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Cyclic-Prešić–Ćirić operators in metric-like spaces and fixed point theorems

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Abstract. In this paper, we define the cyclic-Prešić–Ćirić operators in metric-like spaces and prove some fixed point results for such operators. Our results generalize that of S.B. Prešić [Sur une classe d'inéquations aux différences finite et sur la convergence de certaines suites, *Publications de l'Institut Mathématique* (*N.S.*), 5(19):75–78, 1965] and several later results. An example is given which shows that the results proved herein are the proper generalizations of existing ones.

Keywords: Prešić type mapping, cyclic mapping, metric-like space, partial metric space, fixed point.

1 Introduction

The celebrated Banach contraction principle is a fundamental piece in several branches of functional analysis as well as in many applications. Due to its relevance, generalizations of Banach's fixed point theorem have been studied by many authors.

Let (X, d) be a metric space, A and B are two nonempty closed subsets of X. A mapping $T : X \to X$ is called a Banach contraction if the following condition is satisfied:

$$d(Tx, Ty) \leqslant \lambda d(x, y) \tag{1}$$

for all pairs $(x, y) \in X \times X$, where $\lambda \in [0, 1)$. Banach contraction principle states that every Banach contraction on a complete metric space has a unique fixed point $x^* \in X$, that is, $Tx^* = x^*$. An interesting generalization of Banach contraction principle was obtained by Kirk et al. [8]. They introduced the class of mappings $T : A \cup B \to A \cup B$ satisfying the following conditions:

1. $T(A) \subseteq B$ and $T(B) \subseteq A$;
2. $d(Tx, Ty) \leqslant \lambda d(x, y)$ for all $x \in A$ and $y \in B$, where $\lambda \in [0, 1)$.

They named such contractive conditions, *cyclical contractive conditions*. They also obtained a unique fixed point of mappings satisfying cyclical contractive conditions. The mappings satisfying the above conditions are called cyclic contractions. Indeed, the cyclic contractions may not be a contraction on the whole space, that is, may not satisfy the contractive condition (1) for all pairs $(x, y) \in X \times X$ and need not be continuous.

The cyclic representation of a set was defined as follows:

Definition 1. (See [23].) Let X be a nonempty set, m a positive integer and $f : X \to X$ be a mapping. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f if:

1. A_i, $i = 1, \dots, m$, are non-empty subsets of X;
2. $f(A_1) \subset A_2,\ f(A_2) \subset A_3, \dots,\ f(A_{m-1}) \subset A_m,\ f(A_m) \subset A_1$.

Theorem 1. (*See* [8].) *Let (X, d) be a complete metric space and let $A_1, A_2, \dots, A_m$ be nonempty closed subsets of X (also assume that $A_{m+1} = A_1$). Suppose that $f : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ is an operator such that*:

1. $f(A_i) \subseteq A_{i+1}$ *for all* $i \in \{1, 2, \dots, m\}$;
2. *There exists $k \in [0, 1)$ such that*

$$d(fx, fy) \leqslant kd(x, y)$$

for all $x \in A_i$, $y \in A_{i+1}$, $i \in \{1, 2, \dots, m\}$. Then f has exactly one fixed point.

Following [23] and [8], a number of fixed point theorems on cyclic contractions have appeared (see, e.g., [1, 5, 6, 11, 13, 17, 18, 19]). In the recent paper of Nashine et al. [12], various types of cyclic contractions in the setting of partial metric spaces can be seen.

Remark 1. (See [23].) If $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f, then $\mathcal{F}(f) \subset \bigcap_{i=1}^{m} A_i$, where $\mathcal{F}(f)$ is the set of all fixed points of f.

Let $f : X^k \to X$, where k is a positive integer. A point $x \in X$ is called a fixed point of f if $x = f(x, \dots, x)$. Consider the kth order nonlinear difference equation

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), \quad n = 1, 2, \dots, \tag{2}$$

with the initial values $x_1, \dots, x_k \in X$.

Equation (2) can be studied by means of fixed point theory in view of the fact that $x \in X$ is a solution of (2) if and only if x is a fixed point of the self-mapping $F : X \to X$ given by

$$F(x) = f(x, \dots, x) \quad \text{for all } x \in X.$$

One of the most important result in this direction is obtained by Prešić [20, 21]. Prešić proved the following theorem for the mappings defined on the product spaces.

Theorem 2. *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping satisfying*

$$d\big(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})\big) \leqslant \sum_{i=1}^{k} \alpha_i d(x_i, x_{i+1}) \tag{3}$$

for every $x_1, x_2, \dots, x_{k+1} \in X$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i < 1$. Then there exists a unique point $x \in X$ such that $f(x, x, \dots, x) = x$. Moreover, if $x_1, x_2, \dots, x_k$ are arbitrary points in X, then the sequence $\{x_n\}$ generated by

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), \tag{4}$$

is convergent and $\lim x_n = f(\lim x_n, \lim x_n, \dots, \lim x_n)$.

An operator satisfying (3) is called a Prešić type operator. Prešić type operators have applications in solving the nonlinear difference equations and in the convergence of sequences, for example, see [3, 7, 20, 21].

Inspired with the results in Theorem 2, Ćirić and Prešić [4] proved the following theorem:

Theorem 3. *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping satisfying the following contractive type condition*:

$$d\big(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})\big) \leqslant \lambda \max\big\{d(x_i, x_{i+1})\colon 1 \leqslant i \leqslant k\big\}, \tag{5}$$

where $\lambda \in [0, 1)$ is constant and $x_1, x_2, \dots, x_{k+1}$ are arbitrary points in X. Then there exists a point x in X such that $f(x, x, \dots, x) = x$. Moreover, if $x_1, x_2, \dots, x_k$ are arbitrary points in X and, for $n \in \mathbb{N}$, $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = f(\lim x_n, \lim x_n, \dots, \lim x_n)$. If, in addition, we suppose that on diagonal $\Delta \subset X^k$, $d(f(u, u, \dots, u), f(v, v, \dots, v)) < d(u, v)$ holds for $u, v \in X$ with $u \neq v$, then x is unique fixed point satisfying $x = f(x, x, \dots, x)$.

An operator satisfying (5) is called a Prešić–Ćirić operator.

In the recent years, many authors generalize and extend the result of Prešić in different directions, see, e.g., [4,14,15,16,24,25,26,27,29,30,31]. Very recently, Shukla and Abbas [26] introduced the notion of cyclic-Prešić operators which turns into a generalization of the concept of the cyclic contractions.

In [26], the notion of cyclic representation with respect to an operator $f : X^k \to X$ and cyclic-Prešić operator are defined as follows:

Definition 2. (See [26].) Let X be any nonempty set, k a positive integer, $f : X^k \to X$ an operator and $A_1, A_2, \dots, A_m$ be subsets of X. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f if:

1. A_i, $i = 1, 2, \dots, m$, are nonempty sets;
2. $f(A_1 \times A_2 \times \cdots \times A_k) \subseteq A_{k+1}$, $f(A_2 \times A_3 \times \cdots \times A_{k+1}) \subseteq A_{k+2}, \dots,$ $f(A_i \times A_{i+1} \times \cdots \times A_{i+k-1}) \subseteq A_{i+k}, \dots$, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$.

If we take $k = 1$, then the above definition reduces to well known cyclic representation of set X with respect to an operator $f : X \to X$.

Definition 3. (See [26].) Let $A_1, A_2, \ldots, A_m$ be subsets of a metric space (X, d), k a positive integer, and $Y = \bigcup_{i=1}^{m} A_i$. An operator $f : Y^k \to Y$ is called a cyclic-Prešić operator if the following conditions are met:

1. $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;
2. There exist nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i < 1$ and

$$d\big(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})\big) \leqslant \sum_{i=1}^{k} \alpha_i d(x_i, x_{i+1}) \tag{6}$$

   for all $x_1 \in A_i, x_2 \in A_{i+1}, \ldots, x_{k+1} \in A_{i+k}$, $i = 1, 2, \ldots, m$, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$.

If we take $k = 1$, then the above definition reduces to well known cyclic contraction, therefore, the concept of cyclic-Prešić operator is more general than the cyclic contractions.

On the other hands, Matthews [9] introduced the notion of partial metric spaces with an interesting property that the points in such spaces may have a nonzero self distance. This notion is further generalized by Harandi [2] by introducing the notion of metric-like spaces. In metric-like spaces, the assumption of smallest self distance of partial metric spaces was removed and the triangular inequality of partial metric was replaced by a weaker one. Further, Shukla et al. [28] introduced the notion of 0-σ-complete metric-like spaces and generalized the results of Harandi [2].

In this paper, we introduce the cyclic-Prešić–Ćirić operators in metric-like spaces as a generalization of earlier cyclic contraction condition on product spaces. We develop some new fixed point results for such cyclic contraction mappings in 0-σ-complete metric-like spaces. Our results are the extensions or refinements of fixed point theorems of Kirk et al. [8], Prešić [21], Ćirić and Prešić [4], Shukla and Fisher [27], Shukla and Abbas [26] and several other known results of the literature. Examples are given to support the usability of the results and to show that these extensions are proper.

2 Preliminaries

First, we recall some definitions and properties about the partial metric and metric-like spaces.

Definition 4. (See [9].) A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that, for all $x, y, z \in X$:

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leqslant p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that if $p(x, y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Also every metric space is a partial metric space with zero self distance. Each partial metric on X generates a T_0 topology τ_p on X which has a base the family of open p-balls $\{B_p(x, \epsilon)\colon\ x \in X,\ \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X\colon\ p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.
(ii) A sequence $\{x_n\}$ in (X, p) is called Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.
(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(iv) A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X is converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = 0$.

For more details on partial metric spaces, see [9, 22].

Definition 5. (See [2].) A metric-like on a nonempty set X is a function $\sigma : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

1. $\sigma(x, y) = 0$ implies $x = y$;
2. $\sigma(x, y) = \sigma(y, x)$;
3. $\sigma(x, y) \leqslant \sigma(x, z) + \sigma(z, y)$.

A metric-like space is a pair (X, σ) such that X is a nonempty set and σ is a metric-like on X. Note that a metric-like satisfies all the conditions of metric except that $\sigma(x, x)$ may be positive for $x \in X$. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x, \epsilon) = \bigl\{y \in X\colon\ \bigl|\sigma(x, y) - \sigma(x, x)\bigr| < \epsilon\bigr\} \quad \text{for all } x \in X \text{ and } \epsilon > 0.$$

A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$. Sequence $\{x_n\}$ is said to be σ-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. The metric-like space (X, σ) is called complete if, for each σ-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{m,n\to\infty} \sigma(x_n, x_m).$$

It is obvious that every metric space is partial metric space (for definition and properties of partial metric spaces, see [9]) and every partial metric space is a metric-like space, but the converse may not be true.

Example 1. (See [2].) Let $X = \{0, 1\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2 & \text{if } x = y = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Then (X, σ) is metric-like space, but it is neither a metric space nor a partial metric space.

Example 2. Let $X = \mathbb{R}^+$, $a > 0$, $b \geqslant 0$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = a(x + y) + b \quad \text{for all } x, y \in X.$$

Then (X, σ) is metric-like space, but it is neither a metric space nor a partial metric space, because, for $x > 0$, $\sigma(x, x) = 2ax + b > 0$ and $\sigma(1, 1) = 2a + b > \sigma(1, 0) = a + b$.

Definition 6. (See [28].) Let (X, σ) be a metric-like space. A sequence $\{x_n\}$ in X is called 0-σ-Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$. The space (X, σ) is said to be 0-σ-complete if every 0-σ-Cauchy sequence in X converges with respect to τ_σ to a point $x \in X$ such that $\sigma(x, x) = 0$.

Note that the limit of a convergent sequence in a metric like space (X, σ) may not be unique. A subset $A \subseteq X$ is said to be closed if every limit of a convergent sequence in A is in A. It is obvious that every 0-σ-Cauchy sequence is σ-Cauchy sequence in (X, σ) and every σ-complete metric-like space is 0-σ-complete. Also, every 0-complete partial metric space (for details, see [28] and the references therein) is 0-σ-complete metric-like space. The following example shows that the converse assertions of these facts do not hold.

Example 3. Let $X = [0, \infty) \cap \mathbb{Q}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2x & \text{if } x = y; \\ \max\{x, y\} & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then (X, σ) is a metric-like space. Note that (X, σ) is not a partial metric space as $\sigma(1, 1) = 2 \nleq \sigma(1, 0) = 1$ (for details, see [21]). Now, it is easy to see that (X, σ) is a 0-σ-complete metric-like space, while it is not a σ-complete metric-like space.

The following definition will be needed in the sequel and can be found in [26].

Definition 7. (See [21].) Let X be a nonempty set and $A_1, A_2, \dots, A_m$ be nonempty subsets of X. A sequence $\{x_n\}$ in X is called m-cyclic sequence if:

1. There exists $i \in \{1, 2, \dots, m\}$ such that $x_1 \in A_i$;
2. $x_n \in A_i$ for some $n \in \mathbb{N}$, $i \in \{1, 2, \dots, m\}$, implies that $x_{n+1} \in A_{i+1}$, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$.

Now we can state our main results.

3 Cyclic-Prešić–Ćirić operators in metric-like spaces

In this section, we will prove some fixed point theorems for self-mappings defined on a 0-σ-complete metric-like space and satisfying certain cyclic-Prešić–Ćirić operator condition. To achieve our goal, we introduce a new class of cyclic operators.

Definition 8. Let $A_1, A_2, \dots, A_m$ be nonempty subsets of a metric-like space (X, σ), k a positive integer and $Y = \bigcup_{i=1}^m A_i$. An operator $f : Y^k \to Y$ is called a cyclic-Prešić–Ćirić operator if:

1. $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;
2. There exists $\lambda \in [0, 1)$ such that

$$\begin{aligned}\sigma\bigl(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})\bigr) \\ \leqslant \lambda \max\bigl\{\sigma(x_i, x_{i+1})\colon 1 \leqslant i \leqslant k\bigr\}\end{aligned} \tag{7}$$

   for all $x_1 \in A_i$, $x_2 \in A_{i+1}, \dots, x_{k+1} \in A_{i+k}$, $i = 1, 2, \dots, m$, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$.

To prove our main result, we need the following form of proposition of [26] in metric-like spaces. The proof is similar to the metric case, therefore, we omit the proof.

Proposition 1. *Let $A_1, A_2, \dots, A_m$ be closed subsets of a 0-σ-complete metric-like space (X, σ). Suppose that $\{x_n\}$ is an m-cyclic sequence in $Y = \bigcup_{i=1}^m A_i$. If $\{x_n\}$ converges to some $u \in X$, then $u \in \bigcap_{i=1}^m A_i$.*

Our main result is the following:

Theorem 4. *Let $A_1, A_2, \dots, A_m$ be closed subsets of a 0-σ-complete metric-like space (X, σ), k a positive integer, and $Y = \bigcup_{i=1}^m A_i$. Let $f : Y^k \to Y$ be a cyclic-Prešić–Ćirić operator. Then $\bigcap_{i=1}^m A_i \neq \emptyset$ and f has a fixed point $u \in \bigcap_{i=1}^m A_i$ such that $\sigma(u, u) = 0$. Moreover, if $i \in \{1, 2, \dots, m\}$ and $x_1 \in A_i$, $x_2 \in A_{i+1}, \dots, x_k \in A_{i+k-1}$ be arbitrary points, then the sequence $\{x_n\}$ defined by*

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}) \quad \text{for all } n \in \mathbb{N}$$

is an m-cyclic sequence and converges to a fixed point of f.

Proof. Let $i \in \{1, 2, \dots, m\}$ and $x_1 \in A_i$, $x_2 \in A_{i+1}, \dots, x_k \in A_{i+k-1}$, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$. We define a sequence $\{x_n\}$ in Y by

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}) \quad \text{for all } n \in \mathbb{N}.$$

As $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f, so we have $x_n \in A_{i+n-1}$ for all $n \in \mathbb{N}$ and so the sequence $\{x_n\}$ is an m-cyclic sequence. Now we shall show that the sequence $\{x_n\}$ is a 0-σ-Cauchy sequence.

For notational convenience, let $\sigma_n = \sigma(x_n, x_{n+1})$. We shall prove by induction that

$$\sigma_n \leqslant \mu\theta^n, \tag{8}$$

is true for each $n \in \mathbb{N}$, where $\theta = \lambda^{1/k}$ and $\mu = \max\{\sigma_1/\theta, \sigma_2/\theta^2, \ldots, \sigma_k/\theta^k\}$. By definition of μ, (8) is obviously true for $n \in \{1, 2, \ldots, k\}$. Suppose the following k inequalities

$$\sigma_n \leqslant \mu\theta^n, \quad \sigma_{n+1} \leqslant \mu\theta^{n+1}, \ldots, \quad \sigma_{n+k-1} \leqslant \mu\theta^{n+k-1}$$

hold. Since $x_n \in A_{i+n-1}$ for all $n \in \mathbb{N}$, therefore, we obtain from (7) that

$$\begin{aligned}
\sigma_{n+k} &= \sigma(x_{n+k}, x_{n+k+1}) \\
&= \sigma\bigl(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k})\bigr) \\
&\leqslant \lambda \max\bigl\{\sigma(x_n, x_{n+1}), \sigma(x_{n+1}, x_{n+2}), \ldots, \sigma(x_{n+k-1}, x_{n+k})\bigr\} \\
&= \lambda \max\{\sigma_n, \sigma_{n+1}, \ldots, \sigma_{n+k-1}\} \\
&\leqslant \lambda \max\bigl\{\mu\theta^n, \mu\theta^{n+1}, \ldots, \mu\theta^{n+k-1}\bigr\} \\
&= \lambda\mu\theta^n \quad \bigl(\text{as } \theta = \lambda^{1/k} < 1\bigr) \\
&= \mu\theta^{n+k}.
\end{aligned}$$

Hence, by induction, (8) is true for each $n \in \mathbb{N}$.

Now for $n, m \in \mathbb{N}$ with $m > n$, we obtain from (8) that

$$\begin{aligned}
\sigma(x_n, x_m) &\leqslant \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \cdots + \sigma(x_{m-1}, x_m) \\
&= \sigma_n + \sigma_{n+1} + \cdots + \sigma_{m-1} \\
&\leqslant \mu\theta^n + \mu\theta^{n+1} + \mu\theta^{n+2} + \cdots + \mu\theta^{m-1} \\
&\leqslant \mu\theta^n\bigl[1 + \theta + \theta^2 + \cdots\bigr] \\
&= \frac{\mu}{1-\theta}\theta^n.
\end{aligned}$$

As $\theta < 1$, we have $\lim_{m,n\to\infty} \sigma(x_n, x_m) = 0$. Hence, $\{x_n\}$ is a 0-σ-Cauchy sequence. By 0-σ-completeness of (X, σ), there exists $u \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, u) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = \sigma(u, u) = 0. \tag{9}$$

Thus, $\{x_n\}$ is an m-cyclic sequence in $Y = \bigcup_{i=1}^m A_i$ which converges to $u \in X$. Therefore, by Proposition 1, we have $u \in \bigcap_{i=1}^k A_i$ and so $\bigcap_{i=1}^k A_i \neq \emptyset$. Now we shall show that u is a fixed point of f.

For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sigma\bigl(f(u, u, \ldots, u), u\bigr) \\
&\quad\leqslant \sigma\bigl(f(u, u, \ldots, u), x_{n+k}\bigr) + \sigma(x_{n+k}, u) \\
&\quad= \sigma\bigl(f(u, u, \ldots, u), f(x_n, x_{n+1}, \ldots, x_{n+k-1})\bigr) + \sigma(x_{n+k}, u) \\
&\quad\leqslant \sigma\bigl(f(u, u, \ldots, u), f(u, \ldots, u, x_n)\bigr) \\
&\qquad + \sigma\bigl(f(u, \ldots, u, x_n), f(u, \ldots, u, x_n, x_{n+1})\bigr) \\
&\qquad + \cdots + \sigma\bigl(f(u, x_n, \ldots, x_{n+k-2}), f(x_n, x_{n+1}, \ldots, x_{n+k-1})\bigr) \\
&\qquad + \sigma(x_{n+k}, u).
\end{aligned}$$

Since $u \in \bigcap_{i=1}^{m} A_i$ and for every $n \in \mathbb{N}$, there exists $i \in \{1, 2, \dots, m\}$ such that $x_n \in A_i$, $x_{n+1} \in A_{i+1}, \dots, x_{n+k-1} \in A_{n+k-1}$, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$. Therefore, we obtain from (7) and previous inequality that

$$\begin{aligned}
&\sigma\bigl(f(u,u,\dots,u),u\bigr)\\
&\quad\leqslant \lambda \max\bigl\{\sigma(u,u),\dots,\sigma(u,u),\sigma(u,x_n)\bigr\}\\
&\qquad + \lambda \max\bigl\{\sigma(u,u),\dots,\sigma(u,u),\sigma(x_n,x_{n+1}),\sigma(u,x_n)\bigr\}\\
&\qquad + \cdots + \lambda \max\bigl\{\sigma(u,x_n),\sigma(x_n,x_{n+1}),\dots,\sigma(x_{n+k-2},x_{n+k-1})\bigr\}\\
&\qquad + \sigma(x_{n+k},u),
\end{aligned} \tag{10}$$

that is,

$$\begin{aligned}
&\sigma\bigl(f(u,u,\dots,u),u\bigr)\\
&\quad\leqslant \lambda\sigma(u,x_n) + \lambda \max\bigl\{\sigma_n,\sigma(u,x_n)\bigr\}\\
&\qquad + \cdots + \lambda \max\bigl\{\sigma(u,x_n),\sigma_n,\dots,\sigma_{n+k-2}\bigr\} + \sigma(x_{n+k},u).
\end{aligned} \tag{11}$$

Using (9) and passing to the limit as $n \to \infty$ in (11), we obtain that $\sigma(f(u,u,\dots,u),u) = 0$, that is, $f(u,u,\dots,u) = u$. Hence, u is a fixed point of f. $\square$

Next, we give a simple example which illustrate the above theorem and shows that the above theorem is a proper generalization of the main results of Shukla and Fisher [27], Ćirić and Prešić [4] and Shukla and Abbas [26].

Example 4. Let $X = \{0, 1, 2\}$ and define a metric-like σ on X by:

$$\sigma(x,y) = \sigma(y,x) \quad \text{for all } x, y \in X,$$

$$\begin{array}{lll}
\sigma(0,0) = 0, & \sigma(0,1) = 2, & \sigma(0,2) = 4,\\
\sigma(1,1) = 1, & \sigma(1,2) = 3, & \sigma(2,2) = 2.
\end{array}$$

Then (X, σ) is a 0-σ-complete metric-like space. For $k = m = 2$, let $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$ and $Y = A_1 \cup A_2$. Then A_1, A_2 are nonempty, closed subsets of X. Define $f : X^2 \to X$ by:

$$f(x,y) = \min\{x,y\} \quad \text{when } x \neq y,$$

$$f(0,0) = 0, \qquad f(1,1) = 0, \qquad f(2,2) = 2.$$

Then f is not a Prešić–Ćirić operator, neither in the usual metric space $(X, |\cdot|)$ nor in the metric-like space (X, σ). Indeed, at the points $x_1 = x_2 = 1$, $x_3 = 2$, we have $|f(x_1,x_2) - f(x_2,x_3)| = 1$ and $\max\{|x_1 - x_2|, |x_2 - x_3|\} = 1$. Therefore, there exists no $\lambda \in [0, 1)$ such that

$$\bigl|f(x_1,x_2) - f(x_2,x_3)\bigr| \leqslant \lambda \max\bigl\{|x_1 - x_2|, |x_2 - x_3|\bigr\}.$$

So, f fails to be a Prešić–Ćirić operator in usual metric space $(X, |\cdot|)$ and the result of Ćirić and Prešić [4] is not applicable.

At the points $x_1 = x_2 = x_3 = 2$, we have $\sigma(f(x_1, x_2), f(x_2, x_3)) = 2$ and $\max\{\sigma(x_1, x_2), \sigma(x_2, x_3)\} = 2$. Therefore, there exists no $\lambda \in [0, 1)$ such that

$$\sigma\bigl(f(x_1, x_2), f(x_2, x_3)\bigr) \leqslant \lambda \max\bigl\{\sigma(x_1, x_2), \sigma(x_2, x_3)\bigr\}.$$

So, f fails to be a Prešić–Ćirić type operator (therefore, fails to be a Prešić type operator) in the metric-like space (X, σ) and the result of Shukla and Fisher [26] is not applicable.

Also, f is not a cyclic-Prešić operator in the usual metric space $(X, |\cdot|)$. Indeed, at the points $x_1 = x_2 = 1$, $x_3 = 0$, we have $|f(x_1, x_2) - f(x_2, x_3)| = 1$ and $|x_1 - x_2| = 0$, $|x_2 - x_3| = 1$. Therefore, there exist no nonnegative constants α_1, α_2 such that $\alpha_1 + \alpha_2 < 1$ and

$$\bigl|f(x_1, x_2) - f(x_2, x_3)\bigr| \leqslant \alpha_1 |x_1 - x_2| + \alpha_2 |x_2 - x_3|.$$

So, f fails to be a cyclic-Prešić operator in the usual metric space $(X, |\cdot|)$ and the result of Shukla and Abbas [26] is not applicable.

On the other hand, $f(A_1 \times A_2) = \{0, 1\} \subseteq A_1$ and $f(A_2 \times A_1) = \{0, 1\} \subseteq A_2$, therefore, $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f. Now it is easy to see that condition (7) is satisfied with $\lambda \in [2/3, 1)$, therefore, f is a cyclic-Prešić–Ćirić operator. Thus all the conditions of Theorem 4 are satisfied and f has a fixed point $0 \in A_1 \cap A_2$ with $\sigma(0, 0) = 0$.

Remark 2. In the above example, the fixed point of f is not unique. Indeed, $\mathcal{F}(f) = \{0, 2\} \not\subset A_1 \cap A_2$ and $2 \notin A_1 \cap A_2$. Therefore, this example shows that the fixed point of a cyclic-Prešić–Ćirić operators in a metric-like space may not be unique and the fixed point may not be in $\bigcap_{i=1}^m A_i$. Also, $\sigma(2, 2) = 2 \neq 0$, therefore, if $v \in \mathcal{F}(f)$, then we may have $\sigma(v, v) \neq 0$. While, the self distance of the fixed point of a Prešić operator in a metric-like space is always zero (see Lemma 9 of [27]).

In Theorem 4, the self distance of fixed point u is zero, because u is the limits of a 0-σ-Cauchy sequence in a 0-σ-complete metric-like space. The following remark is useful in proving the uniqueness of fixed point of a cyclic-Prešić–Ćirić operator and it shows that if $\mathcal{F}(f) \subset \bigcap_{i=1}^m A_i$, then the self distance of any fixed point of a cyclic-Prešić–Ćirić operator on a metric-like spaces must be zero.

Remark 3. Let $A_1, A_2, \ldots, A_m$ be subsets of a metric-like space (X, σ), k a positive integer, and $Y = \bigcup_{i=1}^m A_i$. Let $f : Y^k \to Y$ be a cyclic-Prešić–Ćirić operator such that $\mathcal{F}(f) \subset \bigcap_{i=1}^m A_i$. Then, for every $v \in \mathcal{F}(f)$, we have $\sigma(v, v) = 0$.

Proof. Let $v \in \mathcal{F}(f)$ and $\sigma(v, v) > 0$. As $\mathcal{F}(f) \subset \bigcap_{i=1}^m A_i$, we obtain from (7) that

$$\begin{aligned}\sigma(v, v) &= \sigma\bigl(f(v, v, \ldots, v), f(v, v, \ldots, v)\bigr)\\ &\leqslant \lambda \max\bigl\{\sigma(v, v), \sigma(v, v), \ldots, \sigma(v, v)\bigr\}\\ &= \lambda \sigma(v, v) < \sigma(v, v).\end{aligned}$$

This contradiction shows that $\sigma(v, v) = 0$. □

Next theorem provides some sufficient conditions for the uniqueness of fixed point of cyclic-Prešić–Ćirić operators.

Theorem 5. *Let $A_1, A_2, \dots, A_m$ be closed subsets of a 0-σ-complete metric-like space (X, σ), k a positive integer, and $Y = \bigcup_{i=1}^m A_i$. Let $f : Y^k \to Y$ be a cyclic-Prešić–Ćirić operator. Then $\bigcap_{i=1}^m A_i \neq \emptyset$ and f has a fixed point $u \in \bigcap_{i=1}^m A_i$ such that $\sigma(u, u) = 0$. If, in addition, the following conditions are satisfied*:

(A) $\mathcal{F}(f) \subset \bigcap_{i=1}^m A_i$;

(B) *one of the following conditions is satisfied*:

(B1) *on the diagonal $\Delta \subset (\bigcap_{i=1}^m A_i)^k$,*

$$\sigma\big(f(x, \dots, x), f(y, \dots, y)\big) < \sigma(x, y)$$

holds for all $x, y \in \bigcap_{i=1}^m A_i$ with $x \neq y$, or

(B2) *in condition* (7)*, the constant $\lambda \in (0, 1/k)$.*

Then the fixed point of f is unique.

Proof. The existence of a fixed point $u \in \bigcap_{i=1}^m A_i$ with $\sigma(u, u) = 0$ follows from Theorem 4. Suppose $\mathcal{F}(f) \subset \bigcap_{i=1}^m A_i$ and the condition (B1) is satisfied. Suppose $v \in \mathcal{F}(f)$ with $u \neq v$. As $u, v \in \bigcap_{i=1}^m A_i$, we have

$$\sigma(u, v) = \sigma\big(f(u, \dots, u), f(v, \dots, v)\big) < \sigma(u, v).$$

This contradiction shows that the fixed point of f is unique.

Now suppose $\mathcal{F}(f) \subset \bigcap_{i=1}^m A_i$ and condition (B2) is satisfied. Suppose $v \in \mathcal{F}(f)$ with $u \neq v$, then by Remark 3 we have $\sigma(v, v) = 0$. Therefore, it follows from (7) that

$$\begin{aligned}
\sigma(u, v) &= \sigma\big(f(u, u, \dots, u), f(v, v, \dots, v)\big) \\
&\leqslant \sigma\big(f(u, u, \dots, u), f(u, \dots, u, v)\big) + \sigma\big(f(u, \dots, u, v), f(u, \dots, u, v, v)\big) \\
&\quad + \dots + \sigma\big(f(u, v, \dots, v), f(v, v, \dots, v)\big) \\
&\leqslant \lambda \max\big\{\sigma(u, u), \dots, \sigma(u, u), \sigma(u, v)\big\} \\
&\quad + \lambda \max\big\{\sigma(u, u), \dots, \sigma(u, u), \sigma(u, v), \sigma(v, v)\big\} \\
&\quad + \dots + \lambda \max\big\{\sigma(u, v), \sigma(v, v), \dots, \sigma(v, v)\big\} \\
&\leqslant \lambda \max\big\{0, \dots, 0, \sigma(u, v)\big\} + \lambda \max\big\{0, \dots, 0, \sigma(u, v), 0\big\} \\
&\quad + \dots + \lambda \max\big\{\sigma(u, v), 0, \dots, 0\big\} \\
&= k\lambda\sigma(u, v) \\
&< \sigma(u, v) \quad \left(\text{as } \lambda \in \left(0, \frac{1}{k}\right)\right).
\end{aligned}$$

This contradiction shows that $\sigma(u, v) = 0$, that is, $u = v$. Thus, the fixed point of f is unique. □

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