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DUALITY FOR A CLASS OF SECOND ORDER SYMMETRIC NONDIFFERENTIABLE FRACTIONAL VARIATIONAL PROBLEMS

Ashish Kumar PRASAD*
Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, India
ashishprasa@gmail.com

Anant Pratap SINGH
Department of Mathematical Sciences, Indian Institute of Technology(BHU), Varanasi-221005, India
anantsingh100396@gmail.com

Sony KHATRI
Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, India
khatrisony44@gmail.com

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Abstract: The present work frames a pair of symmetric dual problems for second order nondifferentiable fractional variational problems over cone constraints with the help of support functions. Weak, strong and converse duality theorems are derived under second order $\mathcal{F}$-convexity assumptions. By removing time dependency, static case of the problem is obtained. Suitable numerical example is constructed.

Keywords: Symmetric Duality, Variational Problem, Second Order $\mathcal{F}$-convexity, Nondifferentiable Programming.

MSC: 90C26, 90C29, 90C30, 90C46.

1. INTRODUCTION

The problems studied by researchers gradually moved to a difficult platform as it was tried to model more general physical laws observed in nature. Consequently, various techniques have been developed and hence new branches of science emerged in order to investigate the problems posed by such laws. One such branch of science is calculus of variations. The development of calculus of variations was triggered by the investigation of physical and mechanical problems. Some typical examples include the problems involving the determination of maxima and minima of functionals, finding the shortest plane curve joining two points, isoperimetric problems, brachistochrone problems. The significance of the concept of the variation of a functional is by no means confined to its applications to the problem of determining the extrema of functionals. Applications of various methods developed impart not only the solution of individual but they explain so-called "variational principles", which are valid in diverse branches of physics, ranging from classical mechanics to the theory of elementary particles.

The alternative way to find the solution leads to the concept of duality. If we closely look at optimization theory, we find that Wolfe and Mond-Weir type duality models captures large space in literature. Several dimensions have emerged in order to broaden the applicability of theory to more sophisticated problems. Dorn [5] introduced the concept of symmetric duality, whereas Mangasarian [12] showed the computational advantage of constructing second and higher order problems. Mond and Schechter [13] constructed the symmetric dual pairs where the objective function was nondifferentiable due to the appearance of support function. Hou and Yang [6] extended the work by Mond and Schechter [13] and Bector and Chandra[4] to study the duality theorems for second order symmetric nondifferentiable programming. Later, Yang [17] formulated Wolfe type nondifferentiable second order symmetric programs and established the usual duality theorems under second order F-convexity. Moreover, they studied minimax mixed order programs. Ahmad and Gulati [3] formulated mixed type dual for multiobjective variational problems. Several duality theorems were established relating properly efficient solutions of the primal and dual variational problems under generalized (F, ρ)-convexity. Saini and Gulati [16] pointed that weak efficiency with respect to a convex cone can be used as a tool to derive weak, strong and converse duality theorems for a pair of Wolfe type nondifferentiable multiobjective second-order symmetric dual programs over arbitrary cones.

Ahmad *et al.* [2] has taken step in the direction of investigating the duality results for a pair of symmetric fractional variational programming problems over cones and established duality theorems under pseudoinvexity. These results were again extended by Ahmad and Sharma [1] for a pair of multiobjective fractional variational symmetric dual problems over cones. But complexity of the problem was enforcing researchers to investigate the problems where the objective function consists of support functions, making it nondifferentiable in nature. Therefore, Kailey and Gupta [11] studied the fractional variational problem over arbitrary cones where the dual problem incorporates the nondifferentiability in the form of support function. Moreover, second and higher order analogue of these problems are also needed as they provide tighter bounds to the values of objective function. In this connection, Jayswal *et al.* [8] introduced a pair of multiobjective second-order symmetric variational control programs over cone constraints and derived weak, strong and converse duality theorems under second-order F-convexity assumptions. The present work is an extension of this work to nondifferentiable case.

In the present paper, we consider second order nondifferentiable fractional symmetric variational programs over cone constraints and establish weak, strong and converse duality theorems under second order $\mathcal{F}$-convexity. The paper is organized as per the following scheme. In Section 2, we design our problem and define basic terms needed in the sequel of the paper. A proper space is given to the numerical example in order to validate the definition used in this paper. In Section 3, we formulate a pair of second order fractional symmetric variational programs over cone constraints along with its equivalent form and derive appropriate duality theorems in Section 4. Section 5 deals with the static case of of the problems studied in this paper followed by conclusions in Section 6.

2. PRELIMINARIES

The present paper considers the following variational problem:

$$\text{(VP)} \qquad \text{Minimize} \qquad \int_a^b \Psi(t, x, \dot{x})\, dt$$

$$\text{subject to} \qquad x(a) = \alpha, \quad x(b) = \beta,$$
$$h(t, x, \dot{x}) \leqq 0, \ t \in I,$$

where $\Psi : I \times R^n \times R^n \to R$ and $h : I \times R^n \times R^n \to R^m$ are continuously differentiable functions and $x(t),\ t \in I = [a, b]$ is an n-dimensional piecewise smooth continuous function the derivative of which is denoted by $\dot{x}(t)$. For notational convenience, x and $\dot{x}$ are written in place of $x(t)$ and $\dot{x}(t)$. The gradient vectors of Ψ with respect to x and $\dot{x}$ are denoted by Ψ_x and $\Psi_{\dot{x}}$, i.e.

$$\Psi_x = (\frac{\partial \Psi}{\partial x^1}, \frac{\partial \Psi}{\partial x^2}, ..., \frac{\partial \Psi}{\partial x^n})^T \quad \text{and} \quad \Psi_{\dot{x}} = (\frac{\partial \Psi}{\partial \dot{x}^1}, \frac{\partial \Psi}{\partial \dot{x}^2}, ..., \frac{\partial \Psi}{\partial \dot{x}^n})^T.$$

Similarly, Ψ_{xx} denotes the $n \times n$ matrix with respect to x. Let $M(t, x, \dot{x}) = \Psi_{xx} - 2D\Psi_{x\dot{x}} + D^2\Psi_{\dot{x}\dot{x}} - D^3\Psi_{\dot{x}\ddot{x}},\ t \in I.$

The space of piecewise smooth functions is denoted by $C(I, R)$, for any $x \in C(I, R)$, we define its norm by

$$\| x \| = \| x \|_\infty + \| Dx \|_\infty,$$

where D stands for the differential operator given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_0^t u(s)\, ds,$$

where α is a specified boundary value. So, $\frac{d}{dt} \equiv D$ (except at discontinuities).

Definition 2.1 A subset $C \subset R^n$ is said to be a cone if it satisfies the following property:

$$0 \leq \lambda \in R,\ x \in C \Rightarrow \lambda x \in C.$$

Definition 2.2 For any cone C, its polar cone C^* is given by

$$C^* = \{z : x^T z \leqq 0 \text{ for all } x \in C\}.$$

Definition 2.3 A functional $\mathcal{F} : I \times X \times X \times X \times X \times R^n \to R$ is sublinear if for any $x, \dot{x}, u, \dot{u} \in X$, we have

(i) $\mathcal{F}(t, x, \dot{x}, u, \dot{u}; \theta_1 + \theta_2) \leqq \mathcal{F}(t, x, \dot{x}, u, \dot{u}; \theta_1) + \mathcal{F}(t, x, \dot{x}, u, \dot{u}; \theta_2)$ for any $\theta_1,\ \theta_2 \in R^n$,

(ii) $\mathcal{F}(t, x, \dot{x}, u, \dot{u}; a\theta) = a\mathcal{F}(t, x, \dot{x}, u, \dot{u}; \theta)$ for any $a \geqq 0$ and $\theta \in R^n$.

From (ii), it is clear that $\mathcal{F}(t, x, \dot{x}, u, \dot{u}; 0) = 0$. For notational convenience, we write $\mathcal{F}(t, x, \dot{x}, u, \dot{u}; \theta) = \mathcal{F}(t, x, u; \theta)$.
Definition 2.4 The support function of a compact convex set $C \subset R^n$ is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

We know that a support function has a subdifferential, *i.e.*, there exits $z \in R^n$ with the following property

$$s(y|C) \geqq s(x|C) + z^T(y - x),\ \forall y \in C.$$

The normal cone to any set $W \subset R^n$ is defined by

$$N_W(x) = \{y \in R^n : y^T(z - x) \leqq 0 \forall z \in W\}.$$

Let us assume that $\mathcal{F}$ and $\mathcal{G}$ are arbitrary sublinear functional with respect to sixth argument. Now, we consider the following definition of second order $\mathcal{F}$-convex function.
Definition 2.6 The functional $\int_a^b \Psi(t, x, \dot{x})\, dt$ is said to be second order $\mathcal{F}$-convex at $u(t) \in R^n$ if

$$\int_a^b \Psi(t, x, \dot{x})\, dt - \int_a^b \Psi(t, u, \dot{u})\, dt + \frac{1}{2}\int_a^b q(t)^T M q(t)\, dt$$
$$\geqq \int_a^b \mathcal{F}(t, x, u; \Psi_x(t, u, \dot{u}) - D\Psi_{\dot{x}}(t, u, \dot{u}) + Mq(t))\, dt,$$

for all $x(t),\ q(t) \in R^n,\ t \in I$ and for some arbitrary sublinear functional $\mathcal{F}$.
Remark 2.1

(i) If $\mathcal{F}(t, x, u; a) = \eta(t, x, u)^T a$, then the above definition reduces to the second order invex with respect to η given in [7].

(ii) In addition to (i) above, if $M(t, x, \dot{x}) = 0$, then we obtain the definition of invexity discussed in Mond *et al.* [14].

(iii) If $M(t, x, \dot{x}) = 0$, then we get the definition of $\mathcal{F}$-convexity as given in Nahak and Nanda [15].

Now, we give an example to show the existence of second order $\mathcal{F}$-convex functions which are not $\mathcal{F}$-convex, as defined in Nahak and Nanda [15].

Example 2.1 Let I=[0,1]. Consider the functional $\phi : I \times R \times R \mapsto R$ be defined by

$$\phi(t, x, \dot{x}) = 2x^2(t) - 3.$$

Suppose $\mathcal{F} : I \times R \times R \times R \times R \times R \mapsto R$ be given by

$$\mathcal{F}(t, x, \dot{x}, u, \dot{u}; a) = -|a|(x^2(t) + u^2(t)).$$

Then the functional $\int_0^1 \phi(t, x, \dot{x})dt$ is second order $\mathcal{F}$-convex at $u(t) = 0$, since

$$\begin{aligned}
&\int_0^1 \phi(t, x, \dot{x})dt - \int_0^1 \phi(t, u, \dot{u})\, dt + \frac{1}{2}\int_0^1 q(t)^T \mathbf{M} q(t)\, dt \\
&\qquad = \int_0^1 (2x^2(t) - 3)\, dt - \int_0^1 (2u^2(t) - 3)\, dt + \frac{1}{2}\int_0^1 4q^2(t)\, dt \\
&\qquad = 2\int_0^1 (x^2(t) + q^2(t))\, dt
\end{aligned}$$

whereas

$$\begin{aligned}
&\int_0^1 \mathcal{F}(t, x, u; \phi_x(t, u, \dot{u}) - D\phi_{\dot{x}}(t, u, \dot{u}) + \mathbf{M}q(t))\, dt \\
&\qquad = \int_0^1 \mathcal{F}(t, x, u; -4x(t) + 4q(t))\, dt \\
&\qquad = -4\int_0^1 |x(t)|x^2(t)\, dt
\end{aligned}$$

From what has been done, it follows that

$$\begin{aligned}
&\int_0^1 \phi(t, x, \dot{x})dt - \int_0^1 \phi(t, u, \dot{u})\, dt + \frac{1}{2}\int_0^1 q(t)^T \mathbf{M} q(t)\, dt \\
&\qquad \geqq \int_0^1 \mathcal{F}(t, x, u; \phi_x(t, u, \dot{u}) - D\phi_{\dot{x}}(t, u, \dot{u}) + \mathbf{M}q(t))\, dt.
\end{aligned}$$

Hence $\int_0^1 \phi(t, x, \dot{x})dt$ is second order $\mathcal{F}$-convex at $u(t) = 0$.

Now,

$$\begin{aligned}
&\int_0^1 \phi(t, x, \dot{x})dt - \int_0^1 \phi(t, u, \dot{u})\, dt \\
&\qquad = \int_0^1 (2x^2(t) - 3)\, dt - \int_0^1 (2u^2(t) - 3)\, dt \\
&\qquad = \int_0^1 (2t^2 - 3 + 3)\, dt \\
&\qquad = \left[\frac{2t^3}{3}\right]_0^1 \\
&\qquad = 0.66
\end{aligned}$$

and
$$\int_0^1 \mathcal{F}(t,x,u;\phi_x(t,u,\dot{u}) - D\phi_{\dot{x}}(t,u,\dot{u}))\,dt$$
$$\begin{aligned} &= \int_0^1 \mathcal{F}(t,x,u;-4t)\,dt \\ &= \int_0^1 4t^2\,dt \\ &= \left[\frac{4t^3}{3}\right]_0^1 \\ &= 1.33 \end{aligned}$$

Therefore,
$$\int_0^1 \phi(t,x,\dot{x})dt - \int_0^1 \phi(t,u,\dot{u})\,dt \ngeqq \int_0^1 \mathcal{F}(t,x,u;\phi_x(t,u,\dot{u}) - D\phi_{\dot{x}}(t,u,\dot{u}).$$

Hence $\int_0^1 \phi(t,x,\dot{x})dt$ is not $\mathcal{F}$-convex at $u(t)=0$.

Let C_1 and C_2 be closed convex cones with nonempty interiors in R^n and R^m, respectively.

3. SECOND ORDER NONDIFFERENTIABLE SYMMETRIC DUALITY

In this paper, we investigate the following second order nondifferentiable symmetric dual variational programs over cone constraints:

Primal (PP) Minimize $\dfrac{\int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Ap(t) + s(x|E) - y^T z_1)\,dt}{\int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Bp(t) - s(x|F) + y^T z_2)\,dt}$

subject to
$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$
$$y(a) = 0 = y(b), \quad \dot{y}(a) = 0 = \dot{y}(b),$$
$$G(x,y)(f_y - Df_{\dot{y}} + Ap(t) - z_1) - F(x,y)(g_y - Dg_{\dot{y}} + Bp(t) + z_2) \in C_2^*,\ t\in I,$$
$$y(t)^T\{G(x,y)(f_y - Df_{\dot{y}} + Ap(t) - z_1) - F(x,y)(g_y - Dg_{\dot{y}} + Bp(t) + z_2)\} \geqq 0,\ t\in I,$$
$$x(t)\in C_1, \quad t\in I,$$
$$z_1 \in J,\ z_2\in K.$$

Dual (DP) Maximize $\dfrac{\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Yq(t) - s(v|J) + u^T r_1)\,dt}{\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Zq(t) + s(v|K) - u^T r_2)\,dt}$

subject to
$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b),$$

$$v(a) = 0 = v(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$-[G(u,v)(f_x - Df_{\dot{x}} + Yq(t) + r_1) - F(u,v)(g_x - Dg_{\dot{x}} + Zq(t) - r_2)] \in C_1^*,\ t \in I,$$

$$u(t)^T\{G(u,v)(f_x - Df_{\dot{x}} + Yq(t) + r_1) - F(u,v)(g_x - Dg_{\dot{x}} + Zq(t) - r_2)\} \leqq 0,\ t \in I,$$

$$v(t) \in C_2,$$

$$r_1 \in E, \qquad r_2 \in F,$$

where
(i) $f : I \times C_1 \times C_1 \times C_2 \times C_2 \to R_+$, and $g : I \times C_1 \times C_1 \times C_2 \times C_2 \to R_+ \setminus \{0\}$,
(ii) $A(t,x,\dot{x},y,\dot{y}) = f_{yy}(t,x,\dot{x},y,\dot{y}) - 2Df_{y\dot{y}}(t,x,\dot{x},y,\dot{y}) + D^2 f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y}) - D^3 f_{\dot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y}),\ t \in I,$
(iii) $B(t,x,\dot{x},y,\dot{y}) = g_{yy}(t,x,\dot{x},y,\dot{y}) - 2Dg_{y\dot{y}}(t,x,\dot{x},y,\dot{y}) + D^2 g_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y}) - D^3 g_{\dot{y}\ddot{y}}(t,x,\dot{x},y,\dot{y}),\ t \in I,$
(iv) $Y(t,u,\dot{u},v,\dot{v}) = f_{xx}(t,u,\dot{u},v,\dot{v}) - 2Df_{x\dot{x}}(t,u,\dot{u},v,\dot{v}) + D^2 f_{\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v}) - D^3 f_{\dot{x}\ddot{x}}(t,u,\dot{u},v,\dot{v}),\ t \in I,$
(v) $Z(t,u,\dot{u},v,\dot{v}) = g_{xx}(t,u,\dot{u},v,\dot{v}) - 2Dg_{x\dot{x}}(t,u,\dot{u},v,\dot{v}) + D^2 g_{\dot{x}\dot{x}}(t,u,\dot{u},v,\dot{v}) - D^3 g_{\dot{x}\ddot{x}}(t,u,\dot{u},v,\dot{v}),\ t \in I,$
(vi) $p : I \to R^m,\ q : I \to R^n,$
(vii) $F(x,y) = \int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Ap(t) + s(x|E) - y^T z_1)\,dt,$
$(viii)$ $G(x,y) = \int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Bp(t) - s(x|F) + y^T z_2)\,dt,$
(ix) E and F are compact convex sets in R^n,
(x) J and K are compact convex sets inR^m.

In order that the problem is suitably defined, we take it for granted that in the primal and the dual problems, defined above, numerators are nonnegative and denominators are positive. First of all, we convert our problem to a parametric problem by introducing l and m, defined below, whose optimal value of the objective function is the same as the optimal value of the objective function in the problem defined earlier. Let us choose

$$l = \frac{\int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Ap(t) + s(x|E) - y^T z_1)\,dt}{\int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Bp(t) - s(x|F) + y^T z_2)\,dt},$$

$$m = \frac{\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Yq(t) - s(x|J) + u^T r_1)\,dt}{\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Zq(t) + s(x|K) - u^T r_2)\,dt}.$$

Equivalently, above problems can be stated as

Primal (PP$'$) Minimize l

subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$

$$y(a) = 0 = y(b), \quad \dot{y}(a) = 0 = \dot{y}(b),$$

$$\int_a^b (f(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Ap(t) + s(x|E) - y^T z_1)\, dt$$

$$-l\int_a^b (g(t,x,\dot{x},y,\dot{y}) - \frac{1}{2}p(t)^T Bp(t) - s(x|F) + y^T z_2)\, dt = 0, \quad (1)$$

$$(f_y - Df_{\dot{y}} + Ap(t) - z_1) - l(g_y - Dg_{\dot{y}} + Bp(t) + z_2) \in C_2^*, \ t \in I, \quad (2)$$

$$y(t)^T\{(f_y - Df_{\dot{y}} + Ap(t) - z_1) - l(g_y - Dg_{\dot{y}} + Bp(t) + z_2)\} \geqq 0, \ t \in I, \quad (3)$$

$$x(t) \in C_1,$$

$$z_1 \in J, \qquad z_2 \in K,$$

Dual (DP$'$) Maximize m

subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b),$$

$$v(a) = 0 = v(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$\int_a^b (f(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Yq(t) - s(v|J) + u^T r_1)\, dt$$

$$-m\int_a^b (g(t,u,\dot{u},v,\dot{v}) - \frac{1}{2}q(t)^T Zq(t) + s(v|K) - u^T r_2)\, dt = 0, \quad (4)$$

$$-[(f_x - Df_{\dot{x}} + Yq(t) + r_1) - m(g_x - Dg_{\dot{x}} + Zq(t) - r_2)] \in C_1^*, \ t \in I, \quad (5)$$

$$u(t)^T\{(f_x - Df_{\dot{x}} + Yq(t) + r_1) - m(g_x - Dg_{\dot{x}} + Zq(t) - r_2)\} \leqq 0, \ t \in I, \quad (6)$$

$$v(t) \in C_2,$$

$$r_1 \in E, \qquad r_2 \in F.$$

Remark 3.1

(a) The problems (PP) and (DP) will reduce to the problem considered by Jayswal and Jha [9], if we take $E = F = J = K = \{0\}$.

(b) In addition, if $A = B = Y = Z = 0$, then we will get the problem studied by Ahmad *et al.* [2].

4. DUALITY THEOREMS

In this section, we derive weak, strong and converse duality theorems. The following weak and strong duality theorems are discussed in terms of (PP$'$) and (DP$'$), but apply equally to (PP) and (DP).

Theorem 1. *(Weak duality). Let (x,y,l,p,z_1,z_2) and (u,v,m,q,r_1,r_2) be feasible solutions to primal (PP$'$) and dual (DP$'$), respectively. Further, assume that*

(a) $\int_a^b (f(t,.,.,v(t),\dot{v}(t)) + (.)^T r_1 - m(g(t,.,.,v(t),\dot{v}(t)) - (.)^T r_2))\, dt$ is second order $\mathcal{F}$-convex at $u(t)$ for fixed $v(t)$,

(b) $-\int_a^b (f(t,x(t),\dot{x}(t),.,.) - (.)^T z_1 - l(g(t,x(t),\dot{x}(t),.,.) - (.)^T z_2))\,dt$ *is second order* $\mathcal{G}$*-convex at* $y(t)$ *for fixed* $x(t)$*,*

(c) $\mathcal{F}(t,x,u;\xi) + u^T\xi \geqq 0,\ \forall x,u \in C_1,\ \ -\xi \in C_1^*,\ t \in I,$

(d) $\mathcal{G}(t,v,y;\zeta) + y^T\zeta \geqq 0,\ \forall v,y \in C_2,\ \ -\zeta \in C_2^*,\ t \in I$ *and*

(e) $\int_a^b (g(t,x,\dot{x},v,\dot{x}) + v^T z_2 - x^T r_2)\,dt \geqq 0.$

Then $l \geqq m$*.*

Proof. From the assumption (c) and constraint (5), we have

$$\mathcal{F}(t,x,u;(f_x - Df_{\dot{x}} + Yq(t) + r_1) - m(g_x - Dg_{\dot{x}} + Zq(t) - r_2)) \\ + u^T\{(f_x - Df_{\dot{x}} + Yq(t) + r_1) - m(g_x - Dg_{\dot{x}} + Zq(t) - r_2)\} \geqq 0,$$

which by the virtue of (6) becomes

$$\mathcal{F}(t,x,u;(f_x - Df_{\dot{x}} + Yq(t) + r_1) - m(g_x - Dg_{\dot{x}} + Zq(t) - r_2)) \geqq 0. \tag{7}$$

Since $\int_a^b (f(t,.,.,v(t),\dot{v}(t)) + (.)^T r_1 - m(g(t,.,.,v(t),\dot{v}(t)) - (.)^T r_2))\,dt$ is second order $\mathcal{F}$-convex at $u(t)$ for fixed $v(t)$, we have

$$\int_a^b (f(t,x,\dot{x},v,\dot{v}) + x^T r_1 - f(t,u,\dot{u},v,\dot{v}) + \frac{1}{2}q(t)^T Y q(t) - u^T r_1) \\ - m(g(t,x,\dot{x},v,\dot{v}) - x^T r_2 - g(t,u,\dot{u},v,\dot{v}) + u^T r_2 + \frac{1}{2}q(t)^T Z q(t)))\,dt \\ \geqq \mathcal{F}(t,x,u;(f_x - Df_{\dot{x}} + Yq(t) + r_1) - m(g_x - Dg_{\dot{x}} + Zq(t) - r_2)),$$

which due to (7) reduces to

$$\int_a^b (f(t,x,\dot{x},v,\dot{v}) + x^T r_1 - f(t,u,\dot{u},v,\dot{v}) + \frac{1}{2}q(t)^T Y q(t) - u^T r_1) \\ - m(g(t,x,\dot{x},v,\dot{v}) - x^T r_2 - g(t,u,\dot{u},v,\dot{v}) + u^T r_2 + \frac{1}{2}q(t)^T Z q(t)))\,dt \geqq 0.$$

This can be rewritten as,

$$\int_a^b (f(t,x,\dot{x},v,\dot{v}) + x^T r_1 - f(t,u,\dot{u},v,\dot{v}) + \frac{1}{2}q(t)^T Y q(t) - u^T r_1) \\ + m(g(t,u,\dot{u},v,\dot{v}) + v^T z_2 - u^T r_2 - \frac{1}{2}q(t)^T Z q(t)) \\ - m(g(t,x,\dot{x},v,\dot{v}) + v^T z_2 - x^T r_2)\,dt \geqq 0.$$

Using (4) together with $v^T z_2 \leqq s(v|K)$ in the above inequality, we get

$$\int_a^b (f(t,x,\dot{x},v,\dot{v}) + x^T r_1 - s(v|J)) - mg(t,x,\dot{x},v,\dot{v}) + v^T z_2 - x^T r_2)\,dt \geqq 0. \tag{8}$$

Similarly, as $-\int_a^b(f(t,x(t),\dot{x}(t),.,.)-(.)^T z_1 - l(g(t,x(t),\dot{x}(t),.,.)-(.)^T z_2))\,dt$ at $y(t)$ is second order $\mathcal{G}$-convex for fixed $x(t)$, we get

$$\int_a^b(-f(t,x,\dot{x},v,\dot{v})+v^T z_1 - s(x|E))+lg(t,x,\dot{x},v,\dot{v})-x^T r_2+v^T z_2)\,dt \geqq 0. \quad (9)$$

On adding (8) and (9), we get

$$\int_a^b((v^T z_1 - s(v|J)+x^T r_1 - s(x|E))+(l-m)(g(t,x,\dot{x},v,\dot{x})+v^T z_2 - x^T r_2))\,dt \geqq 0.$$

Since $v^T z_1 \leqq s(v|J), x^T r_1 \leqq s(x|E)$ the above inequality yields

$$\int_a^b(l-m)(g(t,x,\dot{x},v,\dot{x})+v^T z_2 - x^T r_2)\,dt \geqq 0,$$

which due to (e) gives

$$l \geqq m.$$

This completes the proof. □

Theorem 2. *(Strong Duality). Let us assume that*

(i) $(\bar{x},\bar{y},\bar{l},\bar{p},\bar{z}_1,\bar{z}_2)$ *is an optimal solution of (PP′),*

(ii) the matrix $A-\bar{l}B$ *is nonsingular,*

(iii) $(f_y-\bar{z}_1)-\bar{l}(g_y+\bar{z}_2)-D(f_{\dot{y}}-\bar{l}g_{\dot{y}})+(A-\bar{l}B)\bar{p}(t)\neq 0$ *and*

(iv) the matrix

$$\Big((A\bar{p}(t)_y-\bar{l}(B\bar{p}(t))_y-D(A\bar{p}(t))_{\dot{y}}+\bar{l}D(B\bar{p}(t))_{\dot{y}}+D^2(A\bar{p}(t))_{\ddot{y}}-\bar{l}D^2(B\bar{p}(t))_{\ddot{y}}$$
$$-D^3(A\bar{p}(t))_{\dddot{y}}+\bar{l}D^3(B\bar{p}(t))_{\dddot{y}}+D^4(A\bar{p}(t))_{\ddddot{y}}-\bar{l}D^4(B\bar{p}(t))_{\ddddot{y}}\Big)$$

is positive or negative definite.

Then there exist $\bar{r}_1\in E,\ \bar{r}_2\in F$ *such that* $(\bar{x},\bar{y},\bar{l},\bar{p},\bar{r}_1,\bar{r}_2)$ *is a solution of (DP′). If, in addition, the conditions of Theorem 4.1 are satisfied, then* $(\bar{x},\bar{y},\bar{l},\bar{p}=0,\bar{r_1},\bar{r_2})$ *is an optimal solution of (DP′).*

Proof. Since $(\bar{x},\bar{y},\bar{l},\bar{p},\bar{z}_1,\bar{z}_2)$ is an optimal solution of (PP′), there exist $\alpha\in R,\ \beta\in R,\ \gamma\in C_2$ and $\xi\in R$ satisfying the following Fritz John optimality conditions at the point $(\bar{x}(t),\bar{y}(t),\bar{l},\bar{p}(t))$:

$$\Big[\beta\Big((f_x+\bar{r}_1)-\bar{l}(g_x-\bar{r}_2)-D(f_{\dot{x}}-\bar{l}g_{\dot{x}})-\frac{1}{2}(\bar{p}(t)^T A\bar{p}(t))_x$$
$$+\frac{\bar{l}}{2}(\bar{p}(t)^T B\bar{p}(t))_x+\frac{1}{2}D(\bar{p}(t)^T A\bar{p}(t))_{\dot{x}}$$

$$
\begin{aligned}
&-\frac{\bar{l}}{2}D(\bar{p}(t)^T B\bar{p}(t))_{\dot{x}} - \frac{1}{2}D^2(\bar{p}(t)^T A\bar{p}(t))_{\ddot{x}} + \frac{\bar{l}}{2}D^2(\bar{p}(t)^T B\bar{p}(t))_{\ddot{x}} \\
&+\frac{1}{2}D^3(\bar{p}(t)^T A\bar{p}(t))_{\dddot{x}} - \frac{\bar{l}}{2}D^3(\bar{p}(t)^T B\bar{p}(t))_{\dddot{x}} - \frac{1}{2}D^4(\bar{p}(t)^T A\bar{p}(t))_{\ddddot{x}} \\
&+\frac{\bar{l}}{2}D^4(\bar{p}(t)^T B\bar{p}(t))_{\ddddot{x}}\Big) + (\gamma - \xi\bar{y})\Big(f_{yx} - \bar{l}g_{yx} - D(f_{y\dot{x}} - \bar{l}g_{y\dot{x}}) \\
&-D(f_{\dot{y}x} - \bar{l}g_{\dot{y}x}) + D^2(f_{\dot{y}\dot{x}} - \bar{l}g_{\dot{y}\dot{x}}) - D^3(f_{\dot{y}\ddot{x}} - \bar{l}^3 g_{\dot{y}\ddot{x}}) + (A\bar{p}(t))_x \\
&-\bar{l}(B\bar{p}(t))_x - D((A\bar{p}(t))_{\dot{x}} - \bar{l}(B\bar{p}(t))_{\dot{x}}) + D^2((A\bar{p}(t))_{\ddot{x}} - \bar{l}(B\bar{p}(t))_{\ddot{x}}) \\
&-D^3((A\bar{p}(t))_{\dddot{x}} - \bar{l}(B\bar{p}(t))_{\dddot{x}}) + D^4((A\bar{p}(t))_{\ddddot{x}} - \bar{l}(B\bar{p}^i(t))_{\ddddot{x}}\Big)\Big] \\
&(x(t) - \bar{x}(t)) \geqq 0,\ t \in I, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
&(\beta-\xi)((f_y - Df_{\dot{y}} - z_1) - \bar{l}(g_y - Dg_{\dot{y}} + z_2)) + \beta\big(-\frac{1}{2}(\bar{p}(t)^T A)_y \\
&+\frac{\bar{l}}{2}(\bar{p}(t)^T B)_y + \frac{1}{2}D(\bar{p}(t)^T A\bar{p}(t))_{\dot{y}} \\
&-\frac{\bar{l}}{2}D(\bar{p}^T(t)B\bar{p}(t))_{\dot{y}} - \frac{1}{2}D^2(\bar{p}(t)^T A\bar{p}(t))_{\ddot{y}} + \frac{\bar{l}}{2}D^2(\bar{p}(t)^T B\bar{p}(t))_{\ddot{y}} \\
&+\frac{1}{2}D^3(\bar{p}(t)^T A\bar{p}(t))_{\dddot{y}} - \frac{\bar{l}}{2}D^3(\bar{p}(t)^T B\bar{p}(t))_{\dddot{y}} - \frac{1}{2}D^4(\bar{p}(t)^T A\bar{p}(t))_{\ddddot{y}} \\
&+\frac{\bar{l}}{2}D^4(\bar{p}(t)^T B\bar{p}(t))_{\ddddot{y}}\Big) + (\gamma - \xi\bar{y})\Big(A - \bar{l}B + (A\bar{p}(t))_y - \bar{l}(B\bar{p}(t))_y \\
&-D(A\bar{p}(t))_{\dot{y}} + \bar{l}D(B\bar{p}(t))_{\dot{y}}) + D^2(A\bar{p}(t))_{\ddot{y}} + \bar{l}D(B\bar{p}(t))_{\ddot{y}} - D^3(A\bar{p}(t))_{\dddot{y}} \\
&+\bar{l}D(B\bar{p}(t))_{\dddot{y}} + D^4(A\bar{p}(t))_{\ddddot{y}} - \bar{l}D(B\bar{p}(t))_{\ddddot{y}}\Big) \\
&-\xi(-A\bar{p}(t) + \bar{l}B\bar{p}(t)) = 0,\ t \in I, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\alpha - \beta(g - \frac{\bar{l}}{2}\bar{p}^T(t)Bp(t) - s(\bar{x}|F) + y^T\bar{z}_2) \\
&+(\gamma - \xi\bar{y}(t))(-g_y + Dg_{\dot{y}} - B\bar{p}(t) + \bar{z}_2) = 0,\ t \in I, \qquad (12)
\end{aligned}
$$

$$-\beta(A\bar{p}(t) - \bar{l}B\bar{p}(t)) + (\gamma - \xi\bar{y}(t))(A - \bar{l}B) = 0,\ t \in I, \qquad (13)$$

$$\gamma((f_y - \bar{z}_1) - \bar{l}(g_y + \bar{z}_2) - D(f_{\dot{y}} - \bar{l}g_{\dot{y}}) + Ap(t) - \bar{l}Bp(t)) = 0,\ t \in I, \qquad (14)$$

$$\xi\bar{y}(t)((f_y - \bar{z}_1) - \bar{l}(g_y + \bar{z}_2) - D(f_{\dot{y}} - \bar{l}g_{\dot{y}}) + Ap(t) - \bar{l}Bp(t)) = 0,\ t \in I, \qquad (15)$$

$$s(\bar{x}|E) = \bar{x}^T\bar{r}_1,\ \bar{r}_1 \in E, \qquad (16)$$

$$s(\bar{x}|F) = \bar{x}^T r_2,\ \bar{r}_2 \in F, \qquad (17)$$

$$\beta\bar{y}^T + [\gamma - \xi\bar{y}] \in N_J(z_1), \qquad (18)$$

$$\bar{l}[\beta\bar{y}^T + [\gamma - \xi\bar{y}]] \in N_K(z_2), \qquad (19)$$

$$(\alpha, \beta(t), \gamma, \xi) \neq 0,\ t \in I, \qquad (20)$$

$$(\alpha, \beta(t), \gamma, \xi) \geqq 0, \ t \in I. \tag{21}$$

With the assumption (ii), equation (13) yields

$$\gamma - \xi\bar{y} = \beta\bar{p}(t). \tag{22}$$

Converting (11) into a suitable form, we get

$$(\beta - \xi(t))((f_y - \bar{z}_1) - \bar{l}(g_y + \bar{z}_2) - D(f_{\dot{y}} - \bar{l}g_{\dot{y}})) + (A - \bar{l}B)(\gamma - \xi\bar{y}(t) - \xi\bar{p}(t))$$
$$+\Big((A\bar{p}(t))_y - \bar{l}(B\bar{p}(t))_y - D(A\bar{p}(t))_{\dot{y}} + \bar{l}D(B\bar{p}(t))_{\dot{y}}$$
$$+D^2(A\bar{p}(t))_{\ddot{y}} - \bar{l}D^2(B\bar{p}(t))_{\ddot{y}} - D^3(A\bar{p}(t))_{\dddot{y}}$$
$$+\bar{l}D^3(B\bar{p}(t))_{\dddot{y}} + D^4(A\bar{p}(t))_{\ddddot{y}} - \bar{l}D^4(B\bar{p}(t))_{\ddddot{y}}\Big)$$
$$(\gamma - \xi\bar{y}(t) - \frac{1}{2}\beta\bar{p}(t)) = 0.$$

In the light of (22), the above equation becomes

$$(\beta - \xi(t))((f_y - \bar{z}_1) - \bar{l}(g_y + \bar{z}_2) - D(f_{\dot{y}} - \bar{l}g_{\dot{y}})) + (A - \bar{l}B)\bar{p}(t))$$
$$+\frac{1}{2}(\gamma - \xi\bar{y}(t))((A\bar{p}(t))_y - \bar{l}(B\bar{p}(t))_y$$
$$-D(A\bar{p}(t))_{\dot{y}} + \bar{l}D(B\bar{p}(t))_{\dot{y}} + D^2(A\bar{p}(t))_{\ddot{y}} - \bar{l}D^2(B\bar{p}(t))_{\ddot{y}} - D^3(A\bar{p}(t))_{\dddot{y}}$$
$$+\bar{l}D^3(B\bar{p}(t))_{\dddot{y}} + D^4(A\bar{p}(t))_{\ddddot{y}} - \bar{l}D^4(B\bar{p}(t))_{\ddddot{y}}) = 0. \tag{23}$$

Multiplying $\gamma - \xi\bar{y}(t)$ to both sides of (14) and using (15), the above relation gives

$$\frac{1}{2}(\gamma - \xi\bar{y}(t))\Big((A\bar{p}(t)_y - \bar{l}(B\bar{p}(t))_y - D(A\bar{p}(t))_{\dot{y}} + \bar{l}D(B\bar{p}(t))_{\dot{y}}$$
$$+D^2(A\bar{p}(t))_{\ddot{y}} - \bar{l}D^2(B\bar{p}(t))_{\ddot{y}} - D^3(A\bar{p}(t))_{\dddot{y}} + \bar{l}D^3(B\bar{p}(t))_{\dddot{y}}$$
$$+D^4(A\bar{p}(t))_{\ddddot{y}} - \bar{l}D^4(B\bar{p}(t))_{\ddddot{y}}\Big) = 0,$$

which due to hypothesis (iv) provides

$$\gamma = \xi\bar{y}(t). \tag{24}$$

On substituting (24) in (23), we obtain

$$(\beta - \xi(t))((f_y - \bar{z}_1) - \bar{l}(g_y + \bar{z}_2) - D(f_{\dot{y}} - \bar{l}g_{\dot{y}})) + (A - \bar{l}B)\bar{p}(t) = 0, \tag{25}$$

which by hypothesis (iii) leads to

$$\beta = \xi(t). \tag{26}$$

Now, if we substitute $\xi(t) = 0$ in (26), we get $\beta = 0$ which leads to $\gamma = 0$ on using (24). Moreover, we use (12) to get $\alpha = 0$. Finally, we obtain $(\alpha, \beta(t), \gamma, \xi) \neq 0, \ t \in$

I contradicting (16). Therefore, we take $\xi(t) > 0,\ t \in I$, and hence $\beta > 0$. The fact that $\xi(t) > 0,\ t \in I$ along with (24) yield

$$\bar{y}(t) = \frac{\gamma(t)}{\xi(t)} \in C_2,\ t \in I.$$

Using relations (24) and (26) in (10), we obtain

$$\beta((f_x + \bar{r}_1) - \bar{l}(g_x - \bar{r}_2) - D(f_{\dot{x}} - \bar{l}g_{\dot{x}}))(x(t) - \bar{x}(t)) \geqq 0,\ t \in I. \tag{27}$$

Suppose $x(t) \in C_1$ so that $x(t) + \bar{x}(t) \in C_1$. Replacing $x(t) + \bar{x}(t)$ in place of $x(t)$ in (27), we get

$$x(t)^T((f_x + \bar{r}_1) - \bar{l}(g_x - \bar{r}_2) - D(f_{\dot{x}} - \bar{l}g_{\dot{x}}))(x(t) - \bar{x}(t)) \geqq 0,\ t \in I,$$

which by the property of polar cone gives

$$-((f_x + \bar{r}_1) - \bar{l}(g_x - \bar{r}_2) - D(f_{\dot{x}} - \bar{l}g_{\dot{x}}))(x(t) - \bar{x}(t)) \in C_1^*,\ t \in I.$$

Again, if we take $x(t) = 0$ and $x(t) = 2\bar{x}(t)$ at the same time in equation (27), we have

$$\bar{x}(t)((f_x + \bar{r}_1) - \bar{l}(g_x - \bar{r}_2) - D(f_{\dot{x}} - \bar{l}g_{\dot{x}}))(x(t) - \bar{x}(t)) = 0,\ t \in I.$$

Thus, it becomes clear that $(\bar{x}(t), \bar{y}(t), \bar{l}, \bar{p}(t), \bar{r_1}, \bar{r_2})$ is a feasible solution to (DP′).

Further, with the help of (18), (24) and (26), we have $\bar{y} \in N_J(\bar{z_1})$ and since J is a compact convex set in R^m, one can conclude $\bar{y}^T\bar{z_1} = s(\bar{y}|J)$. Similarly, $\bar{y}^T\bar{z_2} = s(\bar{y}|K)$. So, (PP′) and (DP′) have equal objective function values. The optimality for (DP′) can be seen in the light of weak duality theorem. □

A converse duality theorem can be formulated in the light of above theorem and its proof will be along the same lines as that of proof of Theorem 4.2.

Theorem 3. *(Converse Duality). Let us assume that*

(i) $(\bar{u}, \bar{v}, \bar{m}, \bar{q}(t), \bar{r}_1, \bar{r}_2)$ *is an optimal solution of (DP′),*

(ii) the matrix $Y - \bar{m}Z$ *is nonsingular,*

(iii) $f_x - \bar{r}_1 - \bar{m}(g_x + \bar{r}_2 - D(f_{\dot{x}} - \bar{m}g_{\dot{x}}) + (Y - \bar{m}Z)\bar{q}(t) \neq 0$, *and*

(iv) the matrix

$$\Big((Y\bar{q}(t))_x - \bar{m}(Z\bar{q}(t))_x - D(B\bar{q}(t))_{\dot{x}} + \bar{m}D(Z\bar{q}(t))_{\dot{x}} + D^2(Y\bar{q}(t))_{\ddot{x}} - \bar{m}D^2(Z\bar{q}(t))_{\ddot{x}}$$
$$-D^3(Y\bar{q}(t))_{\dddot{x}} + \bar{m}D^3(Z\bar{q}(t))_{\dddot{x}} + D^4(Y\bar{q}(t))_{\ddddot{x}} - \bar{m}D^4(Z\bar{q}(t))_{\ddddot{x}}\Big)$$

is positive or negative definite.

Then there exist $\bar{z}_1 \in J,\ \bar{z}_2 \in K$ *such that* $(\bar{x}, \bar{y}, \bar{m}, \bar{q}, \bar{z}_1, \bar{z}_2)$ *is a solution of (PP′). If, in addition, the conditions of Theorem 4.1 are satisfied, then* $(\bar{x}, \bar{y}, \bar{m}, \bar{q} = 0, \bar{z_1}, \bar{z_2})$ *is an optimal solution of (PP′).*

5. STATIC SYMMETRIC DUAL PROGRAM

If the time dependency of the problems (PP) and (DP) is waived off then our problems transform into the second order fractional dual symmetric programs over cones given below:

Primal Problem (SPP)

$$\text{Minimize} \qquad \frac{f(x,y) - \frac{1}{2}p^T\nabla_{yy}f(x,y)p + s(x|E) - y^T z_1}{g(x,y) - \frac{1}{2}p^T\nabla_{yy}g(x,y)p - s(x|F) + y^T z_2}$$

subject to

$$(g(x,y) - \frac{1}{2}p^T\nabla_{yy}g(x,y)p - s(x|F) + y^T z_2)(\nabla_y f(x,y) + \nabla_{yy}f(x,y)p - z_1)$$

$$-(f(x,y) - \frac{1}{2}p^T\nabla_{yy}f(x,y)p + s(x|E) - y^T z_1)(\nabla_y g(x,y) + \nabla_{yy}g(x,y)p + z_2) \in C_2^*,$$

$$y^T\Big[(g(x,y) - \frac{1}{2}p^T\nabla_{yy}g(x,y)p - s(x|F) + y^T z_2)(\nabla_y f(x,y) + \nabla_{yy}f(x,y)p - z_1)$$

$$-(f(x,y) - \frac{1}{2}p^T\nabla_{yy}f(x,y)p + s(x|E) - y^T z_1)(\nabla_y g(x,y) + \nabla_{yy}g(x,y)p + z_2)\Big] \geqq 0,$$

$$x \in C_1,$$

$$z_1 \in J, \; z_2 \in K.$$

Dual Problem (SDP)

$$\text{Maximize} \qquad \frac{f(u,v) - \frac{1}{2}q^T\nabla_{xx}f(u,v)q - s(v|J) + u^T r_1}{g(u,v) - \frac{1}{2}q^T\nabla_{xx}g(u,v)q + s(v|K) - u^T r_2}$$

subject to

$$-[(g(u,v) - \frac{1}{2}q^T\nabla_{xx}g(u,v)q + s(v|K) - u^T r_2)(\nabla_x f(u,v) + \nabla_{xx}f(u,v)q + r_1)$$

$$-(f(u,v) - \frac{1}{2}q^T\nabla_{xx}f(u,v)q - s(v|J) + u^T r_1)(\nabla_x g(u,v) + \nabla_{xx}g(u,v)q - r_2)] \in C_1^*,$$

$$u^T\Big[(g(u,v) - \frac{1}{2}q^T\nabla_{xx}g(u,v)q + s(v|K) - u^T r_2)(\nabla_x f(u,v) + \nabla_{xx}f(u,v)q + r_1)$$

$$-(f(u,v) - \frac{1}{2}q^T\nabla_{xx}f(u,v)q - s(v|J) + u^T r_1)(\nabla_x g(u,v) + \nabla_{xx}g(u,v)q - r_2)\Big] \leqq 0,$$

$$v \in C_2,$$

$$r_1 \in E, \qquad r_2 \in F.$$

Equivalent forms of the above problems can be written as

Primal Problem (SPP$'$)

$$\text{Minimize} \qquad l$$

subject to

$$(f(x,y)-\frac{1}{2}p^T\nabla_{yy}f(x,y)p+s(x|E)-y^Tz_1)$$
$$-l(g(x,y)-\frac{1}{2}p^T\nabla_{yy}g(x,y)p-s(x|F)+y^Tz_2)=0,$$
$$(g(x,y)-\frac{1}{2}p^T\nabla_{yy}g(x,y)p-s(x|F)+y^Tz_2)(\nabla_y f(x,y)+\nabla_{yy}f(x,y)p-z_1)$$
$$-(f(x,y)-\frac{1}{2}p^T\nabla_{yy}f(x,y)p+s(x|E)-y^Tz_1)(\nabla_y g(x,y)+\nabla_{yy}g(x,y)p+z_2)\in C_2^*,$$
$$y^T\Big[(g(x,y)-\frac{1}{2}p^T\nabla_{yy}g(x,y)p-s(x|F)+y^Tz_2)(\nabla_y f(x,y)+\nabla_{yy}f(x,y)p-z_1)$$
$$-(f(x,y)-\frac{1}{2}p^T\nabla_{yy}f(x,y)p+s(x|E)-y^Tz_1)(\nabla_y g(x,y)+\nabla_{yy}g(x,y)p+z_2)\Big]\geqq 0,$$
$$x\in C_1,$$
$$z_1\in J,\ z_2\in K.$$

Dual Problem (SDP$^{'}$)

$$\text{Maximize} \qquad m$$

subject to

$$(f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q-s(v|J)+u^Tr_1)$$
$$-l(g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q+s(v|K)-u^Tr_2)=0,$$
$$-[(g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q+s(v|K)-u^Tr_2)(\nabla_x f(u,v)+\nabla_{xx}f(u,v)q+r_1)$$
$$-(f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q-s(v|J)+u^Tr_1)(\nabla_x g(u,v)+\nabla_{xx}g(u,v)q-r_2)]\in C_1^*,$$
$$u^T\Big[(g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q+s(v|K)-u^Tr_2)(\nabla_x f(u,v)+\nabla_{xx}f(u,v)q+r_1)$$
$$-(f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q-s(v|J)+u^Tr_1)(\nabla_x g(u,v)+\nabla_{xx}g(u,v)q-r_2)\Big]\leqq 0,$$
$$v\in C_2,$$
$$r_1\in E,\qquad r_2\in F,$$

where

$$l=\frac{f(x,y)-\frac{1}{2}p^T\nabla_{yy}f(x,y)p+s(x|E)-y^Tz_1}{g(x,y)-\frac{1}{2}p^T\nabla_{yy}g(x,y)p-s(x|F)+y^Tz_2},$$
$$m=\frac{f(u,v)-\frac{1}{2}q^T\nabla_{xx}f(u,v)q-s(v|J)+u^Tr_1}{g(u,v)-\frac{1}{2}q^T\nabla_{xx}g(u,v)q+s(v|K)-u^Tr_2}.$$

The weak and strong duality results can be easily established. For details, one can refer to Jayswal and Prasad [10].

6. CONCLUSIONS

In the present work, we have introduced a pair of second order symmetric nondifferentiable fractional variational problems and constructed equivalent problems through parametric approach. Duality results are derived for these equivalent problems but they apply equally to original problems. Further, we can derive duality theorems for second order multiobjective nondifferentiable fractional symmetric dual problems over cone constraints using the same approach and this we leave for the readers.

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