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Differential Transform – Pade Technique for Treating Non-linear Singular Boundary Value Problems Arising in the Applied Sciences

Abstract: This paper applies differential transform – Pade technique for treating non-linear singular boundary value problems arising in various physical problems of science and engineering. Comparisons are made between the results of the proposed method, and the exact solutions. The results show that the proposed method is an attractive method for solving non-linear singular boundary value problems.

Keywords: non-linear singular boundary value problems, differential transform method, differential transform – Pade approximation

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1 Introduction

The aim of this paper is to introduce differential transform – Pade technique for the numerical solution of the following class of singular boundary value problems:

$$y'' + \frac{\alpha}{x} y' + f(x, y) = 0 \tag{1}$$

subject to the boundary conditions

$$y'(0) = A\,, \quad y(1) = B \text{ (or } \eta y(1) + \beta y'(1) = \mu) \tag{2}$$

If $\alpha=1$, (1) becomes a cylindrical problem and if $\alpha=2$, then it becomes a spherical problem, where A, B, η, β and μ are real constants. It is well known that (1) has a unique solution if $f(x, y)$ continuous function, $\partial f / \partial y$ exists and is continuous and $\partial f / \partial y \geq 0$ [1]. Accurate and fast numerical solution of two-point singular boundary value problems for ordinary differential equations is necessary in many important scientific and engineering applications, e.g. reactant concentration in a chemical reactor, boundary layer theory, control and optimization theory, and flow networks in biology, the study of astrophysics such as the theory of stellar interiors, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents. In recent years, seeking numerical solution of singular differential equations has been the focus of a number of authors. In [2] the original differential equation is modified at singular point then the boundary value problem is treated by using cubic splines. Kamel Al-Khaled [3] used the Sinc-Galerkin method and homotopy perturbation method for finding the approximate solution of a certain class of singular two-point boundary value problems. In [4] a method based on B-splines for solving a class of singular boundary value problems was presented. Sami Bataineh et al. [5] used the modified homotopy analysis method to obtain the approximate solutions of singular two-point boundary value problems. Ravi Kanth and Bhattacharya [6] used a quasilinearization technique to reduce a class of nonlinear singular boundary value problems arising in physiology to a sequence of linear problems; the resulting set of differential equations are modified at the singular point and the spline technique is utilized to obtain a numerical solution. William and Pennline [7] discussed the existence and uniqueness of singular boundary value problems in chemical engineering. Recently, in [8] the application of VIM was extended for non linear singular boundary value problems.

In this paper, we applied differential transform – Pade technique for non-linear singular two-point boundary value problems arising in various physical problems of science and engineering. The concept of differential transform method (DTM) was first proposed by Zhou [9] in solving linear and nonlinear initial valued problems in electrical circuit analysis. The DTM is a semi-numerical-analytic-technique that formalizes the Taylor series in a totally different manner. With this technique, the given differential equation and related initial and boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. Therefore the DTM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. There is no need for linearization or perturbations, large computational work and round-off errors are avoided. This method has been successfully applied to solve many types of linear and nonlinear problems and it has been well addressed in [10–28].

DTM constructs the approximate solutions of differential equations are calculated in the form of truncated series with easily computable terms. The series solutions obtained by DTM have a limited region of convergence even for large number of terms. In order to enlarging the convergence domain of the truncated series solution we applied the Pade' approximant [29], where converting the polynomial approximation into a ratio of two polynomials. Peker et al. [30] have employed the differential transformation method and Pade approximant for a form of Blasius equation. The modified differential transform method for solving MHD boundary layer equations studied in [31]. In [32] used the DTM-Pade modeling of natural convective boundary layer flow of nanofluid past a vertical surface. Rashidi and Keimanesh [33] used differential transform method and Pade approximant for solving MHD flow in a laminar liquid film from a horizontal stretching surface.

2 Differential transform method

An arbitrary function $y(x)$ can be expanded in Taylor series about a point $x=0$ as

$$y(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{3}$$

The differential transform of $y(x)$ is defined [9] as

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{4}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function.

The differential inverse transform is

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^{k}. \tag{5}$$

In actual applications, the function $y(x)$ is expressed by a finite series and equation (5) can be rewritten as follows:

Table 1: The fundamental operations of differential transform method

Original function	Transformed function
$y(x)=g(x)\pm h(x)$	$Y(k)=G(k)\pm H(k)$
$y(x)=\alpha g(x)$	$Y(k)=\alpha G(k)$
$y(x)=\frac{dg(x)}{dx}$	$Y(k)=(k+1)G(k+1)$
$y(x)=\frac{d^2g(x)}{dx^2}$	$Y(k)=(k+1)(k+2)G(k+2)$
$y(x)=g(x)h(x)$	$Y(k)=\sum_{l=0}^{k}G(l)H(k-l)$
$y(x)=x^m$	$Y(k)=\delta(k-m)=\begin{cases}1, & \text{if } k=m\\ 0 & \text{if } k\neq m\end{cases}$
$f(y)=e^{ay(x)}$	$F(k)=\begin{cases}e^{aY(0)}, & k=0\\ a\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-m-1), & k\geq 1\end{cases}$

$$y(x)=\sum_{k=0}^{n}Y(k)x^{k} \tag{6}$$

which means that $y(x)=\sum_{k=n+1}^{\infty}Y(k)x^{k}$ is small, negligibly. Usually the value of n are decided by convergence of the series coefficients. The fundamental mathematical operations performed by differential transform method are listed in Table 1.

3 Pade approximants on the truncated series solution

The series solutions obtained by DTM have a limited region of convergence even if taking n to be as large as possible. We apply the Pade approximant on the truncated series obtained to increase the convergence region. A truncated series solution of order at least $(L+M)$ in x will be used to obtain Pade $[L/M]$, which is an approximate solution for $y(x)$. We denote the L, M Pade approximants to $y(x)$ by

$$y(x)=\sum_{k=0}^{\infty}a_k x^k \tag{7}$$

the pade approximant is a rational fraction and the notation for such a Pade approximant is [29]

$$[L,M]=\frac{P_L(x)}{Q_M(x)} \tag{8}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. We have

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad (9)$$

$$P_L(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \dots + p_L x^L \quad (10)$$

$$Q_M(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \dots + q_M x^M \quad (11)$$

Notice that in Eq. (8) there are $L+1$ numerator coefficients and $M+1$ denominator coefficients. Since we can clearly multiply the numerator and denominator by a constant and leave $[L, M]$ unchanged, we impose the normalization condition

$$Q_M(0) = 1 \quad (12)$$

So there are $L+1$ independent numerator coefficients and M independent denominator coefficients, making $L+M+1$ unknown coefficient in all. This number suggests that normally the $[L, M]$ ought to fit the power series Eq. (7) through the orders $1, x, x^2, \dots, x^{L+M}$. By using the conclusion given in [29], we know that the $[L, M]$ approximant is uniquely determined.

In the notation of formal power series,

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0 + p_1 x + p_2 x^2 + \dots + p_L x^L}{q_0 + q_1 x + q_2 x^2 + \dots + q_M x^M} + O(x^{L+M+1}) \quad (13)$$

By cross-multiplying Eq. (13), we find that

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(q_0 + q_1 x + q_2 x^2 + \dots + q_M x^M) = p_0 + p_1 x + p_2 x^2 + \dots + p_L x^L) + O(x^{L+M+1}) \quad (14)$$

From Eq. (14), one can obtained the set of equations

$$\left.\begin{aligned} a_0 &= p_0, \\ a_1 + a_0 q_1 &= p_1, \\ a_2 + a_1 q_1 + a_0 q_2 &= p_2, \\ &\vdots \\ a_L + a_{L-1} q_1 + \dots + a_0 q_L &= p_L, \end{aligned}\right\} \quad (15)$$

and

$$\left.\begin{aligned} a_{L+1} + a_L q_1 + \dots + a_{L-M+1} q_M &= 0, \\ a_{L+2} + a_{L+1} q_1 + \dots + a_{L-M+2} q_M &= 0, \\ &\vdots \\ a_{L+M} + a_{L+M-1} q_1 + \dots + a_L q_M &= 0, \end{aligned}\right\} \quad (16)$$

where $a_n = 0$ for $n < 0$ and $q_j = 0$ for $j > M$.

If Eqs. (15) and (16) are nonsingular, then we can solve them directly

$$[L/M] = \frac{\begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=0}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{vmatrix}}{\begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{vmatrix}} \quad (17)$$

To obtain diagonal Pade approximants of different order like [2/2], [4/4] or [6/6] MATHEMATICA can be efficiently used.

4 Numerical results

In this section, we have presented DTM and DTM-Pade techniques on four physical model examples: (i) the equilibrium of isothermal gas spheres; (ii) the steady state temperature distribution in the interior of a cylinder of unit radius; (iii) the steady of the distribution of heat sources in the human head; (iv) the steady state oxygen diffusion in spherical cells. The applicability of the results shows that the DTM-Pade technique converges rapidly when compared with the DTM technique.

Example 1: Consider the non-linear singular two-point boundary value problem arising in Astronomy; the equilibrium of isothermal gas spheres can be described by

$$y'' + \frac{2}{x} y' + y^5 = 0 \quad (18)$$

Subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = \frac{\sqrt{3}}{2} \quad (19)$$

This problem has earlier been considered by Russell and Shampine [1] and its exact solution is $y(x) = \dfrac{1}{\sqrt{1 + \frac{x^2}{3}}}$.

The transformed version of (18):

$$\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)Y(k-l+2)+2(k+1)Y(k+1)$$
$$+\sum_{l=0}^{k}Y(k-l)\sum_{m=0}^{l}Y(l-m)\sum_{n=0}^{m}Y(m-n)\sum_{p=0}^{n}Y(n-p)\sum_{q=0}^{p}\delta(q-1)\times Y(p-q)=0 \quad (20)$$

where $Y(k)$ is the differential transform of $y(x)$. The transformed boundary conditions are

$$Y(0)=A, \quad Y(1)=0 \quad (21)$$

where A is a constant. Substituting Eq. (21) in (20), and by recursive method we can calculate all values of $Y(k)$:

$$Y(2)=-\frac{A^5}{6}, \quad Y(3)=0, \quad Y(4)=\frac{A^9}{24}, \quad Y(5)=0,$$
$$Y(6)=-5\frac{A^{13}}{432}, \quad Y(7)=0, \quad Y(8)=35\frac{A^{17}}{10368}, \quad Y(9)=0,$$
$$Y(10)=-7\frac{A^{21}}{6192}, \quad \ldots$$

Hence substituting $Y(k)$'s into Eq. (6), we have a series solution

$$y(x)=A-\frac{A^5x^2}{6}+\frac{A^9x^4}{24}-\frac{5A^{13}x^6}{432}+\frac{35A^{17}x^8}{10368}-\frac{7A^{21}x^{10}}{6912}+\frac{77A^{25}x^{12}}{248832}-\ldots \quad (22)$$

Unknown value A can be determined in two different manners.

Case i: By incorporating the boundary condition $y(1)=\frac{\sqrt{3}}{2}$ in equation (22) we can obtain A. For analytical solution, the convergence analysis was performed for $n=20$, we have $A=0.9999983380238335$. The solution of equation (18) using DTM is as follows:

$$\begin{aligned}y(x)=\;&0.9999983380238335-0.16666528169113154x^2\\&+0.0416660434297474x^4-0.0115738240107093x^6\\&+0.00337567622852322x^8\\&-0.0010126961362216245x^{10}\\&+0.00030943287342337296x^{12}\\&-0.00009577620506017517x^{14}\\&-0.00002992986510958811x^{16}\\&-9.422302303079663\times10^{-6}x^{18}\\&+2.983709227144985410\times10^{-6}x^{20}.\end{aligned} \quad (23)$$

Case ii: We applied the Pade approximation for the equation (22) and then by substituting the boundary condition $y(1)=\frac{\sqrt{3}}{2}$, we obtain A. For analytical solution, the convergence analysis was performed for $n=20$, we have $A=0.99999999999895$. The solution of equation (18) using DTM-Pade is as follows:

$$\begin{aligned}y(x)_{[10/10]}\\=\;&(0.999999999999895+0.74999999999608x^2\\&+0.1944444444426156x^4+0.02025462962935445x^6\\&+0.000723378629616778x^8+4.018775720076\times10^{-6}x^{10})\\&\div(1+0.916666666628348x^2+0.305555555553001x^4\\&+0.04456018518462637x^6+0.002652391975264291 7x^8\\&+0.00004420653292088674x^{10})\end{aligned} \quad (24)$$

Table 2 exhibits the errors obtained by the DTM and DTM-Pade solution. It is observed that the solutions obtained by DTM-Pade are more accurate than the DTM. It is observed that if the order of the Pade-approximation increases the accuracy of the solution increases.

Example 2: Next we consider the steady-state temperature distribution in the interior of a cylinder of unit radius

Table 2: Absolute errors for Example 1

x	DTM solution	DTM-Pade [6/6] solution	DTM-Pade [8/8] solution	DTM-Pade [10/10] solution	Exact solution
0	1.66198×10^{-6}	3.93363×10^{-8}	2.02771×10^{-10}	1.04505×10^{-12}	1
0.1	1.64819×10^{-6}	3.901×10^{-8}	2.01088×10^{-10}	1.03628×10^{-12}	0.99833748845958
0.2	1.60756×10^{-6}	3.80484×10^{-8}	1.96132×10^{-10}	1.01086×10^{-12}	0.99339926779878
0.3	1.5422×10^{-6}	3.65014×10^{-8}	1.88157×10^{-10}	9.6978×10^{-13}	0.98532927816429
0.4	1.45537×10^{-6}	3.44462×10^{-8}	1.77564×10^{-10}	9.15601×10^{-13}	0.97435470369245
0.5	1.35112×10^{-6}	3.19763×10^{-8}	1.64843×10^{-10}	8.49543×10^{-13}	0.96076892283052
0.6	1.23389×10^{-6}	2.91768×10^{-8}	1.50521×10^{-10}	7.75602×10^{-13}	0.94491118252307
0.7	1.10789×10^{-6}	2.60249×10^{-8}	1.34915×10^{-10}	6.96332×10^{-13}	0.92714554082312
0.8	9.72425×10^{-7}	2.20391×10^{-8}	1.16751×10^{-10}	6.08957×10^{-13}	0.90784129900320
0.9	7.73419×10^{-7}	1.52764×10^{-8}	8.63716×10^{-11}	4.72289×10^{-13}	0.88735650941611
1	0	1.66533×10^{-15}	2.22045×10^{-16}	0	0.86602540378444

is described by the solution of the nonlinear two-point boundary value problem:

$$y'' + \frac{1}{x}y' + e^y = 0 \tag{25}$$

subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = 0 \tag{26}$$

This problem has earlier been considered by Russell and Shampine [1] and its exact solution and its exact solutions are $y(x) = 2\log\left(\frac{B+1}{Bx^2+1}\right)$ where $B = 3 \pm 2\sqrt{2}$.

The transformed version of (25) is

$$\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)Y(k-l+2) + (k+1)Y(k+1) + \sum_{l=0}^{k}\delta(l-1)F(k-l) = 0 \tag{27}$$

where

$$F(k) = \begin{cases} e^{Y(0)}, & k = 0 \\ \sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-1-m), & k \geq 1 \end{cases} \tag{28}$$

The transformed boundary conditions are

$$Y(0) = A, \quad Y(1) = 0 \tag{29}$$

where A is a constant. Substituting Eq. (29) in (27)–(28), and by recursive method we can calculate all values of $Y(k)$ as follows:

$$Y(2) = \frac{-e^A}{4}, \quad Y(3) = 0, \quad Y(4) = \frac{e^{2A}}{64}, \quad Y(5) = 0, \quad Y(6) = \frac{e^{3A}}{768}, \quad Y(7) = 0, \quad Y(8) = \frac{e^{4A}}{8192}, \quad \ldots$$

By using inverse transform rule the following solution is obtained:

$$y(x) = A - \frac{e^A}{4}x^2 + \frac{e^{2A}}{64}x^4 - \frac{e^{3A}}{768}x^6 + \frac{e^{4A}}{8192}x^8 - \frac{e^{5A}}{81920}x^{10} + \ldots \tag{30}$$

Following the same procedure given in Example 1 we obtained the DTM and DTM-Pade solution for 20 terms. The order of Pade approximation [10, 10] has sufficient accuracy, on the other hand if the order of Pade approximation increases, the accuracy of the solution increseases. Errors obtained by the DTM, DTM-Pade and exact solution are presented in Table 3. Comparison between the solutions obtained by DTM and the DTM-Pade revealed that the DTM-Pade results are in excellent agreement with the exact solutions.

Table 3: Absolute errors for Example 2

x	DTM solution	DTM-Pade [10/10] solution	Exact solution
0	8.42767×10^{-10}	6.10623×10^{-15}	0.31669436764075
0.1	8.3988×10^{-10}	6.05072×10^{-15}	0.31326585049806
0.2	8.31278×10^{-10}	5.66214×10^{-15}	0.30301542283230
0.3	8.17136×10^{-10}	5.9952×10^{-15}	0.28604726530485
0.4	7.97733×10^{-10}	5.71765×10^{-15}	0.26253112745603
0.5	7.73442×10^{-10}	5.38458×10^{-15}	0.23269678387383
0.6	7.44706×10^{-10}	5.52336×10^{-15}	0.19682680569295
0.7	7.11802×10^{-10}	5.13478×10^{-15}	0.15524810668276
0.8	6.71374×10^{-10}	4.85723×10^{-15}	0.10832276344446
0.9	5.76865×10^{-10}	4.1564×10^{-15}	0.05643860246924
1.0	2.87273×10^{-17}	1.1045×10^{-15}	0

Example 3: Consider the following boundary value problem arises in the study of the distribution of heat sources in the human head [6]:

$$y'' + \frac{2}{x}y' + e^{-y} = 0 \tag{31}$$

subject to the boundary conditions

$$y'(0) = 0, \quad 0.1y(1) + y'(1) = 0 \tag{32}$$

The transformed version of (31) is

$$\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)Y(k-l+2) + 2(k+1)Y(k+1) + \sum_{l=0}^{k}\delta(l-1)F(k-l) = 0 \tag{33}$$

where

$$F(k) = \begin{cases} e^{-Y(0)}, & k = 0 \\ -\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-1-m), & k \geq 1 \end{cases} \tag{34}$$

The transformed boundary conditions are

$$Y(0) = A. \quad Y(1) = 0 \tag{35}$$

where A is a constant. Substituting Eq. (35) in (33)–(34), and by recursive method we can calculate all values of $Y(k)$ as follows:

$$Y(2)=-\frac{1}{6}e^{-A},\quad Y(3)=0,\quad Y(4)=-\frac{1}{120}e^{-2A},\quad Y(5)=0,$$
$$Y(6)=-\frac{1}{1890}e^{-3A},\quad Y(7)=0,\quad \ldots$$

By using inverse transform rule the following solution is obtained:

$$y(x)=A-\frac{e^{-A}}{6}x^2-\frac{e^{-2A}}{120}x^4-\frac{e^{-3A}}{1890}x^6-\frac{e^{-4A}}{1632960}x^8 -\frac{629e^{-5A}}{224532000}x^{10}+\ldots \tag{36}$$

Following the procedure given in Example 1 we obtained the DTM and DTM-Pade solution. Table 4 shows the comparison between the approximate solution obtained by DTM, DTM-Pade and the solution in [6].

Example 4: Consider the boundary value problem arises in the steady state oxygen diffusion in spherical cells [6]:

$$y''+\frac{2}{x}y'=\frac{ny}{y+k}, n>0, k>0\quad n=0.71629\quad k=0.03119 \tag{37}$$

subject to the boundary conditions

$$y'(0)=0,\quad 5y(1)+y'(1)=5 \tag{38}$$

The transformed version of (37) is

$$\begin{aligned}&\sum_{l=0}^{k}(k-l+1)(k-l+2)Y(k-l+2)\sum_{m=0}^{l}Y(l-m)\delta(m-1)\\ &+0.03119\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)Y(k-l+2)\\ &+2\sum_{l=0}^{k}Y(l)(k-l+1)Y(k-l+1)+0.06238(k+1)Y(k+1)\\ &=0.76129\sum_{l=0}^{k}\delta(l-1)Y(k-l)\end{aligned} \tag{39}$$

The transformed boundary conditions are

$$Y(0)=A,\quad Y(1)=0 \tag{40}$$

where A is a constant. Substituting Eq. (40) in (39), and by recursive method we can calculate all $Y(k)$ as follows:

$$Y(2)=\frac{0.12688166666666A}{0.03119+A},\quad Y(3)=0,$$
$$Y(4)=\frac{0.00015063794379399A}{(0.03119+A)^3},\quad Y(5)=0,$$
$$Y(6) =\frac{8.5162928752444\times10^{-8}A-9.1015206532481\times10^{-8}A^2}{(0.03119+A)^5},$$
$$\ldots$$

By using inverse transform rule the following solution is obtained:

$$\begin{aligned}y(x) &=A+\frac{0.126881666666A}{0.03119+A}x^2\\ &+\frac{0.00015063794379399A}{(0.03119+A)^3}x^4\\ &+\frac{8.5162928752444\times10^{-8}A-9.1015206532481\times10^{-8}A^2}{(0.03119+A)^5}x^6\\ &+\ldots\end{aligned} \tag{41}$$

Table 4: Numerical solutions for Example 3

x	DTM solution	DTM-Pade [6/6] solution	DTM-Pade [10/10] solution	Solution in [6]
0	1.147039019329823`	1.1470390193289595	1.1470390193298237	1.14704108351547
0.1	1.1465096424106855	1.1465096424098216	1.1465096424106866	1.14651170579035
0.2	1.1449205020921818	1.1449205020913167	1.1449205020921827	1.14492256347331
0.3	1.1422685635711016	1.1422685635702344	1.1422685635711025	1.14227062156003
0.4	1.1385487483651475	1.1385487483642764	1.1385487483651482	1.13855080147511
0.5	1.1337539033258215	1.1337539033249466	1.1337539033258222	1.13375594993740
0.6	1.127874756707035	1.1278747567061555	1.1278747567070355	1.12787679502618
0.7	1.1208998607256917	1.1208998607248053	1.1208998607256921	1.12090188873858
0.8	1.1128155198684448	1.1128155198675516	1.1128155198684453	1.11281753529231
0.9	1.1036057040000842	1.1036057039991858	1.1036057040000855	1.10360770422896
1.0	1.0932519451087037	1.0932519451078066	1.0932519451087044	1.09325392715304

Table 5: Numerical solutions for Example 4

x	DTM solution	DTM-Pade [6/6] solution	DTM-Pade [10/10] solution	Solution in [6]
0	0.82848329035897	0.8284829954392473	0.8284832893460309	0.82848325128559
0.1	0.82970609243307	0.8297057974975877	0.8297060914200943	0.82970605379966
0.2	0.83337473359029	0.8333744386079305	0.8333747325771332	0.83337469615985
0.3	0.83948991395299	0.8394896189004586	0.8394899129395661	0.83948987847975
0.4	0.84805278499535	0.8480524899082842	0.8480527839815545	0.84805275216454
0.5	0.85906492716851	0.8590646323468222	0.8590649261542664	0.85906489757388
0.6	0.87252831995757	0.8725280265493858	0.8725283189429615	0.87252829408339
0.7	0.88844530562247	0.8884450169575867	0.8884453046089263	0.88844528383088
0.8	0.90681854806609	0.9068182722749433	0.9068185470622508	0.90681853058973
0.9	0.92765098836487	0.9276507429891095	0.9276509874111585	0.92765097530503
1.0	0.95094579849589	0.9509456179544574	0.9509457977493267	0.95094578982530

Following the same procedure given in Example 1 we obtained the DTM and DTM-Pade solution. Table 5 shows the comparison between the approximate solution obtained by DTM, DTM-Pade and the solution in [6].

5 Conclusion

In this paper, we implemented the DTM and DTM-Pade technique for treating non-linear singular boundary value problems arising in various physical problems. Comparison between the solutions obtained by DTM and DTM-Pade with the exact solution remarked that the accuracy of the DTM Pade is very good. The present method can be applied directly without requiring linearization, discretization or perturbation. Also, it shows a promising tool for solving nonlinear two-point boundary value problems.

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References

[1] Russell RD, Shampine LF. Numerical methods for singular boundary value problems. SIAM J. Numer. Anal. 1975; **12**(1):13–36.

[2] Ravi Kanth ASV, Reddy YN. Cubic spline for a class of singular two-point boundary vaue problems. Appl. Math. Comput. 2005; **170**(2):733–740.

[3] Kamel Al-Khaled. Theory and computation in singular boundary value problems. Chaos Solitons & Fractals. 2007; **33**(2):678–684.

[4] Nazan Caglar, Hikmet Caglar. B-spline solution of singular boundary value problems. Appl. Math. Comput. 2006; **182**(2):1509–1513.

[5] Sami Bataineh A, Noorani MSM, Hashim I. Approximate solutions of singular two-point BVPs by modified homotopy analysis method. Phys. Lett. A. 2008; **372**:4062–4066.

[6] Ravi Kanth ASV, Vishnu Bhattacharya. Cubic spline for a class of non-linear singular boundary value problems arising in physiology. Appl. Math. Comput. 2006; **174**(1):768–774.

[7] William FF, Pennline JA. Singular non-linear two-point boundary value problems: Existence and uniqueness. Nonlinear Anal. 2009; **71**(3–4):1059–1072.

[8] Ravi Kanth ASV, Aruna K. He's variational iteration method for treating nonlinear singular boundary value problems. Comput. Math. Appl. 2010; **60**(3):821–829.

[9] Zhou JK. *Differential transform and its Applications for Electrical Circuits*. Huazhong University Press, Wuhan, China, 1986.

[10] Chen CK, Ho SH. Application of differential transformation to eigenvalue problems. Appl. Math. Comput. 1996; **79**(2–3):173–188.

[11] Jang Ming-Jyi, Chen Chieh-Li. Analysis of the response of a strongly nonlinear damped system using a differential transformation technique. Appl. Math. Comput. 1997; **88**(2–3):137–151.

[12] Chen CL, Liu YC. Solution of two-point boundary value problems using the differential transformation method. J. Optim. Theory Appl. 1998; **99**(1):23–25.

[13] Chen CK, Ho SH. Transverse vibration of a rotating twisted Timoshenko beams under axial loading using differential transform. Int. J. Mech. Sci. 1999; **41**(11):1339–1356.

[14] Jang MJ, Chen CL. On solving the initial value problems using the differential transformation method. Appl. Math. Comput. 2000; **115**(2–3):145–160.

[15] Jang MJ, Chen CL, Liu YC. Two-dimensional differential transform for partial differential equations. Appl. Math. Comput. 2001; **121**(2–3):261–270.

[16] Abdel-Halim Hassan IH. On solving some eigenvalue problems by using a differential transformation. Appl. Math. Comput. 2002; **127**(1):1–22.

[17] Ayaz F. On the two-dimensional differential transform method. Appl. Math. Comput. 2003; **143**(2–3):361–374.

[18] Abdel-Halim Hassan IH. Differential transformation technique for solving higher-order initial value problems. Appl. Math. Comput. 2004; **154**(2):299–311.
[19] Aytac Arikaglu, Ibrahim Ozkol. Solution of boundary value problems for integro differential equations by using differential transform method. Appl. Math. Comput. 2005; **168**(2):1145–1158.
[20] Ravi Kanth ASV, Aruna K. Solution of singular two-point boundary value problems using differential transformation method. Phys. Lett. A 2008; **372**(26):4671–4673.
[21] Liu Hong, Song Yongzhong. Differential transform method applied to high index differential-algebraic equations, Appl. Math. Comput. 2007; **184**(2):748–753.
[22] Moustafa El-Shahed. Applications of differential transform method to non-linear oscillatory systems. Commun. Nonlinear Sci. Numer. Simul. 2008; **13**(8):1714–1720.
[23] Ravi Kanth ASV, Aruna K. Differential transform method for solving linear and non-linear systems of partial differential equations. Phys. Lett. A. 2008; **372**:6896–6898.
[24] Chang Shih-Hsiang, Chang I-Ling. A new algorithm for calculating one-dimensional differential transform of nonlinear functions. Appl. Math. Comput. 2008; **195**(2):799–808.
[25] Zaid M. Odibat. Differential transform method for solving Volterra integral equation with separable kernels. Math. Comput. Modelling 2008; **48**:1144–1149.
[26] Figen Kangalgil O, Ayaz F. Solitary wave solutions for the KdV and mKdV equations by differential transform method. Chaos Solitons & Fractals 2009; **41**:464–472.
[27] Biazar J, Eslami M. Analytic solution for telegraph equation by differential transform method. Phys. Lett. A. 2010; **374**(29):2904–2906.
[28] Montri Thongmoon, Sasitorn Pusjuso. The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations. Nonlinear Anal.: Hybrid systems 2010; **4**(3):425–431.
[29] Baker GA. *Essential of Pade Approximants*, Academic Press, London, 1975.
[30] Peker HA, Karaoglu O, Oturanc G. The differential transformation method and Pade approximant for a form of Blasius equation. Math. Comput. Appl. 2011; **16**(2):507–513.
[31] Rashidi MM. The modified differential transform method for solving MHD boundary layer equations. Comput. Phy. Commun. 2009; **180**(11):2210–2217.
[32] Rashidi MM, Anwar Beg O, Asadi M, Rastegari MT. DTM-Pade modeling of natural convective boundary layer flow of nanofluid past a vertical surface. Int. J. Thermal and Environmental Eng. 2012; **4**:13–24.
[33] Rashidi MM, Keimanesh M. Using differential transform method and Pade approximant for solving MHD flow in a laminar liquid film from a horizontal stretching surface. Mathematical Problems in Eng. 2010; 2010 article ID 491319: 14 pages.