_n(p)$ follows by Propositions 4.2 and 4.3. □

We are able now to state the main results of this section.

Theorem 4.2. *Assume that (H_A), (H₁) (H₂), (H_B), (H_W) hold. If (H₄)' there exists a sequence of functions $\{\omega_n\} \subset L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ such that*

$$\sup_{\|c\| \leq n} \|F(t, c)\| \leq \omega_n(t)$$

for a.e. $t \in [0, b]$, $n \in \mathbb{N}$ with

$$\liminf_{n \rightarrow \infty} \frac{\|\omega_n\|_{\frac{1}{q_1}}}{n} = 0, \tag{13}$$

then controllability problem (2) and (9) has a solution.

Proof. We show that there exists $n \in \mathbb{N}$ such that the operator Π_n maps the ball Q_n into itself.

Assume, to the contrary, that there exist sequences $\{z_n\}, \{y_n\}$ such that $z_n \in Q_n, y_n \in \Pi_n(z_n)$ and $y_n \notin Q_n, \forall n \in \mathbb{N}$. Then there exists a sequence $\{f_n\} \subset L^{\frac{1}{q_1}}([0, b]; X), f_n(s) \in F(s, z_n(s))$ such that

$$y_n(t) = \mathcal{T}(t)x_0 + S_1 y_n(t) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) B W^{-1} (x_1 - \mathcal{T}(b)x_0)(s) ds + S_2 y_n(t), \quad \forall t \in [0, b].$$

As the reason for Proposition 4.3, we have

$$\begin{aligned} \|y_n\|_0 &\leq C_1 + C_2 \left(\int_0^b \|f_n(\eta)\|_{\frac{1}{q_1}} d\eta \right)^{q_1} \\ &\leq C_1 + C_2 \left(\int_0^b |\omega_n(\eta)|_{\frac{1}{q_1}} d\eta \right)^{q_1}, \end{aligned}$$

where

$$C_1 = M_1 \|x_0\| + \frac{qM_1M_2M_3}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} (\|x_1\| + M_1 \|x_0\|), \tag{14}$$

$$C_2 = \frac{M_1q}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \left(1 + \frac{qM_1M_2M_3}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \right). \tag{15}$$

Then

$$1 < \frac{\|y\|_0}{n} \leq \frac{C_1}{n} + C_2 \frac{\|\omega_n\|_{\frac{1}{q_1}}}{n}, \quad n \in \mathbb{N},$$

which contracts (13).

Fix $n \in \mathbb{N}$ such that $\Pi_n(Q_n) \subseteq Q_n$. By Proposition 4.3 the set $V_n = \overline{\Pi_n(Q_n)}^w$ is weakly compact. Let now $W_n = \overline{co}(V_n)$, where $\overline{co}(V_n)$ denotes the closed convex hull of V_n . By Theorem 2.3, W_n is a weakly compact set. Moreover from the fact that $\Pi_n(Q_n) \subset Q_n$ and that Q_n is a convex closed set we have that $W_n \subset Q_n$ and hence

$$\Pi_n(W_n) = \Pi_n(\overline{co}(\Pi_n(Q_n))) \subseteq \Pi_n(Q_n) \subseteq \overline{\Pi_n(Q_n)}^w = V_n \subset W_n.$$

In view of Proposition 4.2, Π_n has a weakly sequentially closed graph. Thus from Theorem 2.1, the system (1) has a solution. The proof is now completed. \square

We are able to prove the controllability results also under less restrictive growth assumptions, for instance sublinearity.

Theorem 4.3. *Assume that (H_A), (H₁), (H₂), (H_B), (H_W) hold. If*

(H₅)' there exists $\alpha \in L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ such that

$$\|F(t, c)\| \leq \alpha(t)(1 + \|c\|), \text{ for a.e. } t \in [0, b], \forall c \in X$$

and

$$\frac{M_1q}{\Gamma(1+q)} \|\alpha\|_{\frac{1}{q_1}} \left(1 + \frac{qM_1M_2M_3}{\Gamma(1+q)} \left[\left(\frac{1-q_1}{q-q_1} \right) b^{\frac{q-q_1}{1-q_1}} \right]^{1-q_1} \right) < 1, \tag{16}$$

then controllability problem (2) and (9) has a solution.

Proof. As the reason for Theorem 4.2. Assume that there exist $\{z_n\}, \{y_n\}$ such that $z_n \in Q_n, y_n \in \Pi_n(z_n)$ and $y_n \notin Q_n, \forall n \in \mathbb{N}$, we get

$$\begin{aligned} n < \|y_n\|_0 &\leq C_1 + C_2 \left(\int_0^b |\alpha(\eta)|^{\frac{1}{q_1}} (1 + \|z_n(\eta)\|)^{\frac{1}{q_1}} d\eta \right)^{q_1} \\ &\leq C_1 + C_2(1+n)\|\alpha\|_{\frac{1}{q_1}}, \quad n \in \mathbb{N}, \end{aligned}$$

which contracts (16).

The conclusion then follows by Theorem 2.1, like Theorem 4.6. □

Furthermore we are able to consider also superlinear growth condition, as next theorem shows.

Theorem 4.4. *Assume that $(H_A), (H_1), (H_2), (H_B)$ and (H_W) hold. If*

$(H_6)'$ *there exists $\alpha \in L^{\frac{1}{q_1}}([0, b]; \mathbb{R}_+)$ and a nondecreasing function $\beta : [0, +\infty) \rightarrow [0, +\infty)$ such that*

$$\|F(t, c)\| \leq \alpha(t)\beta(\|c\|), \text{ for a.e. } t \in [0, b], \forall c \in X,$$

and $L > 0$ such that

$$\frac{L}{C_1 + C_2\|\alpha\|_{\frac{1}{q_1}}\beta(L)} > 1,$$

where C_1 and C_2 are the positive constants defined in (14) and (15), then the controllability problem (2) and (9) has a solution.

Proof. It is sufficient to prove that the operator Π maps the ball Q_L into itself. In fact, given any $z \in Q_L$ and $y \in \Pi(z)$, we have

$$\|y_n\|_0 \leq C_1 + C_2 \left(\int_0^b |\alpha(\eta)|^{\frac{1}{q_1}} (\beta(\|z_n(\eta)\|))^{\frac{1}{q_1}} d\eta \right)^{q_1} \leq C_1 + C_2\|\alpha\|_{\frac{1}{q_1}}\beta(L) < L,$$

The conclusion then follows by Theorem 2.1, like in Theorem 4.6. □

5. Examples. As applications we give two examples to illustrate our theoretical results.

Example 5.1. Consider the following fractional differential inclusion of the form

$$\begin{cases} \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}}x(t, y) \in x_{yy}(t, y) + P(t, x(t, y)), \quad t \in J = [0, 1], \\ x(t, 0) = x(t, 1) = 0, \\ x(0, y) = 0, \quad 0 < y < 1, \end{cases} \tag{17}$$

where $P : J \times X \rightarrow 2^X \setminus \{\Omega\}$.

Let $X = L^2(0, 1)$ and define $A : D(A) \subset X \rightarrow X$ by $Ax = x_{yy}$, where domain $D(A)$ is given by $\{x \in X : x, x_y \text{ are absolutely continuous, } x_{yy} \in X, x(t, 0) = x(t, 1) = 0\}$. A can be written as

$$Ax = \sum_{n=1}^{\infty} n^2 \langle x, x_n \rangle x_n, \quad x \in D(A),$$

where $x_n(y) = \sqrt{2} \sin ny, n = 1, 2, \dots$ is the orthonormal set of eigenfunctions of A . Moreover, for any $x \in X$ we have

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2t} \langle x, x_n \rangle x_n.$$

Clearly, A generates the above strongly continuous semigroup $T(t)$ on Y , here, $T(t)$ satisfies the hypotheses (H_A) .

Define $x(t)(y) = x(t, y)$, $F(t, x(t))(y) = P(t, x(t, y))$.

Here $F : J \times X \rightarrow 2^X \setminus \{\Omega\}$. With the choice of A and F , system (17) can be rewritten as

$$\begin{cases} {}^C D_t^q x(t) \in Ax(t) + F(t, x(t)), & t \in J, \quad q = \frac{2}{3} \in (0, 1), \\ x(0) = 0. \end{cases}$$

Assume that F satisfies (H_1) , (H_2) and one of (H_4) - (H_6) . Thus all the conditions of Theorem 3.3, 3.4 and 3.5 are satisfied. Hence, system (17) has at least a mild solution on J .

Example 5.2. Let $X = U = L^2(0, 1)$. Consider the following fractional differential inclusion with control

$$\begin{cases} {}^C D_t^{\frac{3}{4}} x(t, y) \in x_{yy}(t, y) + F(t, x(t, y)) + Bu(t, y), & y \in [0, 1], t \in J = [0, 1], \\ x(t, 0) = x(t, 1) = 0, & t \geq 0, \\ x(0, y) = \phi(y), & 0 \leq y \leq 1. \end{cases} \quad (18)$$

Similarly, A and $T(t)$ are defined as in Example 1. Then the operator $\mathcal{S}(\cdot)$ can be written as

$$\mathcal{S}(t) = \frac{3}{4} \int_0^\infty \theta \xi_{\frac{3}{4}}(\theta) T(t^{\frac{3}{4}} \theta) d\theta.$$

Define

$$Bu = \sum_{n=1}^\infty e^{-\frac{1}{1+n^2}} \langle u, x_n \rangle x_n,$$

and $W : \bar{U} \rightarrow X$ as follows:

$$Wu := \int_0^1 (1-s)^{-\frac{1}{4}} \mathcal{S}(1-s) Bu(s, y) ds.$$

Since

$$\|u\| = \sqrt{\sum_{n=1}^\infty \langle u, x_n \rangle^2},$$

for $u \in U$, we have

$$\|Bu\| = \sqrt{\sum_{n=1}^\infty e^{-\frac{2}{1+n^2}} \langle u, x_n \rangle^2} \leq \sqrt{\sum_{n=1}^\infty \langle u, x_n \rangle^2} = \|u\|,$$

which implies $\|B\| \leq 1$. Hence, (H_B) holds.

Since $q = \frac{3}{4} > \frac{1}{2} = q_1$, we take $\bar{U} = L^{\frac{1}{q_1}}(J, U) = L^2(J, U)$. Next, let $u(s, y) = x(y) \in U$. Then

$$\begin{aligned} Wu &= \int_0^1 (1-s)^{-\frac{1}{4}} \frac{3}{4} \int_0^\infty \theta \xi_{\frac{3}{4}}(\theta) T((1-s)^{\frac{3}{4}} \theta) Bx d\theta ds \\ &= \int_0^1 (1-s)^{-\frac{1}{4}} \frac{3}{4} \int_0^\infty \theta \xi_{\frac{3}{4}}(\theta) \sum_{n=1}^\infty e^{-\frac{1}{1+n^2}} e^{-n^2(1-s)^{\frac{3}{4}} \theta} \langle x, x_n \rangle x_n d\theta ds \\ &= \int_0^\infty \xi_{\frac{3}{4}}(\theta) \sum_{n=1}^\infty e^{-\frac{1}{1+n^2}} \int_0^1 \frac{3}{4} \theta (1-s)^{-\frac{1}{4}} e^{-n^2(1-s)^{\frac{3}{4}} \theta} ds \langle x, x_n \rangle x_n d\theta \end{aligned}$$

$$\begin{aligned} &= \int_0^\infty \xi_{\frac{3}{4}}(\theta) \sum_{n=1}^\infty \int_0^1 n^{-2} e^{-\frac{1}{1+n^2}} \frac{d}{ds} \left(e^{-n^2(1-s)^{\frac{3}{4}}\theta} \right) ds \langle x, x_n \rangle x_n d\theta \\ &= \int_0^\infty \xi_{\frac{3}{4}}(\theta) \sum_{n=1}^\infty n^{-2} e^{-\frac{1}{1+n^2}} \left(1 - e^{-n^2\theta} \right) \langle x, x_n \rangle x_n d\theta \\ &= \sum_{n=1}^\infty n^{-2} e^{-\frac{1}{1+n^2}} \left(1 - \mathbb{E}_{\frac{3}{4}}(-n^2) \right) \langle x, x_n \rangle x_n, \end{aligned}$$

where

$$\mathbb{E}_{\frac{3}{4}}(-n^2) := \int_0^\infty e^{-n^2\theta} \xi_{\frac{3}{4}}(\theta) d\theta$$

is a Mittag-Leffler function (for more details, see [14]). Note that $0 < 1 - e^{-\theta} < 1 - e^{-n^2\theta} < 1$ for any $\theta > 0$. So $1 - \mathbb{E}_{\frac{3}{4}}(-1) \leq 1 - \mathbb{E}_{\frac{3}{4}}(-n^2) \leq 1$. From the above computations we know that W is surjective. So we may define a right inverse $W^{-1} : X \rightarrow \bar{U}$ by

$$(W^{-1}x)(t, y) := \sum_{n=1}^\infty \frac{n^2 e^{\frac{1}{1+n^2}} \langle x, x_n \rangle x_n}{1 - \mathbb{E}_{\frac{3}{4}}(-n^2)},$$

for $x = \sum_{n=1}^\infty \langle x, x_n \rangle x_n$. Since

$$\|x\|_{D(A)} := \|Ax\| := \sqrt{\sum_{n=1}^\infty n^2 \langle u, x_n \rangle^2},$$

for $x \in D(A)$, we derive

$$\begin{aligned} \|(W^{-1}x)(t, \cdot)\| &= \sqrt{\sum_{n=1}^\infty \frac{n^4 e^{\frac{1}{1+n^2}} \langle x, x_n \rangle^2}{\left(1 - \mathbb{E}_{\frac{3}{4}}(-n^2)\right)^2}} \\ &\leq \frac{e^{\frac{1}{2}}}{1 - \mathbb{E}_{\frac{3}{4}}(-1)} \sqrt{\sum_{n=1}^\infty n^2 \langle x, x_n \rangle^2} \\ &= \frac{e^{\frac{1}{2}}}{1 - \mathbb{E}_{\frac{3}{4}}(-1)} \|x\|_{D(A)}. \end{aligned}$$

Note that $W^{-1}x$ is independent of $t \in J$. Consequently, we obtain

$$\|W^{-1}\| \leq \frac{e^{\frac{1}{2}}}{1 - \mathbb{E}_{\frac{3}{4}}(-1)} =: M_3.$$

Hence, condition (H_W) is satisfied.

Next, we suppose that $F : J \times X \rightarrow 2^X$ satisfies (H_1) , (H_2) and $(H_4)'$. Now, the system (18) can be abstracted as

$$\begin{cases} {}_0^C D_t^{\frac{3}{4}} x(t) \in Ax(t) + F(t, x(t)) + Bu(t), \quad t \in J, \\ x(0) = \phi. \end{cases}$$

Clearly, all the assumptions in Theorem 4.2 are satisfied. Then the system (18) is controllable on J .

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